CHAPTER VI

SOME STATISTICAL INFERENCE PROBLEMS AS MARKOVIAN DECISION PROCESSES

6.1 INTRODUCTION

Sequential and Bayesian decision procedures have been used for a long time (Blackwell and Girshick (1954)) to find solutions to classical statistical inference problems. Some of these inference problems can also be posed as M.D.P. problems and solutions attempted.

In this chapter, we consider two inference problems. One problem is that of testing of more than two simple hypotheses simultaneously, while the other one is the quickest detection of disorder. These problems arise naturally in many practical situations, in general, and the models considered in this thesis (Chapters II to V), in particular.

Ross (1970) has presented the problem of testing of two simple hypotheses, sequentially as an optimal stopping problem, and found an optimal solution using M.D.P. approach. Our discussion is a direct generalization of his work. In section 6.3.1, we give a review of literature on disorder detection problem. Whittle (1993) has also dealt with
6.2 SEQUENTIAL TESTING OF HYPOTHESES

In the inventory problem considered in Chapter IV, suppose that the distribution of supply is not known exactly, but is known to be one out of a given finite set of \( m \) distributions. Before beginning the decision process, it is advisable to test for the hypotheses regarding the distributions. Observations may be made sequentially, one by one, each at some cost, and if the correct distribution is not chosen, there will be loss in future which depends upon the true distribution. Since the true distribution is not known, we may consider the expected loss (risk) under each decision, and use a sequential decision procedure which minimizes this risk.

This is a general problem, which may arise in many situations of control of stochastic processes, where distribution of an uncontrolled random variable is not known. If the random variable affects the control policy, and its distribution affects the future costs, it becomes essential to decide which of the distributions is the most likely distribution.
0.2.1 The Testing Problem As an M.D.P.

We first state the testing problem in the general context:
Let $X$ be a random variable, and let $f(x)$ be its true probability density function (p.d.f.). We then wish to test the hypotheses:

$$H_i : f(x) = f_i(x); \quad i = 1, 2, \ldots, m$$

We assume that any desired number of values of $X$ can be observed sequentially. Each observation costs $C$ units. After observing $x_1, x_2, \ldots, x_k$ ($k \geq 1$), we may either stop observing, and decide upon one of $H_1, H_2, \ldots, H_m$ as true, or pay an additional cost $C$, and observe $x_{k+1}$.

We deal only with those situations where the delay due to waiting to arrive at a conclusion regarding the true hypothesis results in a negligible loss. If we stop taking observations, and choose $H_i$, no loss is incurred if our choice is correct. There is an expected loss $L_i$, if the choice is an incorrect hypothesis $H_i$.

Let

$$p(0) = (p_1(0), p_2(0), \ldots, p_m(0))$$

be a given initial probability set, where

$$p_1(0) = \text{a priori probability of } H_1 \text{ being true.}$$

$$\sum_{i=1}^{m} p_i(0) = 1, \quad 0 \leq p_i(0) \leq 1; \quad i = 1, \ldots, m.$$.

We define the state at the $k^{th}$ observation as
\[ p^{(K)} = (p_1^{(K)}, p_2^{(K)}, \ldots, p_m^{(K)}) \tag{6.2.3} \]

Where,
\[ p_1^{(K)} = \text{posterior probability of } H_1 \text{ being true after } K \text{ observations}, \]
\[ \sum_{j=1}^m p_j^{(K)} = 1 \]

Then the state-space is
\[ S = \{ (p_1, p_2, \ldots, p_m) \mid p_1 \in \{0, 1\}, \quad i = 0, 1, \ldots, m; \quad \sum_{i=1}^m p_i = 1 \} \tag{6.2.4} \]

If we take another observation when the system is in state \( p^{(K)} \), then the value observed will be \( x \) with probability density function
\[ p_1^{(K)} f_1(x) + p_2^{(K)} f_2(x) + \ldots + p_m^{(K)} f_m(x) \tag{6.2.5} \]

