CHAPTER I
FUZZY SUBSETS AND FUZZY TOPOLOGY

1. INTRODUCTION:

In the year 1965, Prof. L.A. Zadeh introduced the concept of a Fuzzy Subset as a generalization of that of an ordinary subset. Fuzzy subsets are the classes of objects with grades of membership ranging between 0 and 1. The introduction of fuzzy subsets paved the way for rapid research work in many areas of Mathematical and Computer Sciences.


The present chapter is intended to provide a brief introduction to fuzzy subsets and fuzzy topology. The concept of a fuzzy subset, operations on fuzzy subsets, fuzzy subsets induced by mappings and fuzzy topological spaces are
discussed in this chapter. More details and other results are contained in Zadeh [56], Kaufmann [25], Chang [9], Warren [47 & 50] and Klir and Yuan [17].

2. THE CONCEPT OF A FUZZY SUBSET

An ordinary subset $A$ of a set $X$ having some elements be characterised by a function called characteristic function $\mu_A : X \rightarrow \{0, 1\}$ of $A$, defined by

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \not\in A. \end{cases}$$

Thus an element $x \in X$ is in $A$ if $\mu_A(x) = 1$ and is not in $A$ if $\mu_A(x) = 0$. Hence $A$ is characterized by its characteristic function $\mu_A : X \rightarrow \{0, 1\}$. In general, if $X$ is a set and $A$ is a subset of $X$ then $A$ has the following representation.

$$A = \{ (x, \mu_A(x)) : x \in X \}$$

Here $\mu_A(x)$, may be regarded as the degree of belongingness of $x$ to $A$, which is either 0 or 1. Hence $A$ is a class of objects with degree of belongingness either 0 or 1 only.

Prof L. A. Zadeh [56] introduced class of objects with continuum grades of belongingness ranging between 0 and 1. He called such a class as a fuzzy subset.

Let $X$ be a set and $\mu_A : X \rightarrow [0, 1]$ be a function from $X$ into the closed unit interval $[0, 1]$, which may take any value between 0 and 1 for an element of $X$. Such a function is called a membership function or membership characteristic function. A fuzzy subset $A$ in $X$ is characterized by a membership function
μₐ : X → [0, 1] which associates with each point in X, a real number μₐ(x) between 0 and 1 which represents the degree or grade of membership or belongingness of x to A. If A is an ordinary subset of X then μₐ can take an either 0 or 1 according as x does or does not belong to A. Then in this case μₐ reduces to the usual characteristic function of A.

Thus a fuzzy subset A of a set X can be written as

A = { (x, μₐ(x)) : x ∈ X }, where μₐ : X → [0, 1] is the membership function.

Equivalently a fuzzy subset A in X is defined as a function from X into closed unit interval [0,1], since A is characterized by its membership characteristic function.

2.1 Definition: [56] A fuzzy subset A in a set X is a function A : X → [0, 1].

2.2 Example: Let X = {a, b, c, d} and μₐ : X → [0, 1] be a function defined by

μₐ(a) = 1, μₐ(b) = 0.666, μₐ(c) = 0, μₐ(d) = 0.4.

Then A = { (a, 1), (b, 0.666), (c, 0), (d, 0.4) } is a fuzzy subset of X.

2.3 Example: Let N be the set of all natural numbers. Define a function μₐ : N → [0, 1] by

μₐ(n) = 0.2 if n is odd

= 0.77 if n is even

Then A = { (1, 0.2), (2, 0.77), (3, 0.2), (4, 0.77), ..........} is a fuzzy subset of N.
A fuzzy subset in $X$ is empty iff its membership function is identically zero on $X$ and it is denoted by $0$ or $\mu_\emptyset$. The set $X$ can be considered a fuzzy subset of $X$ whose membership function is identically $1$ on $X$ and is usually denoted by $1$ or $\mu_X$ or $1_X$.

In fact every subset of $X$ is a fuzzy subset of $X$ but not conversely. Hence the concept of a "fuzzy subset" is a generalization of the concept of a "subset".

3. OPERATIONS ON FUZZY SUBSETS

In this section, the extension of the notions of inclusion, union, intersection and complementation of ordinary subsets to fuzzy subsets, and some of their properties, are given. The definitions and properties contained in this section are from Zadeh [56] and Kaufmann [25].

