1. INTRODUCTION

In a multicomponent system, if a component fails, usually it affects the functioning of the remaining components, and hence the assumption of independence of components is not realistic. So it is reasonable to assume that the failure of a component changes the life distribution of the remaining components. Incorporating this fact, the following survival model is considered.

Consider a system which functions if at least one of \( k \) components functions (parallel system). Let \( Z_1, Z_2, \ldots, Z_k \) be the strength of \( k \) components, each of them is subjected to a random stress \( Y \). Initially all the random variables (r.v.'s) \( Z_1, Z_2, \ldots, Z_k \) are independently and identically distributed (i.i.d.) having exponential distribution with mean \( \theta (e(1/\theta)), \theta > 0 \). The failure of one of the \( k \) components changes the life distribution of each of the remaining \( (k-1) \) components to exponential distribution with mean \( \alpha_i \theta, \alpha_i > 0 \). The failure of the next component changes the life
distribution of each of the remaining \((k-2)\) components to exponential with mean \(\alpha_2 \theta\), \(\alpha_2 > 0\), and so on. Lastly, the failure of \((k-1)\) components changes the life distribution of the remaining one component to exponential distribution with mean \(\alpha_{k-1} \theta\), \(\alpha_{k-1} > 0\). Let \(X_1 < X_2 < \ldots < X_k\) denote the ordered life times (strengths) observed on the \(k\) components of the system. Then \(W_i = X_i - X_{i-1}\) has the distribution of minimum of \((k-i+1)\) identically distributed exponential random variables with mean \(\alpha_i \theta\). By the univariate loss of memory property (LMP),

\[
W_i = X_i - X_{i-1}, i=1, \ldots, k
\]

are i.i.d. exponential r.v.'s with mean \(\alpha_i \theta / (k-i+1)\), with \(X_0 = 0\) and \(\alpha_0 = 1\). Suppose that stress \(Y\) has the exponential distribution with mean 1. Then, the reliability of the system is given by

\[
R_k = P(X_k > Y)
\]

\[
= P(Y < W_1 + W_2 + \ldots + W_k)
\]

\[
= \int_0^\infty \int_0^\infty \cdots \int_0^\infty (1-e^{-\sum_{i=1}^k w_i}) \left( \prod_{i=1}^{k-i+1} \frac{\alpha_i \theta}{\sum_{j=1}^{i-1} \theta} \right) dw_1 \cdots dw_k
\]

\[
= 1 - \prod_{i=1}^k \int_0^{\frac{\alpha_i \theta}{\sum_{j=1}^{i-1} \theta}} e^{-\frac{(k-i+1) \omega_i}{\alpha_i \theta} + 1} \omega_i \ d\omega_i.
\]
Solving the integrals and simplifying, we obtain

$$R_k = 1 - \frac{k!}{k} \prod_{i=1}^{\infty} \frac{(k-i+1+\alpha)}{1-\theta}$$  \hspace{1cm} (2.2)

Weier (1981) has obtained the Bayes estimator of the reliability of the two-component system without taking into consideration of stress-strength relationship. Church and Harris (1970) have obtained the estimator of reliability of one-component system for i.i.d. normal variates having stress-strength relationship with completely known stress distribution. Bhattacharya and Johnson (1974) have obtained the UMVUE of reliability for s-out-of-k system for i.i.d. exponential r.v.'s having stress-strength relationship. Freund (1961) has obtained the maximum likelihood estimators (MLEs) of parameters of bivariate extension of exponential distribution without considering stress-strength relationship. Weinman (1966) has discussed a multivariate exponential distribution which is the extension of Freund bivariate exponential distribution (see Johnson and Kotz (1972)). The results of this chapter are due to Kunchur and Munoli (1992a).

In this chapter, the UMVUE and MLE of reliability $R_k$ are obtained in Section 2. Section 3 deals with UMVUE and MLE of
reliability \( R'_2 \) in case of bivariate survival stress-strength model when stress is also exponential r.v. with unknown mean. The efficiencies of UMVUE and MLE of \( R'_2 \) as compared to Cramer-Rao lower bound (CRB) for the variance of unbiased estimator of \( R'_2 \) are computed in Section 4 by means of Monte Carlo simulation study.

2. UMVUE AND MLE OF \( R_k \)

In this section at first the life testing experiment is described. The UMVUE of \( R_k \) is obtained using the Rao-Blackwell and Lehmann-Sheffe theorems. The MLE of \( R_k \) is also obtained by substituting the MLEs of \( \alpha_1, \ldots, \alpha_{k-1} \) and \( \theta \) in \( R_k \).