The next state will be
\[ p^{(K+1)} = (p_1^{(K+1)}, \ldots, p_m^{(K+1)}) \tag{6.2.6} \]

Where
\[ p_1^{(K+1)} = \frac{p_1^{(K)} f_1(x_{K+1})}{\sum_{j=1}^m p_j^{(K)} f_j(x_{K+1})}, \quad i = 1, \ldots, m \tag{6.2.7} \]

where \( x_{K+1} \) is the observed value leading to stage \( K+1 \).
As described above, our testing procedure can be considered as an m+1 action M.D.P., with non-negative costs, an n-dimensional uncountable state space, considered over an infinite horizon.

(Unlike the previous chapters, we count the stage forward, defining stage K as the decision epoch after K observations. Then \( p^{(K)} \) is the state of the system at stage K).

6.2.2. Expected Losses. The losses incurred, could be possibly due to reduction in the optimum value, consequent on taking a wrong decision. In this section, we consider expectations with respect to the posterior probability distributions of \( X \) at each stage.

If after K observations, we stop and choose \( H_i \), the expected loss (risk) incurred is

\[
(1 - p_i^{(K)}) L_i, \quad i = 1, 2, \ldots, m
\]  

(6.2.8)

If, on the other hand, we decide to take another observation, then an immediate cost 'C' is incurred, followed by the costs incurred in future. We wish to choose our decisions so as to minimize the 'risk' due to wrong decision.

6.2.3 Optimality Considerations. If

\[
p = (p_1, \ldots, p_m)
\]

is the state of the system at any/ stage, and \( V(p) \) the minimum
future risk, then \( V(p) \) must satisfy the relation

\[
V(p) = \min \left\{ (1-p_1)L_1, (1-p_2)L_2, \ldots, (1-p_m)L_m \right\}
\]

\[
= C + \int \sum_{j=1}^{\infty} p_j f_j(x) \sum_{j=1}^{m} p_j f_j(x) \text{ dx } \]

\[
\leq \sum_{j=1}^{\infty} p_j f_j(x) \sum_{j=1}^{m} p_j f_j(x)
\]

Since the cost \( C \) of taking an observation is positive, any policy of continuing observations indefinitely will result in an infinite cost, and hence is not optimal, if the \( L_j \) are all finite and \( p_1 \rightarrow 1 \) as \( K \rightarrow \infty \) and \( H_1 \) is true.

Let \( \Delta \) denote a subclass of the class of all policies, whose action after any \( K \) observations is a function only of \( x_1, \ldots, x_k \), and not of the initial state \( p(0) \). Since the posterior probability \( p(0) \) at stage \( K \) is dependent only on \( p(0) \) and \( x_1, \ldots, x_k \), it follows that given \( p(0) \), any Markovian, i.e., state dependent policy depends only on the observed values. Thus, given \( p(0) \),

\[
V(p) = \min_{n \in \Delta} V_n(p) \quad (6.2.10)
\]

**Lemma 64.** \( V(p) \) is a concave function of \( p \).

**Proof.** Let
\[ p(1) = (p_1(1), \ldots, p_m(1)) \]

and

\[ p(2) = (p_1(2), \ldots, p_m(2)) \]

be any two states. Then, for \( \lambda \in (0,1) \),

\[
p = \lambda p(1) + (1-\lambda) p(2)
\]

\[
= (\lambda p_1(1) + (1-\lambda) p_1(2), \ldots, \lambda p_m(1) + (1-\lambda) p_m(2)) \in S
\]

It follows from (6.2.10) that given \( p(1), p(2) \),

\[
V(\lambda p(1) + (1-\lambda) p(2)) = \min_{\pi \in \Delta} V_{\pi}(\lambda p(1) + (1-\lambda) p(2))
\]  

Since policies in \( \Delta \) are independent of the initial probability we have for \( \pi \in \Delta \),

\[
V_{\pi}(\lambda p(1) + (1-\lambda) p(2)) = \lambda V_{\pi}(p(1)) + (1-\lambda) V_{\pi}(p(2))
\]