Throughout this thesis the phrases "fuzzy subset" and "fuzzy set" are interchangeably used.

3.1 Definition: If $A$ and $B$ are any two fuzzy subsets of a set $X$, then "$A$ is said to be included in $B$" or "$A$ is contained in $B$" or "$A$ is less than or equal to $B$" iff

$$A(x) \leq B(x) \text{ for all } x \text{ in } X \text{ and is denoted by } A \leq B.$$ 

Equivalently, $A \leq B$ iff $\mu_A(x) \leq \mu_B(x)$ for all $x$ in $X$.

Every fuzzy subset is included itself and Empty fuzzy subset is included in every fuzzy subset.
3.2 Definition: Two fuzzy subsets A and B of a set X are said to be equal, written
A = B, if A(x) = B(x) for every x in X.

Equivalently, A = B if \( \mu_A(x) = \mu_B(x) \) for every x in X.

Note that A \( \neq \) B if there exists at least one x in X for which \( \mu_A(x) \neq \mu_B(x) \).

3.3 Definition: The complement of a fuzzy subset A in a set X, denoted by A' or 1 - A, is the fuzzy subset of X defined by

\[ A'(x) = 1 - A(x) \] for all x in X

That is \( \mu_A'(x) = 1 - \mu_A(x) \) for all x in X

Note that \( (A')' = A \).

3.4 Definition: The union of two fuzzy subsets A and B in X, denoted by A v B, is a fuzzy subset in X defined by

\[ (A v B)(x) = \text{Max} \{ A(x), B(x) \} \] for all x in X.

Equivalently, \( \mu_{A \cup B}(x) = \text{Max} \{ \mu_A(x), \mu_B(x) \} \) for all x in X.

One can easily show that \( (A v B) v C = A v (B v C) \) for any three fuzzy subsets A, B, C of X.

In general, the union of a family of fuzzy subsets \( \{ A_\lambda : \lambda \in \Lambda \} \) is a fuzzy subset denoted by \( \bigvee_{\lambda \in \Lambda} A_\lambda \) and defined by

\[ (\bigvee_{\lambda \in \Lambda} A_\lambda)(x) = \text{Sup} \{ A_\lambda(x) \} \] for all x in X.
3.5 Definition: The interaction of two fuzzy subsets $A$ and $B$ in $X$, denoted by $A \wedge B$, is a fuzzy subset in $X$ defined by

$$(A \wedge B)(x) = \min \{ A(x), B(x) \} \text{ for all } x \text{ in } X.$$ 

Equivalently, $\mu_{A\wedge B}(x) = \min \{ \mu_A(x), \mu_B(x) \} \text{ for all } x \text{ in } X.$

It can be shown that $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ for any three fuzzy subsist $A$, $B$, $C$ of $X$.

In general, the intersection of a family of fuzzy subsets $\{ A_\lambda : \lambda \in \Lambda \}$ is a fuzzy subset denoted by $\bigwedge_{\lambda \in \Lambda} A_\lambda$ and defined by

$$(\bigwedge_{\lambda \in \Lambda} A_\lambda)(x) = \inf_{\lambda \in \Lambda} \{ A_\lambda(x) \} \text{ for all } x \text{ in } X.$$ 

3.6 Some properties of union and intersection: The following are some basic properties of union and intersection.

Let $X$ be any set and $A$, $B$, $C$ be fuzzy subsets of $X$.

1) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$. 
2) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$. 
3) $A \wedge 0 = 0$ where $0$ is the empty fuzzy subset. 
4) $A \vee 0 = A$ where $0$ is the empty fuzzy subset. 
5) $A \wedge X = A$. 
6) $A \vee X = X$. 
7) $1 - (A \vee B) = (1 - A) \wedge (1 - B)$. 

8) \(1 - (A \land B) = (1 - A) \lor (1 - B)\).

9) \(A - B = A \land (1 - B)\).

Thus the above properties are clear extensions of the basic set theoretic properties to fuzzy subsets.

Note that, the following properties which are true in the case of set theory are no longer true for fuzzy subsets, in general.

1) \(A \land (1 - A) = 0\) except for \(A = 0\) or \(A = X\).