Suppose \( n \) systems of \( k \) components, each system with life distribution (2.1) are put on life testing experiment, and \( X_{1j} < X_{2j} < \ldots < X_{kj} \) denote the ordered life times (strengths) observed on \( j \)-th system, \( j=1,2,\ldots,n \). Following on the lines of Section 1, we have

\[
W_{ij} = X_{ij}' - X_{i-1,j}' \sim e^{\left( \frac{k-i+1}{\alpha_{i-1}\theta} \right)}, \tag{2.3}
\]

with \( X_{0j} = 0 \) and \( \alpha_0 = 1 \) for \( i=1,2,\ldots,k; j=1,2,\ldots,n \).
The joint probability density function (pdf) of $W_{ij}$'s 
(i=1,...,k; j=1,...,n) is given by

$$
\prod_{j=1}^{n} \prod_{i=1}^{k} \frac{k-i+1}{\alpha_{i-1}} e^{-\left(\frac{k-i+1}{\alpha_{i-1}}\right)w_{ij}}. \quad (2.4)
$$

On simplifying, we have the joint pdf of $W_{ij}$'s as

$$
\frac{(k!)^n}{\prod_{i=1}^{k-1} \alpha_{i}^n} e^{-\frac{1}{\theta} \left(kw_{1} + \frac{k-1}{\alpha_{1}}w_{2} + \cdots + \frac{1}{\alpha_{k-1}}w_{k}\right)}, \quad (2.5)
$$

where $w_i = \sum_{j=1}^{n} w_{ij}$, i=1,2,...,k. The pdf (2.5) belongs to exponential family of distributions with k parameters, $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ and $\theta$. By Theorem 1 of Lehmann (1959, p.132), it follows that $w_1, w_2, \ldots, w_k$ are jointly complete sufficient statistics for the family of distributions (2.5).

In order to obtain the UMVUE of $R_k$, define

$$
\phi(W_{11}, W_{21}, \ldots, W_{k1}) = \begin{cases} 
1, & \text{if } \sum_{i=1}^{k} W_{i1} > y, \\
0, & \text{otherwise},
\end{cases} \quad (2.6)
$$

where $y$ stands for an observation from exponential distribution with mean 1. Then, $\phi(W_{11}, W_{21}, \ldots, W_{k1})$ is an
unbiased estimator of $R_k$. By the Rao-Blackwell and Lehmann-Scheffe theorems, the UMVUE of $R_k$ is given by

$$R_k = \frac{1}{\prod_{i=1}^{k} f(w_{i1} | w_{i1})}\int_{0}^{\infty} \int_{w_{i1}}^{y} f(w_{11}, \ldots, w_{k1} | w_{11}, \ldots, w_{k1}) dw_{11} \ldots dw_{k1}$$

$$= 1 - \prod_{i=1}^{k} e^{-w_{i1}} f(w_{i1} | w_{i1}) dw_{i1}, \quad (2.7)$$

where $f(w_{i1} | w_{i1})$ is the conditional distribution of $w_{i1}$ given $w_{i1}$, $i=1, \ldots, k$. Following on the lines of Basu (1964), these distributions are obtained as follows:

Consider the two independent r.v.'s $w_{i1}$ and $w'_i = \sum_{j=2}^{n} w_{ij}$. $w_{i1}$ is exponential r.v. with mean $\alpha_i^{-1}/(k-i+1)$ and $w'_i$ is gamma variate with parameters $(n-1)$ and $(k-i+1)/\alpha_i^{-1}/\theta_i$, $i=1, \ldots, k$. In the joint pdf of $w_{i1}$ and $w'_i$, making the transformation $w_{i1} = w_{i1}$ and $w_i = w_{i1} + w'_i$, we obtain the joint pdf of $w_{i1}$ and $w_i$. Dividing this joint pdf by the marginal pdf of $w_{i1}$, which is gamma variate with parameters $n$ and $(k-i+1)/\alpha_i^{-1}/\theta_i$, we obtain conditional pdf of $w_{i1}$ given $w_i$ as

$$f(w_{i1} | w_i) = \frac{(n-1)}{w_i} (1 - \frac{w_{i1}}{w_i})^{n-2}, i=1,2,\ldots,k \quad (2.8)$$
with $0 < w_1 < w_i$. Substituting this pdf in (2.7) and simplifying, we obtain the UMVUE of $R_k$ as

$$R_k^* = 1 - \frac{(n-1)^k}{k} \prod_{i=1}^{k} \left[ \sum_{j=0}^{n-2} \frac{(-1)^j (n-2)!}{w_j^i} e^{-w_i} \right].$$