Hence,

\[
V(\lambda p(1) + (1-\lambda) p(2)) = \min_{\pi \in \Delta} V_{\pi}(\lambda p(1) + (1-\lambda) p(2))
\]

\[
= \min_{\pi \in \Delta} (\lambda V_{\pi}(p(1)) + (1-\lambda) V_{\pi}(p(2)))
\]

\[
\geq \lambda \min_{\pi \in \Delta} V_{\pi}(p(1)) + (1-\lambda) \min_{\pi \in \Delta} V_{\pi}(p(2))
\]

\[
= \lambda \ V(p(1)) + (1 - \lambda) \ V(p(2)).
\]

Thus, \( V(p) \) is a concave function of \( p \).

\textit{Note 6.1.} Alternative proofs of this result have been given by Blackwell and Girsichik (1954), Whittle (1963).
THEOREM 6.1. There exist convex sets say $p^*_i$, $i = 1, 2, \ldots, m$ such that when the state of the system is $p$, the optimal policy stops and chooses $H_i$ at stage $K$, if $p \not\in P_i^*$ and continues to take further observations otherwise, if $p_{ii} \rightarrow 1$ as $n \rightarrow \infty$ when $H_i$ is true. $P_i^*$ contains, and lies in the vicinity of the point $E_i$, which is a vector with $i$th component unity, and all other components 0.

PROOF. We require to prove that for the optimal policy, the region of states $p$ which call for stopping and deciding to choose hypothesis $H_i$, is convex, and includes the point $E_i$.

Let $p(1)$ and $p(2)$ as defined in (6.2.11) be any two states which call for stopping and selecting $H_i$. Equivalently

$$V(p(1)) = (1 - p_1(1)) L_1$$

and

$$V(p(2)) = (1 - p_1(2)) L_1$$  \hspace{1cm} (6.2.15)

Then, for $p$ defined in (6.2.12), by Lemma 6.1,

$$V(p) \geq \lambda V(p(1)) + (1-\lambda) V(p(2))$$

$$= \lambda (1-p_1(1)) L_1 + (1-\lambda) (1-p_1(2)) L_1 \text{ by (6.2.15)}$$

$$= L_1 - (\lambda p_1(1) + (1-\lambda) p_1(2)) L_1$$

$$= (1-p_1) L_1$$  \hspace{1cm} (6.2.16)

But, by definition of $V(p)$,
Thus, \( V(p) = (1 - p_1) L_1 \) leads to \( V(p) = 0 \), which is an optimum value since all costs are non-negative.

6.3 THE DISORDER PROBLEM

Once again, we consider, as an illustrative example, the inventory model of Chapter IV. If the distribution of supply changes during the period for which the optimal policy has been determined, lack of knowledge about the exact point of change may result in a decrease in the optimal profit, since the strategy applied would correspond to the original distribution. In the simplest situation, there may be exactly one change, with the distribution before and after the change known completely, while the point of change may be a random variable. In this chapter, we consider this problem in a more general framework as a disorder problem, which we now state:

'Suppose that during the period of observation of a
sequence of independent random variables, the probability distribution governing them undergoes a sudden change, called a 'disorder' or a 'disruption'—at some unknown instant. It is desired to detect the change quickly, in order to minimize the expected loss associated with the positive or negative delay, where negative delay corresponds to a 'false-signal' of change before the actual occurrence of the change.

6.3.1. Brief Literature Review. The problem of change of distribution of one of the variables contributing an element of change in the resulting stochastic process has been dealt with, among others, by Yadain and Zacks (1978), Vaman (1983), Zacks (1983), Whittle (1983), and in a sequence of papers by Bhat, Rao and Harishchandra.