2) \(A \lor (1 - A) = X\) except for \(A = 0\) or \(A = X\).

4. FUZZY SUBSETS INDUCED BY MAPPINGS

The image and inverse image of a fuzzy subset under a mapping were defined by Zadeh [56]. This section includes the two definitions and some of the related properties proved by C. L. Chang [9] and R. H. Wareen [49].

4.1 Definition [56]: Let \(f: X \rightarrow Y\) be a mapping from a set \(X\) into a set \(Y\). Let \(A\) be a fuzzy set in \(X\) and \(B\) be a fuzzy set in \(Y\).

1) The inverse image of \(B\) under \(f\), written \(f^{-1}(B)\), is a fuzzy set in \(X\), defined by

\[
[f^{-1}(B)](x) = B(f(x)) = (B \circ f)(x) \text{ for each } x \text{ in } X.
\]

2) The image of \(A\) under \(f\), written \(f(A)\), is a fuzzy set in \(Y\), defined by
\[
[f( A )](y) = \text{Sup} \{ A(z) : z \in f^{-1}(y) \}
\]
for each \( y \in Y \),
where \( f^{-1}(y) = \{ x \in X : f(x) = y \} \).

The following properties have been proved by Chang [9].

**4.2 Theorem [9]**: Let \( f \) be a mapping from a set \( X \) into a set \( Y \). The following are true.

1) \( f^{-1}(1 - B) = 1 - f^{-1}(B) \) for any fuzzy set \( B \) in \( Y \).

2) \( f(1 - A) \geq 1 - f(A) \) for any fuzzy set \( A \) in \( X \).

3) \( A \leq B \) implies \( f(A) \leq f(B) \) for any two fuzzy sets \( A, B \) in \( X \).

4) \( C \leq D \) implies \( f^{-1}(C) \leq f^{-1}(D) \) for any two fuzzy sets \( C, D \) in \( Y \).

5) \( A \leq f^{-1}[f(A)] \) for any fuzzy set \( A \) in \( X \).

6) \( B \geq f[f^{-1}(B)] \) for any fuzzy set \( B \) in \( Y \).

7) Let \( g \) be a function from \( Y \) to \( Z \). Then \( (g \circ f)^{-1}(C) = f^{-1}[g^{-1}(C)] \) for any fuzzy set \( C \) in \( Z \).

In addition to above properties R. H. Warren [49] proved the following properties.

**4.3 Theorem [49]**: Let \( f \) be a function from a set \( X \) into a set \( Y \). If \( A, A_i, i \in I \) are fuzzy sets in \( X \), and \( B, B_j, j \in J \) are fuzzy sets in \( Y \), then the following results are true.

1) \( f[f^{-1}(B)] = B \), when \( f \) is onto.

2) \( f(\bigwedge_{i \in I} A_i) \leq \bigwedge_{i \in I} f(A_i) \).
3) \( f^{-1}( \bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} f^{-1}(B_j). \)

4) \( f(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} f(A_i). \)

5) \( f^{-1}(\bigvee_{j \in J} B_j) = \bigvee_{j \in J} f^{-1}(B_j). \)

6) \( f[f^{-1}(B) \wedge A] = B \wedge f(A). \)

In this thesis, the word ‘Crisp’ has been used frequently. According to Goguen [18], things unfuzzified or trivially fuzzified are ‘Crisp’. Crispness is the qualitative opposite of fuzziness, although technically, it is a spacial case. The following definition is from Goguen [18].

**4.4 Definition [18]:** A fuzzy set on \( X \) is ‘Crisp’ if it takes only the values 0 and 1 on \( X \).

**4.5 Example:** Let \( X = \{a, b, c\} \) be any set and \( A \) be a fuzzy subset of \( X \) defined by

\[ \mu_A(a) = 1, \mu_A(b) = 0, \mu_A(c) = 1. \]

Then \( A = \{(a, 1), (b, 0), (c, 1)\} = \{a, c\} \) is a crisp fuzzy subset of \( X \).

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5. FUZZY TOPOLOGICAL SPACES

In the year 1968, C. L. Chang [9] introduced the notion of fuzzy topological spaces as an application of fuzzy sets to general topological spaces. Since then several researchers have contributed to the development of fuzzy
topological spaces. In this section, some basic concepts on fuzzy topological space which may be used in the sequel, are included.

