CHAPTER 3

FLOW OF A COUPLE STRESS FLUID THROUGH A POROUS MEDIUM WITH FLUCTUATING SUCTION
CHAPTER 3

Flow of a couple stress fluid through a porous medium with fluctuating suction.

3.1 Introduction:

The class of oscillatory boundary layer flows has technological importance. Lighthill [35] initiated the study of such flows dealing with the response of the boundary layer to external unsteady fluctuations about a mean value. Stuart [94] examined the interesting features of an oscillatory flow over an infinite plate with constant suction. Messiha [43] investigated the laminar boundary layer flow in the presence of fluctuating suction in phase with the fluctuations of free stream over a non-zero mean. In these studies only viscous fluids are considered. However, in many transport processes occurring in industries, the fluids are non-Newtonian in nature. Soundalgekar and Puri [92] have examined the fluctuating flow of an elastico-viscous fluid past an infinite plate with variable suction.

The literature on the study of oscillatory flow of fluids, which sustain couple stresses, through porous medium is scarce. The effects of couple stresses on the flow through a porous medium have been investigated by Raptis [64] and Patil and
Hiremath [57], and the results have been discussed in the chapter 2. In the present chapter, the objective of the study is to extend the analysis of the flow problem dealt in the chapter 2 to include the effects of variable suction on the flow characteristics. Such an analysis is relevant to the study of mobility control in oil displacement mechanism which improves the efficiency of oil recovery. The crude oil can be modelled as a couple stress fluid. The other applications of the couple stress fluid models have been to suspensions, to slurries, to blood flow and to mean turbulent flow [20].

In the present analysis, the constant porosity medium is assumed and the generalised Darcy equation accounting for the couple stress effects is employed. In the special case of constant suction, the results obtained reduce to those presented in the chapter 2. The analysis reveals the multiple boundary layer structure for velocity and angular velocity fields near the wall. There exist Stokes sublayers which are essentially due to the fluctuations in the suction velocity. There is clear evidence of significant dependence of boundary layer thicknesses on the material parameters of the fluid, permeability of the porous medium and the frequency of the perturbations over the non-zero mean suction velocity. It is observed that both velocity and angular velocity increase as the frequency of the perturbations increase.
3.2 Mathematical Formulation:

An unsteady two-dimensional flow of a viscous incompressible fluid with couple stresses through a porous medium bounded by an infinite horizontal porous plate with fluctuating suction at the wall is considered. The $x'$-axis is chosen along the plate and $y'$-axis is normal to it. Let $(u', v', 0)$ and $(0, 0, \omega')$ be components of the velocity and the angular velocity fields. Due to infinite plate assumption, the flow variables except pressure $p$ are functions of $y'$ and $t'$ only. The governing equations, for the boundary layer flow considered, are given by [section 1.5 in chapter 1].

Continuity:

\[
\frac{\partial v'}{\partial y'} = 0, \tag{3.2.1}
\]

Momentum:

\[
\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} = -\frac{1}{\rho'} \frac{\partial p}{\partial x'} + (\nu' + \nu_r) \frac{\partial^2 u'}{\partial y'^2} + 2 \nu' \frac{\partial \omega'}{\partial y'} - \frac{(\nu' + \nu_r)}{k'} \frac{\partial}{\partial y'} u', \tag{3.2.2}
\]

Angular momentum:

\[
\frac{\partial \omega'}{\partial t'} + v' \frac{\partial \omega'}{\partial y'} = \frac{\gamma}{t} \frac{\partial^2 \omega'}{\partial y'^2}, \tag{3.2.3}
\]

where the notations are as described in the chapter 2. The geometry of the flow problem is shown in the Fig.3.1. The boundary conditions are derived by D'ep [22] as
\[ y' = 0 : u' = 0, \quad \frac{\partial \omega'}{\partial y'} = - \frac{\partial^2 u'}{\partial y'^2}, \quad (3.2.4) \]

\[ y' \rightarrow \infty : u' \rightarrow U_\infty, \quad \omega' \rightarrow 0, \]

where \( U_\infty \) is the free-stream velocity. These boundary conditions are derived from the assumption that couple stresses are dominant during the rotation of the particles.

