3.1 Introduction

Almost every phase of economic behaviour is affected by uncertainty. The economic system has adapted to uncertainty by methods that facilitate the reallocation of risk among individuals and firms. The most apparent and familiar method of shifting the risks is the insurance method. This chapter studies one aspect of this - the premium control.

Depending upon the object or phenomenon under risk, there are various types of insurance coverage, viz., life-insurance, health insurance, fire insurance, vehicle insurance, etc. Two basic elements present in any insurance contract are the 'premium' and the 'claim'. When a client pays a certain sum to the insurance company as the 'premium', he is buying an insurance protection. This protection varies in amount, duration, and occurrence of the contingent event from one type of insurance to another. Quite often, the coverage is for a one year period. The insurance company has an obligation to provide such an insurance protection,
which is equivalent to the 'debt' the company will have to pay if the client makes a valid 'claim'. In some insurance contracts, there is a clause providing the client with a lumpsum amount of money at the end of the period of coverage if there have been no claims during the period. We shall not consider this kind of policies or contracts.

An insurance contract is generally characterized by the premium paid by the insured and a coverage function specifying the transfer from the insurer to the insured for each possible loss. The admissible coverage functions are restricted to be non-negative and less than the size of the loss. The premium is decided upon by the insurance company. Given the terms of the contract, i.e., the premium and the coverage rules, the individual has an option to enter into a contract or not. In the presence of competing insurance companies, he also has a choice over a wide range of coverage functions. His decision usually depends upon his risk aversion and to the extent that the premium and the coverage functions balance each other. For the same coverage function, if the premium is increased, he may be prompted not to enter into the contract. His inclination may also tilt towards a well established company, or any company with low pecuniary liabilities and high reserve of funds.

When a company commences its operation, it must decide upon the premium rates by combining trial and error with the
experience of other insurance companies and the information regarding mortality available from relevant organisations like the police, traffic inspector’s office, hospitals, etc.

In this chapter, we consider an insurance company which considers a finite time horizon for its operation, within which, it has to ‘prove’ itself by adopting various strategies, and earn a stamp of ‘stability’ and trustworthiness. Its reserves at the end of this period form a decisive factor in determining its future profits. The problem considered is the decision about premium rates, at the beginning of each year.

3.1: Review. How did ‘they’ tackle the insurance risk problem

Before presenting our model, analysis and optimization, we give a brief review of earlier relevant papers. Most of these papers deal with insurance risk from the point of view of the insured. We consider the insurer’s viewpoint.

Raviv (1979), finds the premium P and the coverage function I(x) (corresponding to a loss x) that maximizes the expected utility of the final wealth of the insured subject to the constraint that the insurer’s expected utility is a constant.

Martin Löff (1983) deals with the problem of determining
the premium for successive years in a branch of insurance. He uses methods from mathematical control theory to find suitable premium control methods which meet demands for equity, solvency, and sufficient smoothing of the premium variations in time.

Norberg (1985) extends the result of Lidstone (1905). Considering the variation in the 'net level premium reserve' with time, as a continuous decision process, he investigates the effect on the reserve of the changes in the force of interest. He shows that if under the original conditions, the net reserve increases with the duration, then an increase in the rate of interest produces a decrease in reserves, while a digressive increase in mortality produces a decrease in reserve.

3.2 The Model

We consider an insurance company, which, at the end of each year, can decide upon the premium rates as one out of a finite set of possible values \( R(0), R(1), R(2), \ldots, R(m) \). This option is available for a finite horizon of \( N \) years. The units insured form a homogeneous class in the sense that each unit is exposed to the same class of risks, pays premia at the same rate and is eligible for the same range of risk claims depending upon the loss in the event of a mishap. Each year, units are insured up to the end of that year (calendar or financial). This assumption is not very
restrictive, since if the arrivals of policy holders is uniformly distributed over the year, then, except for the first and the last year, approximately half of the policies carry over from the previous year, and half to the next year.

We make the following simplifying assumptions:

(A1) The premium is to be paid effectively at the beginning of the year; and

(A2) Claims, irrespective of the amount are to be paid at the end of the year, without compensating for the lapsed time between the actual occurrence of the mishap and the payment.

The company wishes to choose its premium rates each year in such a way, that its expected discounted reserve at the end of the horizon is maximized, assuming that money is discounted at the rate $\alpha$ per year, and the number of units insured in any year depends upon the premium rate and amount of effective reserve the company has at the beginning of the year.

