CHAPTER - 6

ESTIMATION OF RELIABILITY FOR TWO - COMPONENT LIGHTLY LOADED STANDBY SYSTEM

1 INTRODUCTION

In case of loaded standbys, failure in the standby components do not provide the desired gain in reliability. At the same time, the conditions under which we are working do not allow any cessation of operation of our system. In such cases we are unable to use an unloaded standby, because a certain amount of time lapses between the instant it becomes in an operating condition (i.e. the component has to warm up). In such cases the lightly loaded standbys are useful (Gnedenko et al. (1969), p.303). Fujii and Sandoh (1984) have obtained Bayes interval limits for the reliability of the 2-out-n hot-standby redundant system.

In a lightly loaded standby system the standby component is in lightly loaded state until it is put into operation as a basic component. Here, the component can fail when it is in a lightly loaded state, but the probability of this is less than the probability of failure when it is functioning as basic component. Section 2 deals about the maximum
estimator (MLE) and Bayes estimator of reliability of a two component lightly loaded standby system without repair facility. In Section 3, the estimators of reliability of a two component lightly loaded standby system with single repair facility is considered.

2. ESTIMATION OF RELIABILITY FOR TWO COMPONENT SYSTEM WITHOUT REPAIR

In this section at first, reliability function is obtained, the life testing experiment is described, MLE and Bayes estimators of reliability are obtained. Lastly, the comparison of these estimators is done.

2.1 Reliability function

Consider a system of two identical and independent components and a switch. One component serves as a basic component (say A) and the other one functions as standby component (say B), which is lightly loaded. The switchover is instantaneous when the basic component fails. The probability that the switchover is successful is \( \varphi \). The switch tries to operate when basic component is functioning. The times to failure of Basic Component (A) and standby component (B) are exponentially distributed with parameters \( \theta_A \) and \( \theta_B \).
respectively with \( \theta > 0, 0 < \alpha < 1 \). Here it is assumed that expected life of component B is more than expected life of component A when component B is working as a lightly loaded standby. When component A fails and B is functioning as standby then the instantaneous switch will operate. If switch operates successfully, then component B will start working as basic component and its remaining life will have exponential distribution with parameter \( \theta \). If switchover is not successful, then the system fails. Let \( X \) and \( Y \) denote the life times of components A and B respectively. When component B fails as standby component while A is working as basic component, the system functions until component A is working. The probability that the component B fails first at \( y \) (i.e., when it is lightly loaded) and then component A (basic component) fails at \( x \) is

\[
\alpha \theta e^{-\alpha y} e^{-\theta x}.
\]

The probability that the component A (basic component) fails first at \( x \), the switchover is successful and component B starts working as basic component and fails at \( y \) is

\[
\theta e^{-\theta x} e^{-\theta y} e^{-\theta(y-x)}.
\]
Similarly the probability that the component A fails at t, switchover is not successful is
\[ \theta e^{-\theta t} e^{-\omega x} (1-\rho). \]  \hfill (6.3)

Let
\[ Z = \begin{cases} 1, & \text{if switchover is successful} \\ 0, & \text{otherwise} \end{cases} \]  \hfill (6.4)

and \( P(z=1)=\rho, P(z=0)=1-\rho \).

It follows from (6.1) to (6.4) that the joint probability density function of \( X \) and \( Y \) is
\[
f(x,y) = \begin{cases} \omega e^{2} e^{-\omega x} e^{-\theta x}, & \text{if } 0 \leq y \leq x \\ \rho e^{2} e^{-\omega x} e^{-\theta y}, & \text{if } 0 \leq x < y, z=1 \\ \theta (1-\rho) e^{-\theta (1+\alpha) x}, & \text{if } 0 \leq x \leq 0. \end{cases} \]  \hfill (6.5)

The reliability of this system at mission time \( T \) is
\[ R(T) = P(X>T) + P(0<X<T<Y/Z=1) \]
\[
= e^{-\theta T} \int_0^T \int_0^x \rho e^{2} e^{-\omega x} e^{-\theta y} dxdy 
\]  \hfill (6.6)

Solving the integrals and simplifying, we have
\[ R(T) = e^{-\theta T} + \frac{\rho}{\alpha} e^{-\theta T} - \frac{\rho}{\alpha} e^{-\theta (1+\alpha) T} \]  \hfill (6.7)
\[ = R_1(T) + R_2(T) - R_3(T), \text{ (say).} \]  \hfill (6.8)

One can easily verify that \( R(0)=1, R(\infty)=0 \) and \( R(1) \neq 1 \).
2.2 Life testing experiment

Suppose \( n \) systems having the life density \((6.5)\) are put on life test and the life testing experiment is conducted until all the systems fail. Out of \( n \) systems, say for \( \beta \) systems the component B fails before the component A fails (i.e. B fails while functioning as lightly loaded standby). For remaining \( n-\beta = m \) systems, the basic component fails first and out of \( m \) systems for \( r \) systems the switchover is successful. Let

\[
(Y_1, Y_2, \ldots, Y_r) \quad \text{and} \quad (Y_1, Y_2, \ldots, Y_s)
\]

denote the life times of B components when \( \beta \) of them have failed in lightly loaded standby state and remaining \( m \) have failed while working as basic components respectively, where \( (r_1, r_2, \ldots, r_\beta, s_1, s_2, \ldots, s_m) \) is permutation of \((1,2,\ldots,n)\).