For fluctuating suction velocity at the wall, we assume

\[ v' = - v_0 (1 + \epsilon A e^{i \omega t'}), \quad \text{at } y' = 0, \quad (3.2.5a) \]

where, \( v_0 \) is a non-zero constant mean suction velocity, \( \epsilon \) is small and \( A \) is a real positive constant such that \( \epsilon A \ll 1 \). Only the real part of the physical quantities involved in the mathematical analysis has physical meaning. The negative sign in the equation (3.2.5a) indicates that the suction velocity is directed towards the plate.

The continuity equation (3.2.1) yields the solution

\[ v'(y',t') = - v_0 (1 + \epsilon A e^{i \omega t'}), \quad (3.2.5b) \]

satisfying the suction condition (3.2.5a) at the wall.

In view of the equation (3.2.5b), the equations (3.2.2) and (3.2.3) become

\[ \frac{\partial^2 u'}{\partial y'^2} - v_0 (1 + \epsilon A e^{i \omega t'}) \frac{\partial u'}{\partial y'} = - \frac{1}{\rho} \frac{\partial \rho}{\partial x'} + 
+ (\nu + \nu_r) \frac{\partial^2 u'}{\partial y'^2} + 2 \nu_r \frac{\partial \omega'}{\partial y'} - \frac{(\nu + \nu_r)}{K} \frac{\partial \rho}{\partial x'}, \quad (3.2.6) \]
\[ \frac{\partial \omega'}{\partial t'} - v_o \left(1 + \epsilon A e^{i \omega t'} \right) \frac{\partial \omega'}{\partial y'} = \frac{\gamma}{f} \frac{\partial^2 \omega'}{\partial y'^2}. \]  

(3.2.7)

For the free-stream, the equation (3.2.2) reduces to

\[ 0 = -\frac{1}{\rho} \frac{\partial \rho}{\partial x'} - \frac{(\nu + \nu_r)}{K} U_\infty. \]  

(3.2.8)

Eliminating pressure gradient between the equations (3.2.6) and (3.2.8), we obtain

\[ \frac{\partial u'}{\partial t'} - v_o \left(1 + \epsilon A e^{i \omega t'} \right) \frac{\partial u'}{\partial y'} = (\nu + \nu_r) \frac{\partial^2 u'}{\partial y'^2} + \]

\[ + 2 \nu_r \frac{\partial \omega'}{\partial y'} + \frac{(\nu + \nu_r)}{K} (U_\infty - u'). \]  

(3.2.9)

Introducing the non-dimensional quantities, defined by

\[ y = y'/v_0, \quad t = v_0^2 t'/\nu, \quad n = \nu n'/v_0^2, \]

\[ u = u'/U_\infty, \quad \omega = \omega'/U_\infty v_0, \quad a = \nu r'/\nu, \]

\[ \beta = I \nu/\gamma, \quad K = v_0^2 K'/\nu^2, \]  

(3.2.10)

the equations (3.2.9) and (3.2.7) are reduced to following non-dimensional equations:

\[ \frac{\partial u}{\partial t} - (1 + \epsilon A e^{i \omega t}) \frac{\partial u}{\partial y} = (1 + a) \frac{\partial^2 u}{\partial y^2} + 2a \frac{\partial \omega}{\partial y} + \]

\[ + \frac{(1 + a)}{K} (1 - u), \]  

(3.2.11)

\[ \frac{\partial \omega}{\partial t} - (1 + \epsilon A e^{i \omega t}) \frac{\partial \omega}{\partial y} = \frac{1}{\beta} \frac{\partial^2 \omega}{\partial y^2}, \]  

(3.2.12)
and the boundary conditions are reduced to

\[ y = 0 : u = 0, \quad \frac{\partial \omega}{\partial y} = -\frac{\partial^2 u}{\partial y^2}, \]  
\[ y \to \infty : u \to 1, \quad \omega \to 0. \]  
\[ (3.2.13) \]

The equations (3.2.11) and (3.2.12) are solved subject to the boundary conditions (3.2.13).