We first consider the choice between only two premium rates $R(0)$ and $R(1)$.

**Notations**

Decisions regarding premium rates are made at the beginning of each of the $N$ years. The effective value of an amount $j$ at the beginning of a year is an amount $\alpha^{-1}j$ at the end of the year.
The system is said to be in state $i$ at stage $K$, if its effective amount of reserve $S_K$ is $i$. The state space is $S \subset (-\infty, \infty)$ or $S \in \mathbb{I}$, where $\mathbb{I}$ is the set of integers. We have assumed that the number of units insured in any year depends only on the premium rate and the reserves of the company, and not on the number of years elapsed since its inception. Let

$$A^K(i) = A(i),$$

be the random variable denoting the number of units insured during the year beginning with state $i$, stage $K$, premium is charged at the rate $R(l)$; $l = 1, 0$.

Let $B^K(A(i)) = R_1(i)$ be the random variable denoting the total amount of claims during the year beginning with stage $K$, state $i$.

Let,

$$\lambda_1(i) = E(A_1(i)); \quad l = 1, 0, \quad i \in S,$$

and

$$\mu_1(i) = E(B_1(i)); \quad l = 1, 0, \quad i \in S.$$

If the system is in state $j$ at the end of the finite horizon, i.e., at stage $0$, a terminal reward $f(j)$ is received, $j \geq 0$.

$$A = \{0, 1\},$$

where '0' stands for premium rate $R(0)$, and '1' for premium rate $R(1)$ defines the action space.

The effective amount of reserve $S_K$ at stage $K$, given that
$S_{K+1} = i$, is given by:

$$S_K = \frac{1}{\alpha} \left( i + A_1 K+1(i) R(1) \right) - B_1 K+1(i)$$

This follows from assumptions (A1) and (A2), if premium rate charged in the year beginning with stage $K+1$ is $R(1)$, which we denote by action $R^{K+1}(1)$. Thus, the transition probability $p_{i,j}^1 = P\left\{ S_K = \sum_{j=1}^{N} i_j R^{K+1}(1) \right\}$ is a function of $i$ and $R(1)$ only. Hence, the effective amount of reserves forms a Markov chain for each action $1 \in \{0,1\}$. If state $S$ is discrete,

$$p_{i,j}^1 = P \left\{ \frac{1}{\alpha} A_1(i) R(1) - B_1(i) = j - \frac{1}{\alpha} i \right\} = \sum_{K \in I} P \left\{ A_1(i) = K \right\} P \left\{ B_1(i) = j + \frac{1}{\alpha} (i + A_1(i) R(1))/A_1(i) \right\}$$

The insurance company wishes to maximize the present worth of its expected reserve at the terminal epoch. Equivalently, the criterion of optimality is the expected discounted value of total future increments.

The one step expected increment in the reserves (at the end of the $n$th period) is

$$\frac{1}{\alpha} \lambda_1(i) R(1) - \mu_1(i); 1 = 0,1; \text{ all } n.$$

Then, the expected discounted increments satisfy the
recurrence relation
\[ V_n^{1}(i) = \lambda_1(i) R(1) - \alpha \omega_1(i) + \alpha \sum_j p^1(i,j) V_{n-1}(j) \] (3.2.2)
\[ V_n(i) = \text{Max} \left\{ V_n^{0}(i), V_n^{1}(i) \right\} . \]

Note. The summation operator \( \Sigma \) is to be replaced by \( \int \) and
\( p^1(i,j) \) by the transition p.d.f. if the variables under
consideration are continuous.

3.3 Optimal Policy

THEOREM3.1. Let,
(i) \( A_1(i) R(1) - \alpha B_1(i) \) be stochastically increasing in
\( i, \ 1 = 0,1. \)
(ii) \( p^0(i,j) - p^1(i,j) = 0 \) for \( j < j_1(i) \)
\( < 0 \) for \( j_1(i) \leq j < j_2(i) \)
\( > 0 \) for \( j_2(i) < j \leq j_3(i) \)
\( = 0 \) for \( j_3(i) < j. \)

Where \( j_1(i), j_2(i), j_3(i) \) are non-decreasing functions of
\( i, \) and \( (j_3(i) - j_2(i)) - (j_2(i) - j_1(i)) \) increases with \( i. \)