5.1 Definition [9]: Let $X$ be a set and $T$ be a family of fuzzy subsets of $X$. $T$ is called a fuzzy topology on $X$ iff $T$ satisfies the following conditions.

1) $\mu_\emptyset$, $\mu_X \in T$. That is $0, 1 \in T$.

2) If $G_i \in T$ for $i \in I$ then $\bigvee_{i \in I} G_i \in T$.

3) If $G, H \in T$ then $G \wedge H \in T$.

The pair $(X, T)$ is called a fuzzy topological (abbreviated as fts). The members of $T$ are called open fuzzy sets and a fuzzy set $A$ in $X$ is said to be closed iff $1 - A$ is an open fuzzy set in $X$.

5.2 Remark: Every topological space is a fuzzy topological space but not conversely.

For example, let $X = \{a, b, c\}$ be a set and $A = \{(a, 0.5), (b, 0.8), (c, 1)\}$ be a fuzzy set in $X$. Let $T = \{1, 0, A\}$. Then $(X, T)$ is a fts which is not a topological space.

The concept of closure of a fuzzy set was introduced in [38].

5.3 Definition [38]: Let $X$ be a fts and $A$ be a fuzzy subset in $X$. Then

$\wedge \{B: B$ is a closed fuzzy set in $X$ and $B \supseteq A\}$ is called the closure of $A$ and is denoted by $\overline{A}$ or $\text{Cl}(A)$

The properties of closure of a fuzzy set are included in the following.
5.4 Theorem [48 and 49]: Let A and B be two fuzzy sets in a fts (X, T). Then the following results are true.

1) $\bar{A}$ is a closed fuzzy set in X.

2) $\bar{A}$ is the least closed fuzzy set in X which is greater than or equal to A.

3) A is closed iff $\bar{A} = \bar{A}$.

4) $0 = 0$, where 0 is the empty fuzzy set.

5) $\bar{A} = \bar{A}$.

6) $\bar{A} \lor \bar{B} = \bar{A} \lor \bar{B}$.

7) $\bar{A} \land \bar{B} \geq \bar{A} \land \bar{B}$.

8) $A \leq B$ implies $\bar{A} \leq \bar{B}$.

The interior of a fuzzy set was defined by Chang as follows.

5.5 Definition [9]: Let A and B be two fuzzy sets in a fuzzy topological space (X, T) and let $A \geq B$. Then B is called an interior fuzzy set of A if there exists $G \in T$ such that $A \geq G \geq B$. The least upper bound of all interior fuzzy sets of A is called the interior of A and is denoted by $A^\circ$.

Some basic properties which are extension of the corresponding results in general topology, are given below.

5.6 Theorem [9, 48 and 49]: Let $X$ be a fts and A and B be two fuzzy sets in X. The following results hold good.
1) $A^0$ is an open fuzzy set in $X$.

2) $A^0$ is the largest open fuzzy set in $X$ which is less than or equal to $A$.

3) $A$ is open iff $A = A^0$.

4) $A \leq B$ iff $A^0 \leq B^0$.

5) $(A^0)^0 = A^0$.

6) $A^0 \land B^0 = (A \land B)^0$.

7) $A^0 \lor B^0 \leq (A \lor B)^0$.

8) $(1 - A)^0 = 1 - \overline{A}$.

9) $\overline{1 - A} = 1 - A^0$.

The concept of a base and a subbase for fuzzy topology were introduced by Goguen.

5.7 Definition [19]: Let $(X, T)$ be a fts. A subfamily $\mathcal{B}$ of $T$ is called a base for $T$ iff for each $A$ in $T$, there exists a subfamily $\mathcal{B}_A$ of $\mathcal{B}$ such that $A = \bigvee \mathcal{B}_A$.

A subfamily $\mathcal{S}$ of $T$ is called a subbase for $T$ iff the family $\mathcal{S} = \{ \land \mathcal{J} : \mathcal{J} \text{ is a finite subfamily of } \mathcal{S} \}$ is a base for $T$.

The following characterization of a base is due to R. H. Warren.