3.3 Solution of the problem:

In order to solve the equations (3.2.11) and (3.2.12), we assume the velocity and the angular velocity in the following form:

\[ u(y, t) = u_0(y) + e^{i\alpha t} u_1(y), \quad (3.3.1) \]
\[ \omega(y, t) = \omega_0(y) + e^{i\alpha t} \omega_1(y). \quad (3.3.2) \]

On substituting equations (3.3.1) and (3.3.2) into equations (3.2.11) and (3.2.12), respectively, we get the following system of equations:

**Steady basic flow:**

\[ (1+\alpha) u_0'' + u_0' - \frac{(1+\alpha)}{K} u_0 = - \left[ 2a \omega_0' + \frac{1+\alpha}{K} \right], \quad (3.3.3) \]
\[ \omega_0'' + \beta \omega_0' = 0, \quad (3.3.4) \]

subject to the corresponding boundary conditions (reduced from the conditions (3.2.13)) given by
\begin{align*}
y = 0 : & \quad u_0 = 0, \quad \omega_0^i = -u_0'' , \quad (3.3.5) \\
y \to \infty : & \quad u_0 \to 1, \quad \omega_0 \to 0 ,
\end{align*}

Unsteady flow :

\begin{align*}
(1+a) u_1'' + u_1' - \left[ \frac{1+a}{k} + i n \right] u_1 = - [A u_0' + 2 a \omega_1'] , \quad (3.3.6) \\
\omega_1'' + \beta \omega_1' - \beta \omega_1 = - \beta A \omega_0' , \quad (3.3.7)
\end{align*}

subject to the corresponding boundary conditions (reduced from the conditions (3.2.13)) given by

\begin{align*}
\begin{align*}
y = 0 : & \quad u_1 = 0, \quad \omega_1^i = -u_1'' , \quad (3.3.8) \\
y \to \infty : & \quad u_1 \to 0, \quad \omega_1 \to 0 ,
\end{align*}
\end{align*}

where primes denote the differentiation with respect to \( y \).

On solving the equations (3.3.3) and (3.3.4) subject to the conditions (3.3.5), and the equations (3.3.6) and (3.3.7) subject to the conditions (3.3.8), and then substituting the solutions so obtained into the equations (3.3.1) and (3.3.2), we get

\begin{align*}
u = 1 + C_2 e^{R_1 y} + A_1 C_1 e^{-\beta y} + \\
&+ e^{int} \left[ C_4 e^{R_5 y} + A_4 C_3 e^{R_3 y} + A_3 e^{R_3 y} + A_5 e^{-\beta y} \right], (3.3.9) \\
\omega = C_1 e^{-\beta y} + e^{int} \left[ C_3 e^{R_3 y} + A_2 e^{-\beta y} \right], \quad (3.3.10)
\end{align*}

where,

\begin{align*}
R_{1,2} = \frac{(-1 \mp \sqrt{1 + 4(1+\omega)^2/k})}{2(1+\omega)},
\end{align*}
\[ R_{3,4} = \left( -\beta + [\beta^2 + 4\ln \beta]^{\frac{1}{3}} \right) / 2, \]
\[ R_{5,6} = \left( -1 + [1 + 4(1+\alpha)((1+\alpha)/K + \ln)]^{\frac{1}{2}} \right) / (2(1+\alpha)), \]

and the expressions for the constants \( C_1, C_2, C_3, C_4, A_1, A_2, A_3, A_4 \) and \( A_5 \) are given in the Appendix A.

The expressions for the velocity (3.3.9) and the angular velocity (3.3.10) can be written in the form

\[ u = u_0 + \varepsilon e^{i\int (M_r + iM_i)}, \tag{3.3.11} \]
\[ \omega = \omega_0 + \varepsilon e^{i\int (W_r + iW_i)}, \tag{3.3.12} \]

where,

\[ M_r + iM_i = u_1, \tag{3.3.13} \]
\[ W_r + iW_i = \omega_1, \tag{3.3.14} \]

and the expressions for \( M_r, M_i, W_r \) and \( W_i \) are given in the Appendix A. When \( nt = \pi/2 \), the equations (3.3.12) and (3.3.13) yield

\[ u = u_0 - \varepsilon M_i, \tag{3.3.15} \]
\[ \omega = \omega_0 - \varepsilon W_i, \tag{3.3.16} \]

and when \( nt = \pi \), we obtain

\[ u = u_0 - \varepsilon M_r, \tag{3.3.17} \]
\[ \omega = \omega_0 - \varepsilon W_r. \tag{3.3.18} \]
3.4 Results and Discussions:

To gain physical insight into the flow problem considered herein, the numerical computation of the distributions of velocity $u$ and angular velocity $\omega$ of polar fluid is done for various values of the material parameters $\alpha$ and $\beta$, permeability parameter $K$, frequency parameter $n$ and suction parameter $A$. The numerical results are shown in the Figs.3.2-3.6. From the Fig.3.2, we notice that the velocity $u$ increases as $n$ and $\alpha$ increase. For a given $n$, e.g. $n=1.0$, the velocity is found to increase as compared to the Newtonian case. As $\beta$ increases, the velocity decreases (Fig.3.3). In the Fig.3.4, we observe that the velocity decreases as $K$ increases. The suction parameter $A$, which is such that $\varepsilon A \ll 1$, represents the order of magnitude of the amplitude of fluctuations in suction velocity with non-zero mean. From the Fig.3.5, it is noticed that velocity decreases as $A$ increases in the case of polar fluid contrary to the observation that it increases as $A$ increases in the Newtonian case.

The Fig.3.6 shows the angular velocity distribution $\omega$. It is observed that the magnitude of $\omega$ increases as $n$ and $\alpha$ increase whereas it decreases as $K$ and $\beta$ increase. The negative values of $\omega$ indicate that the micro-rotation of substructures in the polar fluid is clock-wise.

The equations (3.3.9) and (3.3.10) reveal that the velocity and angular velocity fields exhibit multiple boundary layer structure.
In the basic steady part of the flow, the boundary layer thicknesses of order $|R_1|^{-1}$ and $\beta^{-1}$ exist. Predominantly, the diffusion of angular momentum takes place in the sublayer of thickness $\beta^{-1}$ under the action of couple stresses. In the unsteady part of the flow, there exist the Stokes layers of thicknesses of orders $|R_{3r}|^{-1}$ and $|R_{5r}|^{-1}$, where the subscript $r$ indicates the real part. The dependence of $R_3$ and $R_5$ on $n$ indicates that the corresponding Stokes sublayers are essentially due to the fluctuations in suction velocity.

In conclusion, the mathematical analysis of the flow problem dealt herein reveals that the characteristics of the polar fluid profoundly influence the boundary layer structure and its control by suction applied at the bounding wall. Hence, these results are helpful in designing oil-wells with enhanced throughout of crude oil.
FREE STREAM $U_\infty$ →

POROUS MEDIUM

$V = -v_0(1 + \epsilon A e^{i \omega t})$

(\(y > 0\))

FLUCTUATING SUCTION $V = -v_0(1 + \epsilon A e^{i \omega t})$

Fig. 3.1: Schematic diagram.
Fig. 3.2: Velocity profiles for different values of material parameter \( n \).
Fig. 3.3: Velocity profiles for different values of material parameter $\beta$. 

- $n = 1.08$
- $n = 3.0$
- $K = 0.5$
- $\gamma = 0.2$
- $A = 0.2$
- $\varepsilon = 0.2$

Values of $\beta$: $\beta = 40$, $\beta = 60$, $\beta = 80$
Fig. 3.4: Velocity profiles for different values of permeability $K$. 

<table>
<thead>
<tr>
<th>No.</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>3.0</td>
</tr>
</tbody>
</table>

$\alpha = 0.2$
$\beta = 4.0$
$\lambda = 0.2$
$\epsilon = 0.2$
Fig. 3.4a Velocity distribution obtained for different values of K using unused Engine
Oil data
Fig. 3.5: Velocity profiles for the suction parameter $A = 0.2$ and $0.8$. 
Fig. 3.6: Angular velocity profiles.