While studying the problem of optimal adaptation of an M/M/1 queueing station, when the arrival rate \( \lambda_0 \) of customers shifts at an unknown epoch \( \tau \) to a known value \( \lambda_1 \), Yadain and Zacks (1978) derive a Bayes solution with the prior assumption that \( \tau \) has an exponential distribution. For minimizing the total expected discounted cost, they present a piecewise linear approximation and computing algorithm for recursive solution of the functional equation.

Vaman (1983), on the other hand, considers the problem of optimal detection of the disorder point as an optimal
Bhat and Rao (1992) present a statistical technique for the control of traffic intensity in a single server queueing system. Their procedure is such that the system is readjusted only if the number of customers either increases, or stays beyond the upper control limit longer than a preassigned number of consecutive transitions. In a sequel, Rao et al. (1994) present a method for detecting changes in traffic intensity of queueing systems of M/G/1 and GI/M/S types, which is based on sequential probability ratio tests for queueing systems with embedded Markov chains.

Zacks (1983) is a review article. Zacks and Barzil (1981) study Bayes procedures for detecting shift in the probability of success of Bernoulli trials, when the values of the parameter before and after the shift are unknown, while the system can be inspected after a shift has been declared, to find out if the shift has occurred.

Van Dijk and Puterman (1988) deal with the case where the system is subject to relatively small uncertainties or random fluctuations. They show that under certain conditions, the unperturbed optimal value differs from the perturbed optimal value by a function which is bounded.
In this chapter, we have considered the situation where the changes in reward and transition structure is substantial, and disorder takes place exactly once. Also, unlike Zacks and Barzily, even after the declaration of disorder, inspection of the system provides no immediate evidence regarding the change. Zacks (1983) has considered prior distribution of disorder point as geometric, while we consider a general discrete distribution.

6.3.2. How We View It. We view the disorder problems from two different angles.

(i) We can consider the problem of detection of disorder point as an optimal stopping problem, and formulate it as a Markov decision process.

(ii) For a one stage control process, in which, given an underlying distribution, the optimal decisions are uniquely determined, an 'optimal-stopping' decision procedure regarding the declaration of disorder is superimposed. An existing result from sequential Bayesian procedures is used in order to minimize the expected loss.

The assumption that a change may occur only at the epochs of observation is implicit in our treatment of the problem. We now present the disorder detection problem as a
stopping problem (ref. Yaman (1983)): "Suppose that a process \( \{Y_n, n \in \mathbb{N}\} \) of independent random variables is being observed, and that it consists of

\[
Y_n = \begin{cases} 
Y_n(0) & \text{if } n < t_d \\
Y_n(1) & \text{if } n \geq t_d
\end{cases}
\]  

(6.3.1)

where \( t_d \) is the disorder instant, and \( \mathbb{N} \) the set of natural numbers. The probability distribution of \( Y_n(0) \), \( \ldots \), \( Y_n(t_d-1) \) is \( F_0 \), while that of \( Y_n(t_d+j) \), \( j = 0,1,2, \ldots \), is \( F_1 \), i.e., the disorder at epoch \( t_d \) is assumed to cause an instantaneous switch of the probability distribution from a known \( F_0 \) to a known \( F_1 \).

Let,

\[
\xi_n = \begin{cases} 
0 & \text{if } n < t_d \\
1 & \text{if } n \geq t_d
\end{cases}
\]  

(6.3.2)

The process \( \{\xi_n, n \in \mathbb{N} \cup \{0\}\} \) is such that it indicates the probability distribution of \( Y_n \).

We assume that \( t_d \) is random and has a known prior. Based on the realization of \( Y \) upto the decision epoch, we wish to stop the process \( \{\xi_n\} \) soon after disorder occurs, so as to minimize an associated loss. The decision problem is thus an optimal stopping problem."
6.3.2.1 Presentation as a Two Action M.D.P.

Decision regarding whether or not to stop the process of detection of disorder, is taken after each observation. We call the decision epoch after \( n \) observations as the \( n \)th stage for an infinite horizon problem unlike that in the finite horizon problem (when \( \pi_0 (t_0 = k) = 0 \) for \( K > M \), \( \pi_0 \) giving the prior probability distribution).