5.8 Theorem [49]: Let $T$ be a fuzzy topology on $X$ and $\mathcal{S}$ be a subfamily of $T$. Then the following two properties are equivalent.

1) $\mathcal{S}$ is a base for $T$. 

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2) For each \( G \in T \), for each \( x \in X \), such that \( G(x) > 0 \) and for each real number \( \varepsilon > 0 \), there is \( B \) in \( \emptyset \) such that \( B \leq G \) and \( G(x) - B(x) < \varepsilon \).

The concept of relative fuzzy topology is due to R. H. Warren.

5.9 Definition [49]: Let \((X, T)\) be a fts and let \( A \) be a crisp subset of \( X \). Then the family \( T_A = \{ G|_A : G \in T \} \) is a fuzzy topology on \( A \), where \( G|_A \) is the restriction of \( G \) to \( A \). The fuzzy topology \( T_A \) is called the relative fuzzy topology on \( A \) or the fuzzy topology on \( A \) induced by the fuzzy topology \( T \) on \( X \). Also \( (A, T_A) \) is called the fuzzy subspace of \((X, T)\).

The concept of continuity of maps for fuzzy topological spaces was defined by Chang C. L.

5.10 Definition [9]: Let \( f : X \to Y \) be a function from a fts \((X, T)\) to a fts \((Y, S)\). Then \( f \) is said to be fuzzy continuous (F-continuous) iff for each \( B \in S \), \( f^{-1}(B) \in T \).

Some basic characterizations of F-continuous maps established by R. H. Warren are included in the following.

5.11 Theorem [49]: Let \( f \) be a function from a fts \((X, T)\) into a fts \((Y, S)\). Then the following statements are equivalent.

1) \( f \) is F-continuous.

2) The inverse image of every closed fuzzy set in \( Y \) is closed in \( X \).

3) The inverse image of every element of a subbase for \( S \) is in \( T \).
4) For every fuzzy set $A$ in $X$, $f(\overline{A}) \leq f(A)$.

5) For every fuzzy set $B$ in $Y$, $f(\overline{B}) \leq f(\overline{B})$.

6) If the set $G = \{(x, f(x)) : x \in X\}$ has the fuzzy topology inherited as a subspace of $(X \times Y, T \times S)$, then the function $g : X \rightarrow G$ given by $g(x) = (x, f(x))$, is $F$-continuous.

The concept Fuzzy open map and Fuzzy closed map were defined by C. K. Wong [53], and some characterizations given below are due to S. R. Malghan and S. S. Benchalli [33 and 34].

5.12 Definition [53]: A function $f : X \rightarrow Y$ from a fts $X$ into a fts $Y$ is said to be $F$-open (resp. $F$-closed) iff for each open (resp. closed) fuzzy set $A$ in $X$, $f(A)$ is an open (resp. closed) fuzzy set in $Y$.

5.13 Theorem [33 and 34]: Let $(X, T)$ and $(Y, S)$ be two fuzzy topological spaces and $f : X \rightarrow Y$ be a function. Then the following statements are equivalent.

1) $f$ is $F$-open.

2) $f(A^0) \leq f(A)^0$ for each fuzzy set $A$ in $X$.

3) If $B$ is a base for $T$, then $f(A)$ is an open fuzzy set in $Y$ for each $A$ in $B$.

4) $f^{-1}(\overline{B}) \leq f^{-1}(\overline{B})$ for each fuzzy set $B$ in $Y$.

5) $f^{-1}(B^0) \leq f^{-1}(B^0)$ for each fuzzy set $B$ in $Y$.

5.14 Theorem [33]: Let $f$ be a map from a fts $(X, T)$ into a fts $(Y, S)$, then $f$ is $F$-closed iff $f(A) \leq f(A)$ for each fuzzy set $A$ in $X$. 

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5.15 Theorem [34]: Let $f$ be a map from a fts $(X, T)$ into a fts $(Y, S)$. Then $f$ is $F$-closed (F-open), iff for each fuzzy set $A$ in $Y$, and for any open (closed) fuzzy set $B$ in $X$ such that $f^{-1}(A) \neq B$, there is an open (closed) fuzzy set $C$ in $Y$ such that $A \leq C$ and $f^{-1}(C) \leq B$. 