Then, if \( f(j) \) is an increasing function of \( j, \)
(a) there exists \( i^n, n = 1, \ldots, N \) such that at the \( n \)th
stage, the optimal decision is to use
\[ \begin{cases} R(0), & \text{if } i > i^n, \\ R(1), & \text{if } i \leq i^n; \end{cases} \]
(b) \( V_n(i) \) defined in section3.2 is a non-decreasing function
of \( i, \) for \( n = 0,1, \ldots, N. \)
PROOF. By condition (ii), since the difference between the length of intervals of $j'$ containing the positive and the negative values of $p^0(i,j) - p^1(i,j)$ increases with $i$, and higher values of $j$ fall in the positive interval, we have, for any non-decreasing function $\phi(j)$,

\[ \sum_j \phi(j) (p^0(i,j) - p^1(i,j)) \text{ increases in } i. \quad (3.3.1) \]

Hence,

\[ \sum_j j (p^0(i,j) - p^1(i,j)) \text{ is increasing in } i \quad (3.3.2) \]

But,

\[ \sum_j j p^1(i,j) = \sum_j \mathbb{P} \{ S_k = j \mid S_{k+1} = i, l \} \]

is the expected value of the state at the next decision epoch, given that the state at the present decision epoch is $i$, and action $l$ is taken, $l \in \{0,1\}$.

Therefore, \( (\sum_j j p^1(i,j) - \frac{i}{\sum_j j}) \) is the expected one step increment given that the present state is $i$, i.e.,

\[ \sum_j j p^1(i,j) = \alpha^{-1} i + \alpha^{-1} \lambda_1(i) R(l) - \mu_1(i), \quad 1 = 0,1. \quad (3.3.3) \]

From (3.3.2) and (3.3.4) it is inferred that \[ (\lambda_0(i) R(0) - \alpha \mu_0(i)) - (\lambda_1(i) R(1) - \alpha \mu_1(i)) \] is a non-decreasing function of $i$.

\[ (3.3.4) \]

Again, since $f(j)$ is a non-decreasing function of $j$,

\[ \sum_j f(j) (p^0(i,j) - p^1(i,j)) \text{ is a non-decreasing function of } i. \]
Thus,
\begin{align*}
V_1^0(i) - V_1^1(i) &= \lambda_0^0(i) R(\emptyset) - \lambda_1^1(i) R(1) \\
&\quad - \alpha(\mu_0(i) - \mu_1(i)) \\
&\quad + \alpha \sum_j f(j) (p_0^0(i,j) - p_1^1(i,j))
\end{align*}
is a non-decreasing function of \( i \).

Hence, for \( n = 1 \), there exists a point, say \( i_1^* \) such that
\begin{align*}
V_1^0(i) &= \begin{cases} \\
V_1^0(i), & i > i_1^* \\
V_1^1(i), & i \leq i_1^*
\end{cases}
\end{align*}
Thus (a) holds for \( n = 1 \). We now prove part (b) of the Theorem for \( n = 1 \).

By assumption (i) of the Theorem, \( \lambda_1(i) R(1) - \alpha \mu_1(i) \) is a non-decreasing function of \( i \), for each \( l \). Hence, \( V_1^1(i) \), being the sum of two non-decreasing functions of \( i \) is non-decreasing, and \( V_1^0(i) \) being the maximum of two non-decreasing functions is again non-decreasing.

Now suppose the Theorem is true for \( n = m \). Then, by (b), \( V_m(i) \) is a non-decreasing function of \( i \). By assumption (ii),
\begin{align*}
\sum_j (p_0^0(i,j) - p_1^1(i,j)) V_m(j)
\end{align*}
is a non-decreasing function of \( i \).

\( \uparrow \) stands for a non-decreasing function.
Therefore,
\[
\mathcal{V}_{m+1} (i) - \mathcal{V}_{m+1} (i) = \lambda_{0} (i) R(0) - \lambda_{1} (i) R(1)
\]
\[
- \alpha (\mu_{1} (i) - \mu_{2} (i))
\]
\[
+ \alpha \sum_{j} (p_{0} (i,j) - p_{1} (i,j)) \mathcal{V}_{m} (j)
\]
is a non-decreasing function of \( i \), and
\[
\mathcal{V}_{m+1} (i) = \lambda_{1} (i) R(1) - \alpha \mu_{1} (i)
\]
\[
+ \alpha \sum_{j} p_{1} (i,j) \mathcal{V}_{m} (j)
\]
is non-decreasing function of \( i \). But we have already seen that

**Remark 3.1.** Condition (ii) of the Theorem holds if \( p_{0} (i,j) \) and \( p_{1} (i,j) \) (as also \( p_{0} (i,j) \) and \( p_{1} (i,j) \) \( i \neq j \)), are mere translations of each other and are unimodal such that the distributions are spread about \( i \), and are shifted to the right as \( i \) increases more if \( i \) increases. This condition could be replaced by the conditions of Heyman and Sobel (1984), viz., that \( \sum_{j \geq S} p_{0} (i,j) - \sum_{j \geq S} p_{1} (i,j) \) is non-decreasing functions of \( i \) for each \( S \), and \( \sum_{j \geq S} p_{1} (i,j) \) is non-decreasing function of \( i \) for \( 1 = 0,1, \ldots \).