The option available to the decision maker is between the following two actions:

'\( 0 \)' \( \rightarrow \) Do not stop, i.e. continue taking further observations.

'\( 1 \)' \( \rightarrow \) Stop taking further observations and declare the disorder.

The action space is thus a two-tuple

\[ A = \{ 0, 1 \} \]  \hspace{1cm} (6.3.3)

The stopping problem can be alternately considered as the problem of testing at each stage, one of the two hypotheses

\[ H_0 : F(y) = F_0(y) \]

\[ \text{or} \quad H_1 : F(y) = F_1(y) \]

is true. This testing problem differs from the ordinary problem of testing of a simple \( H_0 \) versus a simple \( H_1 \) where
one of the hypotheses is true throughout. In that $H_0$ is true up to $t_d - 1$, and $H_1$ is true from the $t_d$th observation onwards. Once $H_1$ is true, $H_0$ can not be true thereafter, and given $t_d$, it is known exactly whether $H_0$ is true or $H_1$, for the given stage.

Let the prior probability mass function of $t_d$ be given by

$$ P(t_d = K) = \pi_0(K); \quad K = 1, 2, \ldots $$

(6.3.4)

We consider the posterior probability distribution $\pi_n = (\pi_n(t_d); \quad t_d = 0, 1, \ldots, K)$ of the disorder time $t_d$ after $n$ observations as the state at stage $n$. Then $\pi_n$ is Markovian as seen below.

Suppose $Y$ is a discrete random variable; i.e., $F_0$ and $F_1$ are discrete distributions and given $t_d = K$, $Y_1, Y_2, \ldots, Y_n$ are independent.

**Lemma:** Let $\pi_0$ and $\pi_1$ be the probability mass functions corresponding to distribution functions $F_0$ and $F_1$ respectively. Let

$$ \pi_n(K) = P(t_d = K \mid \sigma(Y_1, \ldots, Y_n)) $$

where $\sigma(.)$ is the sigma field generated by $Y_1, \ldots, Y_n$.

Then $\{\pi_n(K); K \geq 1\}$ is a Markov process.
PROOF. For $K = n+1$, 

\[ \pi_{n+1}(K) = \frac{\sum_{j=1}^{n+1} \pi_n(j) f_1(y_{n+1}) + \sum_{j=n+2}^{\infty} \pi_n(j) f_0(y_{n+1})}{\pi_n(K) f_0(y_{n+1})} \]

Similarly, for $t \leq n+1$,

\[ \pi_{n+1}(K) = \frac{\sum_{j=1}^{n+1} \pi_n(j) f_1(y_{n+1}) + \sum_{j=n+2}^{\infty} \pi_n(j) f_0(y_{n+1})}{\pi_n(K) f_0(y_{n+1})} \]

Thus, $\pi_{n+1}$ can be expressed in terms of $\pi_n(\cdot)$ and $y_{n+1}$.

Further,

\[ \mathbb{P}(Y_{n+1} = y_{n+1}, \gamma_1 = y_1, \ldots, \gamma_n = y_n) = \frac{\pi_n(K)}{\pi_n(K) f_0(y_{n+1})} \cdot \mathbb{P}(Y_{n+1} = y_{n+1}) \]

$+$ $\mathbb{P}(Y_{n+1} = y_{n+1}, t_d \leq n+1, \gamma_1 = y_1, \ldots, \gamma_n = y_n)$
Thus conditional distribution of \( Y_{n+1} \) given \( Y_1, \ldots, Y_n \) depends only on \( \pi_n(\cdot) \).

Hence, \( \pi_n \) forms a Markov chain. \((*)\)

(The result can be similarly proved for the case of general distribution function \( \omega(\cdot) \).)