<table>
<thead>
<tr>
<th>N₀</th>
<th>K</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>4.0</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>4.0</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>4.0</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>6.0</td>
</tr>
</tbody>
</table>

--- n = 1.0
--- n = 30
Fig. 3.6a  Angular velocity distribution obtained using unused Engine Oil data
The expressions for the constants $C_1, C_2, C_3, C_4, A_1, A_2, A_3, A_4$ and $A_5$ appearing in the equations (3.3.9) and (3.3.10) are:

\[
C_1 = - \frac{\beta^2 A_1 - \beta}{(A_1(\beta^2 - R_1^2) - \beta)},
\]
\[
C_2 = \frac{R_1^2}{(A_1(\beta^2 - R_1^2) - \beta)},
\]
\[
A_1 = 2 \alpha \beta / ((1 + \alpha)(\beta + R_1)(\beta + R_2)),
\]
\[
A_2 = \frac{A \beta^2 C_1}{((\beta + R_3)(\beta + R_4))},
\]
\[
A_3 = - A R_1 C_2 / ((1 + \alpha)(R_1 - R_5)(R_1 - R_6)),
\]
\[
A_4 = - 2 \alpha R_3 / ((1 + \alpha)(R_3 - R_5)(R_3 - R_6)),
\]
\[
A_5 = (A \beta A_1 C_1 + 2 \alpha \beta A_2) / ((1 + \alpha)(\beta + R_5)(\beta + R_6)),
\]
\[
H_1 = -(A_3 + A_5),
\]
\[
H_2 = -(A_3 R_1^2 + A_5 \beta^2 - A_2 \beta),
\]
\[
C_3 = (H_2 - H_1 R_5^2) / (R_3 + A_4 (R_3^2 - R_5^2)),
\]
\[
C_4 = (H_1 (R_3 + R_3 A_4) - H_2 A_4) / (R_3 + A_4 (R_3^2 - R_5^2)).
\]

The expressions for the quantities $M_r, M_1, W_r$ and $W_1$ appearing in the equations (3.3.11) and (3.3.12) are:

\[
M_r = B_3 e^{R_1 y} + B_{11} e^{-\beta y} + e^{R_5 r y} [A_{17} \cos(R_{51} y) - A_{18} \sin(R_{51} y)] + e^{R_{31} y} [A_{23} B_7 - A_{24} B_8 \cos(R_{31} y) - (B_8 A_{23} + B_7 A_{24}) \sin(R_{31} y)],
\]
\[M_1 = B_e^{R_1y} + B_{12} e^{-\beta y} + e^{R_5r y} \left[A_{18} \cos(R_{51} y) + A_{17} \sin(R_{51} y)\right] + e^{R_3r y} \left[(A_{23} B_8 + A_{24} B_7) \cos(R_{31} y) + (B_7 A_{23} - B_8 A_{24}) \sin(R_{31} y)\right],\]

\[W_r = e^{R_3r y} \left[A_{23} \cos(R_{31} y) - A_{24} \sin(R_{31} y)\right] + B^*_3 e^{-\beta y},\]

\[W_1 = e^{R_3r y} \left[A_{24} \cos(R_{31} y) + A_{23} \sin(R_{31} y)\right] + B^*_4 e^{-\beta y},\]

where,

\[R_{3r} = (-\beta - (\beta^2 + [\beta^4 + 16 \beta^2 n^2])^k / 2)^k / 2,\]

\[R_{3i} = 1 / 2,\]

\[R_{4r} = (-\beta + (\beta^2 + [\beta^4 + 16 \beta^2 n^2])^k / 2)^k / 2,\]

\[R_{4i} = (\beta^2 + [\beta^4 + 16 \beta^2 n^2])^k / 2)^k / 2,\]

\[R_{5r} = [-1 - (1+4 (1+\alpha)^2/K + [(1+4(1+\alpha)^2/K)^2 + (4n(1+\alpha))^2]^k / 2^k / (2(1+\alpha)),\]

\[R_{5i} = 1 / 2,\]

\[R_{6r} = [-1 + (1+4(1+\alpha)^2/K + [(1+4(1+\alpha)^2/K)^2 + (4n(1+\alpha))^2]^k / 2^k / (2(1+\alpha)),\]

\[R_{6i} = (\beta^2 + [\beta^4 + 16 \beta^2 n^2])^k / 2)^k / (2(1+\alpha)),\]

\[B_1 = (1+\alpha) \left[(R_1-R_{5r}) (R_1-R_{6r}) - R_{5i} R_{6i}\right],\]

\[B_2 = -(1+\alpha) \left[R_{6i}(R_1-R_{5r}) + R_{5i}(R_1-R_{6r})\right],\]

\[B_3 = -A R_1 C_2 B_1 / (B_1^2 + B_2^2),\]

\[B_4 = A R_1 C_2 B_2 / (B_1^2 + B_2^2),\]