Since convolution of symmetric unimodal distributions is unimodal (see Feller 1966), a sufficient condition for \( p_{1} (i,j) \) to be unimodal is that \( A_{1} (i) \) and \( B_{1} (i) \) are independent random variables and have symmetric unimodal distribution (p.d.f's., p.m.f.'s.).
This may hold for some kind of insurance.

**REMARKS.** In practical situations condition (i) is true if

\[ A_1(i)R(1) - B_1(i) \text{ is an increasing function of } j \text{ almost surely, for } i = 0, 1; \]

i.e. though the amount of claims may increase with the number of units insured, the \( R(l) \) amount of claims is slower than that of \( A_1(i) \), as \( i \) increases, with probability 1.

**THEOREM 32.** If

\[ A_1(i)R(1) - \alpha B_1(i) \geq (1 - \alpha) \beta(i) \geq 0 \]

almost surely, then \( V_n(i) \) is a non-decreasing function of \( n \), for each \( i \).

**PROOF.**

\[
V_n(i) = \max_{l \in \{0, 1\}} \left\{ \lambda_1(i)R(l) - \alpha \mu_1(i) + \alpha \sum_j p(i, j) V_{n-1}(j) \right\}
\]

Since,

\[
j = \frac{1}{\alpha} (i + a_1(i)R(1)) - b_1(i)
\]

if action (i) is applied, and \( A_1(i) = a_1(i), B_1(i) = b_1(i) \)

are the observed values of the random variables, by the assumption of this theorem, \( p(i, j) = 0 \), for \( j < i \). Thus,

\[
\sum_j p(i, j) V_{n-1}(j) \geq V_{n-1}(i)
\]

if \( V_{n-1}(i) \) is increasing in \( i \). Also, \( \lambda_1(i)R(1) - \alpha \mu_1(i) \) is non negative, being expectation of non-negative random variable.
In particular, for \( n = 1 \),
\[
V_1(i) = \lambda_1(i) R(1) - \alpha \mu_1(i) + \sum_{j \geq i} p_1(i,j) f(j),
\]
\[
= \left( 1 - \alpha F(i) + \alpha f(i) \right) - a \mu_1(i) + \sum_{j \geq i} p_1(i,j) V_{n-2}(j),
\]
\[
= V_0(i).
\]

If for all \( i \), \( V_{n-1}(i) \geq f(i) \), then
\[
V_n(i) = \lambda_1(i) R(1) - \alpha \mu_1(i) + \sum_{j \geq i} p_1(i,j) V_{n-2}(j),
\]
\[
= V_0(i).
\]

Since we have already shown that \( V_{n-1}(i) \geq f(i) \) for each \( i \) under the given conditions, the result follows from the above by applying induction arguments.

**Remark.** If only those premium rates are chosen, and the kind of units envisaged to the insured satisfy the conditions of this Theorem, then the company will have increasing effective reserves as the length of the horizon increases. Since as proved in the Theorem, \( V_n(i) \) is a non-decreasing function of \( n \).

### 3.4 Discussion

We discuss special cases now.

(i) Independent Increments. If the amount of increment is a random variable, which depends only on the premium rate, and not upon the amount of reserve of the company, i.e., for \( l = 0,1 \), \( A_l(i) R(l) - \alpha B_l(i) \), is independent of \( i \), then
\[
\lambda_l(i) R(l) - \alpha \mu_l(i) \text{ is a constant for all } i.
\]
\[ p^1(i,j) = p^1(j-i), \quad (j > i) \]

where \( p^1(j-i) \) is the probability of increment \( j-i \) in one year, when action '1' is taken.