(2.9)

Using (2.5), the MLEs of $\theta$ and $\alpha_i$'s are obtained as

$$\hat{\theta} = \frac{kw_1}{n}$$

(2.10)

and

$$\hat{\alpha}_i = \frac{(k-i)w_{i+1}}{kw_1}, \quad i=1, \ldots, k-1.$$  

(2.11)

Substituting $\hat{\theta}$ and $\hat{\alpha}_i$ from (2.10) in (2.2), we obtain the MLE $R_k^*$ of $R_k$.

3. UMVUE OF RELIABILITY IN BIVARIATE SURVIVAL STRESS-STRENGTH MODEL WITH UNKNOWN STRESS

Consider a system of two components with strength $X$ and $X'$, which are subjected to stress $Y$ and $Y'$, respectively. The system survives until at least one of the two components survives (parallel system). Initially $X$ and $X'$ are independent exponential r.v.'s with mean $\theta$, and $Y$ and $Y'$ are independent exponential r.v.'s with mean $\lambda$, $\theta > 0$, $\lambda > 0$. After
failure of one of the two components, the strength and stress of the other component will follow exponential distribution with mean $\alpha \Theta$ and $\beta \Lambda$ respectively, $\alpha > 0$ and $\beta > 0$. This model is similar to Weler (1981) model except the stress-strength relationship. One can quote several examples of this model, like pair of kidneys, pair of eyes, pair of lifts, etc. In these cases, the working capacity of a component is considered as its strength and the workload on it as the stress. Then, the failure of one of the two components changes the working capacity (strength) of the other component and workload (stress) on it.

Let $U = \min(X, X')$, $V = \max(X, X')$, $U' = \min(Y, Y')$, $V' = \max(Y, Y')$ and $W = V - U$, $W' = V' - U'$. As in previous section by the univariate LMP $U$ and $W$ are independent and $U \sim e(\frac{1}{\Theta})$, $W \sim e(\frac{1}{\Lambda \Theta})$. Similarly, $U'$ and $W'$ are independent and $U' \sim e(\frac{2}{\Lambda})$, $W' \sim e(\frac{1}{\beta \Lambda})$.

The reliability of this system is given by

$$R_2' = P \left[ \max(X, X') > \max(Y, Y') \right]$$

$$= P \left[ U' < U + W - W' \right]$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (1 - e^{-\frac{u+w}{\lambda}}) f_1(w') f_2(u) f_3(w) dw' du \, dw.$$
After substituting pdf's and integrating, we obtain

\[
R'_2 = \left\{ \begin{array}{l}
\frac{\lambda^2}{(\lambda+\theta)(\lambda+2\alpha \theta)(2\beta-1)} - \frac{4\beta^3 \lambda^2}{(\theta+2\alpha \theta)(\alpha \theta+\beta \lambda)(2\beta-1)} & \text{for } \alpha \neq \frac{1}{2}, \beta \neq \frac{1}{2} \\
\frac{\theta^2}{(\theta+3\lambda)} & \text{for } \alpha = \beta = \frac{1}{2}.
\end{array} \right.
\]

(2.10)

In order to obtain the estimators of \( R'_2 \), suppose \( n \) systems of the type discussed above are put on life testing experiment and \( U_i = \min(X_i, X'_i) \) and \( V_i = \max(X_i, X'_i) \), \( i=1,2,\ldots,n \) are observed and \( U_i \sim e\left(\frac{2}{\lambda}\right) \), \( W_i = V_i - U_i \sim e\left(\frac{1}{\alpha \theta}\right) \). Also the data of stress \( U'_j = \min(Y_j, Y'_j) \) and \( V'_j = \max(Y_j, Y'_j) \), \( j=1,2,\ldots,m \), are obtained separately from a simulation of conditions of the operating environment and \( U'_j \sim e\left(\frac{2}{\lambda}\right) \), \( W'_j = (V'_j - U'_j) \sim e\left(\frac{1}{\beta \lambda}\right) \).