Let \( X_1, X_2, \ldots, X_r, X_s, \ldots, X_m \) be the corresponding observations on component A. Let \( Z_1, Z_2, \ldots, Z_s \) be the observations of switchover. The likelihood function of the sample is

\[
L = n! [\beta! r!(m-r)!]^{-1} \frac{\beta^2}{\theta^2} 2^{\beta+m+r} \frac{r}{\phi} (1-\phi)^{m-r} e^{-\theta t_1 - 2\theta t_2},
\]

where

\[
t_1 = \left[ \sum_{i=1}^{\beta} x_i + \sum_{i=1}^{m} z_i y_i + \sum_{i=1}^{m} (1-z_i) x_i \right]
\]
and

\[ t_2 = \left[ \sum_{i=1}^{\beta} y_{r_i} + \sum_{i=1}^{m} x_s_i \right]. \tag{6.11} \]

Differentiating logL partially with respect to \( \varphi \), \( \alpha \) and \( \beta \), equating these derivatives to 0, and solving for the parameters, we have the following MLEs of \( \varphi, \theta \) and \( \alpha \).

\( \hat{\varphi} = \frac{r}{m} \)

\( \hat{\alpha} = \beta + m + r \)

\( \hat{\alpha} = \frac{r}{m} \)

and

\( \hat{\alpha} = \frac{r}{m} \alpha \). \tag{6.12} \)

2.3 MLE and Bayes estimator of \( R(\tau) \)

The maximum likelihood estimator \( \hat{R}(\tau) \) of \( R(\tau) \) is obtained by replacing the parameters with their MLEs in \( R(\tau) \), given by (6.7).

To obtain the Bayes estimator of \( R(\tau) \); following the idea of Box and Tio (1973), the prior distribution of \( \theta_1 = \theta \) and \( \theta \) are taken as exponential distribution with parameter \( \lambda \). Since \( \theta_1 < \theta \), the prior distribution of \( \theta_1 \) and \( \theta \) is given by

\[ P(\theta_1, \theta) = c e^{-\theta_1}, \quad 0 < \theta_1 < \theta \]. \tag{5.12} \]

where \( c \) is determined such that total probability is one.

After substituting the value of \( c \) and making the
transformation \( \theta_1 = \alpha \theta \) and \( \theta = \theta \), we obtain the joint prior distribution of \( \alpha \) and \( \theta \) as

\[
P(\alpha, \theta) = 2 \theta e^{-(1+\alpha)\theta}, \quad 0 < \alpha < 1, \quad 0 < \theta < \infty.
\] (6.14)

Now, the joint pdf of sample observations is

\[
f(x, y, z | \alpha, \theta, \rho) = n! [\beta! r!(m-r)!]^{-1} \alpha^\beta \theta^{2\beta + m + r} \rho^r (1-\rho)^{m-r} e^{-(\alpha \theta + m + r)z}
\]

with \( x = (x_1, x_2, \ldots, x_r, x_{s_1}, \ldots, x_{s_m}) \)

\( y = (y_1, y_2, \ldots, y_r, y_{s_1}, \ldots, y_{s_m}) \)

\( z = (z_{s_1}, \ldots, z_{s_m}) \). (6.17)

Using (6.13) to (6.15), the posterior probability density function of \((\alpha, \theta, \rho)\) is obtained as

\[
P(\alpha, \theta, \rho | x, y, z) = \alpha^\beta \theta^{2\beta + m + r + 1} \rho^r (1-\rho)^{m-r} e^{-(\alpha \theta + m + r + 1)z}
\]

\[
= \frac{(m+1)! (t_1 + 1)^{\beta + m + r + 1} (t_2 + 2)^{2\beta + m + r + 1}}{(\beta + m + r)! \beta! r! (m-r)! (t_2 + 1)^{\beta + m + r} \sum_{i=0}^{t_2} \left( \frac{t_1 + 1}{t_2 + 1} \right)^i}
\]