\[B_5 = (1+\alpha) \left[(R_{3r}-R_{5r}) (R_{3r}-R_{6r}) - (R_{3i}-R_{5i}) (R_{3i}-R_{6i})\right].\]
\[ B_6 = (1 + a) \left( R_{3r} - R_{5r} \right) \left( R_{3i} - R_{6i} \right) + \left( R_{3i} - R_{5i} \right) \left( R_{3r} - R_{6r} \right), \]
\[ B_7 = -2a \left( R_{3r} B_5 + B_6 R_{3i} \right) / (B_5^2 + B_6^2), \]
\[ B_8 = -2a \left( R_{3i} B_5 - B_6 R_{3r} \right) / (B_5^2 + B_6^2), \]
\[ B_1^* = (\beta + R_{3r}) \left( \beta + R_{4r} \right) - R_{3i} R_{4i}, \]
\[ B_2^* = R_{3i} (\beta + R_{4r}) + R_{4i} (\beta + R_{3r}), \]
\[ B_3^* = \beta^2 C_1 B_1^* / (B_1 + B_2^*), \]
\[ B_4^* = -\beta A C_1 B_2^* / (B_1 + B_2^*), \]
\[ A_6 = \beta A A_1 + 2a \beta B_3^*, \]
\[ B_9 = (1 + a) \left( (\beta + R_{5r}) (\beta + R_{6r}) - R_{5i} R_{6i} \right), \]
\[ B_{10} = (1 + a) \left( (\beta + R_{5r}) R_{6i} + (\beta + R_{6r}) R_{5i} \right), \]
\[ B_{11} = (A_6 B_9 + 2a \beta B_4^* B_{10}) / (B_9^2 + B_{10}^2), \]
\[ B_{12} = (-A_6 B_{10} + 2a \beta B_4^* B_9) / (B_9^2 + B_{10}^2), \]
\[ B_{13} = -(B_3 + B_{11}), \]
\[ B_{14} = -(B_4 + B_{12}), \]
\[ B_{19} = -(B_3 R_{1}^2 + B_{11} \beta^2 - \beta B_3^*), \]
\[ B_{20} = -(B_4 R_{1}^2 + B_{12} \beta^2 - \beta B_4^*), \]
\[ A_7 = R_{3r} + B_7 \left( R_{3r}^2 - R_{3i}^2 \right) - 2 B_8 R_{3r} R_{3i}, \]
\[ A_8 = R_{3i} + B_8 \left( R_{3r}^2 - R_{3i}^2 \right) + 2 B_7 R_{3r} R_{3i}, \]
\[ A_9 = B_7 B_{19} - B_8 B_{20}, \]
\[ A_{10} = B_7 B_{20} + B_8 B_{19}. \]
\[ A_{11} = B_{13} A_7 - B_{14} A_8 - A_9, \]
\[ A_{12} = B_{14} A_7 + B_{13} A_8 - A_{10}, \]
\[ A_{13} = R_{3r}^2 - R_{3i}^2 - R_{5r}^2 + R_{5i}^2, \]
\[ A_{14} = 2(R_{3r} R_{3i} - R_{5r} R_{5i}), \]
\[ A_{15} = R_{3r} + A_{13} B_7 - A_{14} B_8, \]
\[ A_{16} = R_{3i} + A_{14} B_7 + A_{13} B_8, \]
\[ A_{17} = (A_{11} A_{15} + A_{12} A_{16}) / (A_{15} + A_{16}), \]
\[ A_{18} = (A_{12} A_{15} - A_{11} A_{16}) / (A_{15} + A_{16}), \]
\[ A_{19} = B_{13} (R_{5r}^2 - R_{5i}^2) - 2 B_{14} R_{5r} R_{5i}, \]
\[ A_{20} = B_{14} (R_{5r}^2 - R_{5i}^2) + 2 B_{13} R_{5r} R_{5i}, \]
\[ A_{21} = B_{19} - A_{19}, \]
\[ A_{22} = B_{20} - A_{20}, \]
\[ A_{23} = (A_{21} A_{15} + A_{22} A_{16}) / (A_{15} + A_{16}), \]
\[ A_{24} = (A_{22} A_{15} - A_{21} A_{16}) / (A_{15} + A_{16}). \]