Writing the joint pdf of the sample observations \( U_i, W_i, U'_j, W'_j \) for \( i=1,2,\ldots,n \) and \( j=1,2,\ldots,m \), it can be verified that \((u_i, w_i, u'_j, w'_j) = \left( \sum_{i=1}^{n} u_i , \sum_{i=1}^{n} w_i , \sum_{j=1}^{m} u'_j , \sum_{j=1}^{m} w'_j \right)\) is a complete sufficient statistic for \((\theta, \lambda, \alpha, \beta)\).

Now, define

\[
\phi(U_1, W_1, U'_1, W'_1) = \left\{ \begin{array}{ll}
1, & \text{if } U_1 + W_1 > U'_1 + W'_1 \\
0, & \text{otherwise}.
\end{array} \right.
\]

(2.11)
Then $\phi(U_1, W_1, U'_1, W'_1)$ is an unbiased estimator of $R'_2$. Using Rao-Blackwell and Lehmann-Scheffe theorems, the UMVUE of $R'_2$ is given by,

$$R'_2^* = E \left[ \phi(U_1, W_1, U'_1, W'_1 | U_1, W_1, U'_1, W'_1) \right]$$

$$= \int \int \int g(u_1, w_1, u'_1, w'_1 | u_1, w_1, u'_1, w'_1) du'_1 dw'_1 du_1 dw_1.$$  

(2.16)

Once again following on the lines of Basu (1964), the conditional distribution of $u_1, w_1, u'_1, w'_1$ given $u_1, w_1, u'_1, w'_1$ is given by

$$g(u_1, w_1, u'_1, w'_1 | u_1, w_1, u'_1, w'_1)$$

$$= \frac{(n-1)^2(m-1)^2}{u_1 w_1 u'_1 w'_1} \left(1 - \frac{u_1}{u_1}ight)^{n-2} \left(1 - \frac{w_1}{w_1}ight)^{n-2} \left(1 - \frac{u'_1}{w_1}ight)^{m-2} \left(1 - \frac{w'_1}{w_1}ight)^{m-2},$$

(2.16)

with $0<u_1<u_1$, $0<w_1<w_1$, $0<u'_1<u'_1$, $0<w'_1<w'_1$, for $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$.

Substituting this in (2.15) and integrating, we obtain the UMVUE of $R'_2$ for $\alpha \neq \frac{1}{2}$ and $\beta \neq \frac{1}{2}$ as

$$R'_2^* = 1 - (n-1)^2 \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \frac{m-1-l-1}{k+1} \frac{m-1-l}{k+1} \frac{1}{(n+1-1)(n+k-1)(w'_1)^{l+k}} \left(1 - \frac{u_1 + w_1}{w_1}\right)^{m-l-k-1}$$

$$= 1 - (n-1)^2 \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \frac{m-1-l-1}{k+1} \frac{m-1-l}{k+1} \frac{1}{(n+1-1)(n+k-1)(w'_1)^{l+k}} \left(1 - \frac{u_1 + w_1}{w_1}\right)^{m-l-k-1}.$$
\[
\begin{align*}
- \sum_{i=0}^{m-2} \frac{(m-1)}{i+1} \left( \frac{u_i}{w_i} \right)^{i+1} & \left[ \sum_{l=0}^{m-2-i} \frac{m-1-i-2}{l} \sum_{k=0}^{m-l-i-2} \frac{1}{(n+1-l)(n+k-1)} \frac{w_i}{(w_i)^{l+k}} \right] \\
(1 - \frac{u_i + w_i}{w_i})^{m-l-i-k-2} & \sum_{l=0}^{m+i} \sum_{k=0}^{m+i-1} \frac{1}{(n+1-l)(n+k-1)} \\
\frac{w_i}{(u_i)^{l+k}} & \left(1 - \frac{u_i + w_i}{u_i} \right)^{m-l+i-k} \\
\end{align*}
\]

Similarly, for \( \alpha = \beta = \frac{1}{2} \), the UMVUE \( R'_2^* \) is given by

\[
R'_2^* = 1 - 2(n-1)(2n-1) \sum_{i=0}^{2(n-1)-1} \sum_{j=0}^{2m-1} (-1)^{i+j+3} (2n-1)^i (2m-1)^j \frac{v_i}{v'_i} \\
\left[ \frac{v_i}{v'_i (i+j+3)} + \frac{1}{(i+j+2)} \right] \\
\]

where, \( v_i = \sum_{i=1}^{n} v_i, v'_i = \sum_{j=1}^{m} v'_j \).