Then, for each \( l \), the state \( j \) at the next stage is a stochastically increasing function of the state \( i \) at the present epoch, since

\[
\sum_{j \geq K} p^1(i,j) = \sum_{j \geq K} p^1(j-i) \leq \sum_{j \geq K} p^1(j-i) + p^1(K-i-1) = \sum_{j \geq K} p^1(i+1,j)
\]

Then if \( \lambda_1(i)R(1) - \alpha B_1(i) \) is stochastically larger than \( \lambda_0(i)R(0) - \alpha B_0(i) \), then

\[
\lambda_1(i)R(1) - \alpha B_1(i) > \lambda_0(i)R(0) - \alpha B_0(i) \geq 0
\]

is a constant. While \( \sum_j p^1(j-i) \nu_{n-1}(j) \) is a non-decreasing function of \( i \).

Further,

\[
\sum_j p^1(j-i) \nu_{n-1}(j) = \sum_j p^1(j-i) \nu_{n-1}(j) - \sum_j p^0(j-i) \nu_{n-1}(j) = \sum_j (p^1(j-i) - p^0(j-i)) \nu_{n-1}(j),
\]

This is a non-decreasing function of \( i \). Hence, in view of the above observations, \( \nu_{n-1}(i) - \nu_0(i) \) is a non-decreasing function of \( i \). Hence, there exists for each \( n \), a value \( i_n^* \), such that \( R(0) \) is the optimal premium rate for \( i \leq i_n^* \), while for \( i > i_n^* \), \( R(1) \) is the optimal premium rate. Here, \( i_n^* \) is the optimal premium rate.
If, on the other hand, the amount of increment under one of the actions, say \( \mathbf{1} \), is a random variable independent of \( i \), while under action \( \mathbf{0} \), the distribution of the amount of increment increases stochastically with \( i \), then the expected one-step increment is an increasing function of \( i \) under action \( \mathbf{1} \) and \( \mathbf{0} \). Further, for each \( n \)

\[
\sum_{j} p^{1}(i-j) n^{-1} = \sum_{j} p^{1}(j-i) n^{-1} (j)
\]

where, \( p^{1}(j-i) \) is the probability of reaching state \( j \) under action \( 1 \), starting from state \( i \). For independent increments process, \( p^{1}(j-i) = p^{1}(j-i) \).

Now,

\[
\sum_{j} p^{1}(j-i) n^{-1} (j) = \sum_{m} p^{1}(m) n^{-1} (m+i)
\]

is an increasing function of \( i \) for each \( 1 \), if \( f(j) \) and hence \( V_{n}(j) \) is non-decreasing function of \( j \) for each \( n \).

We now consider the expression

\[
\lambda^{1}(i) R(1) - \alpha \mu^{1}(i) + \sum_{m} p^{0}(m) V_{n-1}(m+i)
\]

\[
- (\lambda^{1}(i) R(1) - \alpha \mu^{1}(i) + \sum_{m} p^{0}(m) V_{n-1}(m+i))
\]

Since, under action \( \mathbf{0} \), the amount of increment is stochastically increasing in \( i \), \( \sum_{m} p^{0}(m) V_{n-1}(m+i) \) increases faster than does \( \sum_{m} p^{1}(m) V_{n-1}(m+i) \); also, \( \lambda^{1}(i) R(1) - \alpha \mu^{1}(i) \) is constant with respect to \( i \) while \( \lambda^{0}(i) R(0) - \alpha \mu^{0}(i) \) is increasing.
with \( i \). Thus, the expression in (3.4) is a decreasing function of \( i \), and thus, \( R(0) \) is the preferable premium rate for larger values of \( i \), while \( R(1) \) is preferable for small values of \( i \).

**Extension to Larger Action Space.** Suppose we have to choose optimally between \( K \) premium rates \( R(0) \leq R(1) \leq \ldots \leq R(K-1) \). Then, the premium rates can be compared pairwise, eliminating the non-optimal action each time, so that for any \( i \), at most \( K \) comparisons rather than \( \binom{K}{2} \) comparisons are required, if the ordered premium rates satisfy the stochastic ordering properties mentioned above in section 3.4, and 3.3. The elimination procedure would be as follows: Given \( i \), at any stage, find out if \( R(0) \) or \( R(1) \) is optimal. If \( R(0) \) is optimal, we do not have to proceed further. If \( R(1) \) is optimal, compare \( R(1) \) and \( R(2) \). If \( R(1) \) is optimal, it is the best decision. If \( R(2) \) is optimal, compare \( R(2) \) and \( R(3) \). This procedure is to be continued, until, either the premium rate \( R(1) \) turns out to be better than both \( R(1-1) \), and \( R(1+1) \), or the action space gets exhausted.