Transition probabilities under the two actions '0' and '1' are given by

\[
p^1 (\pi_n, \pi_n) = 1
\]

since no further observations are allowed, and

\[
p^0 (\pi_n, \pi_{n+1}) = p^0 (\pi_n, \pi_{n+1}(\cdot))
\]

where R.H.S. is a vector, with

\[
p^0 (\pi_n, \pi_{n+1}(\cdot)) = \int P (t = \cdot | \pi_n, Y_{n+1}) \cdot P (Y_{n+1} | Y_1, \ldots, Y_n) dy_{n+1}
\]

where

\[
P (Y_{n+1} | Y_1, \ldots, Y_n) = P (Y_{n+1} | Y_{n+1} = \cdot; Y_1 = Y_1, \ldots, Y_n = Y_n)
\]

is given by \((6.3.7)\).

Loss structure.

Let

\[
L_1 (\geq 0) = \text{The expected one step loss due to a false signal},
\]
\( L_2 (\geq 0) \) = The expected ore step loss due to delayed detection of disorder.

\( C (\geq 0) \) = Cost of taking an additional observation.

Then, optimal decision regarding the choice between '0' and '1' will be given by

\[
V_n (\pi_n) = \min \left( \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n (t_d) \right)
\]

\[
+ C + L_2 \sum_{t_d=0}^{n-1} \pi_n (t_d) + \int (V_{n+1}) p^0 (\gamma_n, \pi_{n+1}) \gamma_{n+1} \right) d\gamma_{n+1}
\]

\[
= \min \{ V_n (\pi_n), V_{n+1} (\pi_n) \}
\]  

(6.3.10)

(6.3.11)

since \( \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n (t_d) = L_1 \) (expected number of steps of false alarm if disorder is declared after \( n \) observations)

### 6.3.3 Optimality Considerations

Let us denote by \( x_n \), the posterior probability after \( n \) observations of the event that change has taken place on or before \( (n+1) \)th observation, i.e.,

\[
x_n = \sum_{t_d=0}^{(n+1)} \pi_n (t_d)
\]  

(6.3.12)

Then

\[
p^0 (\gamma_n, \pi_{n+1} | \gamma_{n+1}) = f_1 (\gamma_{n+1}) x_n + f_0 (\gamma_{n+1}) (1-x_n)
\]  

(6.3.13)
We now show that the set of points $\pi_n$ leading to the stopping action '1' at the $n$th stage, is a convex set which includes the set for which \[ \sum_{t_d=n+1}^{\infty} \pi_n(t_d) = 0. \]

**THEOREM 6.1** If $\pi_n^{(1)}$ and $\pi_n^{(2)}$ are two points leading to acceptance of $H_1$, then \[ \pi_n = \lambda \pi_n^{(1)} + (1-\lambda) \pi_n^{(2)} \]
also leads to acceptance of $H_1$ for $0 \leq \lambda \leq 1$.

**PROOF.** $\pi_n^{(1)}$ and $\pi_n^{(2)}$ both satisfy \[ L_1 \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n(t_d) = C + L_1 \sum_{t_d=0}^{n-1} \pi_n(t_d) \]
\[ + \int \left[ \frac{1}{\pi_{n+1}} \left( f_1(y_{n+1}) x_n + f_0(y_{n+1}) (1-x_n) \right) \right] dy_{n+1}; \quad \varepsilon = 1, 2; \]
\[ L_1 \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n\ast(t_d) = L_1 \sum_{t_d=n+1}^{\infty} (t_d-n) (\lambda \pi_n^{(1)}(t_d) \]
\[ + (1-\lambda) \pi_n^{(2)}(t_d)) \]
\[ = \lambda L_1 \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n^{(1)}(t_d) \]
\[ + (1-\lambda) L_1 \sum_{t_d=n+1}^{\infty} (t_d-n) \pi_n^{(2)}(t_d) \]
\[ \leq \lambda C + \lambda L_2 \sum_{t_d=0}^{n-1} \pi_n^{(1)}(t_d) \]
\[ + \int \frac{1}{\pi_{n+1}} \left[ f_1(y_{n+1}) \sum_{t_d=0}^{n+1} \pi_n^{(1)}(t_d) + f_0(y_{n+1}) \sum_{t_d=n+2}^{\infty} \pi_n^{(1)}(t_d) \right] dy_{n+1}. \]
Thus, stopping is the optimal decision for \( \pi_n \) also. (**) 