The MLEs of \( \theta, \lambda, \alpha \) and \( \beta \) are given by

\[
\begin{align*}
\hat{\theta} &= 2u/n \\
\hat{\lambda} &= 2u'/n \\
\hat{\alpha} &= w/2u \\
\hat{\beta} &= w'/2u'.
\end{align*}
\]
Substituting these MLEs in (2.13), the MLE $\hat{R}_2'$ of $R_2'$ is obtained.

4. EFFICIENCY COMPARISON

Monte Carlo simulation experiments were conducted in order to compare the efficiencies of the UMVUE and MLE of $E_2$ (the case $k=2$). The efficiency function considered is the ratio of the CRB to the mean square error (MSE) of the reliability estimator under consideration.

The CRB for the variance of an unbiased estimator of the function $\psi(\theta_1, \theta_2)$ of parameters $\theta_1$ and $\theta_2$ is given by

$$CRB = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial \psi}{\partial \theta_i} \frac{\partial \psi}{\partial \theta_j} I_{ij}^{-1},$$

(2.20)

where $I_{ij}^{-1}$ is $(i,j)$-th element of the inverse of the information matrix

$$I(\theta_1, \theta_2) = \left\{ \frac{1}{L} \frac{\partial L}{\partial \theta_i} \frac{1}{L} \frac{\partial L}{\partial \theta_j} \right\},$$

(2.21)

with $L$ denoting the corresponding likelihood function (Kendall and Stuart (1958)). Letting $\theta_1 = \alpha$ and $\theta_2 = \alpha_2 = \alpha$ (say) we have
\[ I(\theta, \alpha) = \begin{pmatrix} 4n & n \\ \frac{\theta^2}{\alpha} & \frac{n}{\alpha} \end{pmatrix} \]

\[ I^{-1}(\theta, \alpha) = \begin{pmatrix} \frac{\theta^2}{3n} & -\frac{\alpha \theta}{3n} \\ -\frac{\alpha \theta}{3n} & \frac{4\alpha^2}{3n} \end{pmatrix} \]

Letting \( \psi = R = \frac{\theta (1 + 2\alpha + \alpha \theta)}{(2 + \theta)(1 + 3\alpha \theta)} \), which is obtained by putting \( k = 2 \) in (2.2) and using (2.23) in (2.20), we obtain the CRB for the variance of an unbiased estimator of \( R \) as

\[ \text{CRB} = \frac{4\theta^2 (1 + 12\alpha^2 + 4\alpha^2 \theta^2 + 2\alpha \theta + 12\alpha \theta^2)}{3n(2 + \theta)^4 (1 + 3\alpha \theta)^4} \tag{2.24} \]

The simulations were done in the following way:

When \( \theta = 1 \) and 2, for different values of \( \alpha \), 2000 samples each of various sizes \( (n=5, 6, 7, 8, 9, 10) \), were generated from the bivariate model (the case \( k=2 \)) considered in Section 2.

For the \( l \)-th replicate, the realized MLE \( \hat{R}_{2,l} \) and the realized UMVUE \( \hat{R}^{*}_{2,l} \) were computed for \( l=1, 2, \ldots, 2000 \). The simulated MSE's were found from the sum of squares of the differences of the 2000 estimates of \( R \) from the true value \( R \) with the sum divided by 2000.
Table 1 gives the results of a Monte Carlo study of the efficiencies of the MLE $\hat{R}_2$ and UMVUE $\hat{R}_2^*$ relative to CRB of $R_2$. From the table it is clear that $R_2^*$ is more efficient than $R_2$ except for $\alpha = 0.5$ and $\theta = 1$. For $\alpha = 0.5$ and $\theta = 1$, the MLE outperforms the UMVUE. For $n = 8, 9, 10$ (for higher sample size), the efficiencies of $R_2^*$ and $\hat{R}_2$ are greater than or equal to 0.55, and for $\alpha \theta \geq 1$ the efficiency of $R_2^*$ increases as $\alpha \theta$ increases.

Appendix-I outlines the computer program (in basic) for the calculations of efficiencies of $R_2^*$ and $\hat{R}_2$. 
Table 1

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continued,
In each column of efficiencies, first figure shows the efficiency of MLR of $R^2$ relative to CRB and second figure shows the efficiency of UMVUE of $R^2$ relative to CRB.