\[ \text{Note 6.1. Since } C > 0, \text{ if } \sum_{K=0}^{n} \pi_K(K) = 1, \text{ stopping is the optimal decision, since } V_n^1(\pi_n) = 0 \text{ in this case, while } V_n^0(\pi_n) > 0. \]

6.3.4 The Finite Horizon Case. At times, it may be known in advance, that the disorder will definitely occur before \( M \) observations, i.e.,

\[ \pi_0(t_d) = 0, \quad \text{for } t_d \geq M + 1. \]

Then, if disorder has not been declared till \( t_d = M \), it has to be declared at \( t_d = M \). This will lead to no further
losses. Hence,

\[ U_0(\eta_M) = \emptyset = U_{-M}(\eta_M) \quad \text{for all } \eta_M. \]

Here, we retain the notations of the infinite horizon case for ease of comparison, i.e. \( \eta_M \) denotes the state after \( M \) observations, and the notation \( U_0 \) has been used to indicate that \( M \) is the last stage of the finite horizon. Once again, the optimal stopping set at each stage is a convex set. This can be seen as in section 6.3.3.

6.3.5 An Alternate View. Suppose we are faced with a one stage control problem. The optimal decision, given the true distribution of \( Y \), is known uniquely. If the wrong distribution of \( Y \) is applied, it results in a one step expected loss \( L_1 \) if \( F_0 \) is the true distribution, while \( F_1 \) is assumed, and a loss in the opposite case. \( L_1 \) and \( L_2 \) then correspond to one step loss due to false signal, and delayed detection respectively. Here, if there is only one change from \( F_0 \) to \( F_1 \), we can superimpose a sequential decision procedure for detecting the disorder, and apply the unique optimal decision of the control problem, depending upon the decision of the superimposed detection procedure. Since observations are taken at cost of taking an additional observation is nil in this case.

The optimal detection procedure must find \( n \) so as to
minimize

\[ E_n(n) = E \left( \mathbb{L}_n^2 \right) = L_1 P(n < t_d) + L_2 P(n \geq t_d) \]

or equivalently, minimizing

\[ E_n^* \left( \mathbb{L}_n^2 \right) = \frac{1}{L_2} E_n \left( \mathbb{L}_n^2 \right) = \omega P(n < t_d) + \chi(n \geq t_d) \]

where

\[ w = \frac{L_1}{L_2}. \]

Vaman (1983) has proved that, "if \( \eta_0(t_d = 0) = \eta, \eta \not\in (0, 1) \), and

\[ \pi_0(t_d = 1) = p \nu^{-1} \] (0 < p < 1, q = 1 - p, \( \nu \in \mathbb{N} \))

then the \textit{Bayesian} detection rule

\[ n^* = \inf \left\{ n \geq 1: \ z_n = \frac{\omega - 1 - \gamma^*}{p} \right\} \] (6.3.14)

where the constant \( \gamma^* \) is the unique solution of

\[ \gamma^* = q \int \min (\omega - 1 - \gamma^*, \gamma^*) \nu \, dz \] (6.3.15)

minimizes

\[ wP(n < t_d) + P(n = t_d) \]

with \( 1 + \gamma^* \) as its risk, if \( P(n^* = \infty) = 1 \).

Here \( \nu \) is the probability measure induced by

\[ 2 = \frac{1}{q} \frac{f_{\gamma} (\gamma^*)}{\nu (\gamma^*)} \]

Hence, for the above mentioned prior, optimal detection rule exists and the on-line control can be applied as suggested.