(6.16)
Using the squared error loss function, the Bayes estimator of $R(\tau)$ is the posterior expectation of $R(t)$ and is given by

$$\hat{R}_B(\tau) = \hat{R}_1B(\tau) + \hat{R}_2B(\tau) - \hat{R}_3B(\tau),$$

with

$$\hat{R}_1B(\tau) = \frac{t_1+1}{t_1+\tau+1} \left( \frac{t_1+t_2+2}{t_1+t_2+\tau+2} \right)^{2\beta+m+r+1} \left( \sum_{i=0}^{\beta+m+r} \frac{2^{2\beta+m+r+1}}{i!} \left( \frac{t_1+1}{t_2+1} \right)^i \right).$$

$$\hat{R}_2B(\tau) = (r+1)(\beta+m+r+1)(t_2+1)[\beta(m+2)(t_1+\tau+1)]^{-1} \left( \frac{t_1+1}{t_1+\tau+1} \right)^{2\beta+m+r+1} \left( \sum_{i=0}^{\beta+m+r} \frac{2^{2\beta+m+r+1}}{i!} \left( \frac{t_1+1}{t_2+1} \right)^i \right).$$

(6.17)

(6.13)
\[ R_{3B}(\tau) = (r+1)(\beta+m+r+1)(t_2+\tau+1)(m+2)(t_1+\tau+1)^{-1} \]

\[
\left( \frac{t_1+1}{t_1+\tau+1} \right)^{\beta+m+r+1} \left( \frac{t_1+t_2+2}{t_1+t_2+2\tau+2} \right)^{2(\beta+m+r+1)} \left( \frac{t_2+1}{t_2+1} \right)^{\beta+m+r} \]

\[
\sum_{i=0}^{\beta+m+r} \left( \begin{array}{l}
\frac{t_1+1}{t_1+\tau+1} \\
\frac{t_1+t_2+2}{t_1+t_2+2\tau+2}
\end{array} \right)_{i} \left( \begin{array}{l}
\frac{t_2+1}{t_2+1} \\
\frac{t_1+1}{t_1+\tau+1}
\end{array} \right)_{t_1+t_2+2\tau+2} \]

\[ \sum_{i=0}^{\beta+m+r} \left( \begin{array}{l}
\frac{t_1+1}{t_1+\tau+1} \\
\frac{t_1+t_2+2}{t_1+t_2+2\tau+2}
\end{array} \right)_{i} \left( \begin{array}{l}
\frac{t_2+1}{t_2+1} \\
\frac{t_1+1}{t_1+\tau+1}
\end{array} \right)_{t_2+1} \]

2.4 Comparison of estimators

Monte Carlo simulation experiments were conducted in order to compare the MLE and Bayes estimator. Table 1 gives the results of a Monte Carlo study of biases and mean square errors (MSE) of these estimators. Pairs of random numbers were generated from the distribution (6.5) as follows:

For given \( \theta = \theta \) and \( \omega \theta = \omega \theta \) a pair of exponential random numbers \( X' \) and \( Y' \) are generated. If \( Y' \leq X' \), then letting \( X = X' \) and \( Y = Y' \), we obtain a pair of generated exponential observations \( (X, Y) \) from the discussed model with \( Y \leq X \). If \( X' < Y' \), then a uniform random number \( u \) over \((0,1)\) is generated. If \( u \leq \theta \), then for \( \theta = \theta \) another exponential random number \( X' \) is
generated. Letting \( X=X' \) and \( Y=X+\frac{7}{n} \), we obtain pair of generated exponential observations \((X,Y)\) from the discussed model with \( X<Y \) and switchover is successful with \( \rho=\rho_0 \). If \( u>\rho_0 \), then letting \( X=X' \), we get the observation for \( 0<X<u \) and switchover is not successful. In this way we have generated 1000 samples each of size \( n=10, 11, 12, 13, 14, 15 \) for \( \Theta=0.75 \), \( \Theta=0.5 \) and \( \rho=0.7 \) from the distribution (6.5). The true value of reliability \( R(T) \) at mission time \( T=1 \) unit is calculated as 0.7835. The MLE \( \hat{R}(T) \) and the Bayes estimator \( \hat{R}_B(T) \) were evaluated at \( T = 1 \) unit of time and biases and MSEs are calculated (see Table 1).

The comparison shows that the MLE performs uniformly better throughout, both with respect to bias and MSE for \( n=10(1)15 \).

Appendix-II outlines the computer program (basic) for the calculations of biases and MSEs of \( \hat{R}(T) \) and \( \hat{R}_B(T) \).
Table 1
Bias and MSE of reliability estimates

<table>
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<th>MSE Bayes</th>
<th>Bias MLE</th>
<th>MSE Bayes</th>
</tr>
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</tr>
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</table>

3 ESTIMATION OF RELIABILITY FOR TWO-COMPONENT SYSTEM WITH SINGLE REPAIR

In this section, at first the system that we are considering is described. The system reliability function is obtained. The life testing experiment is discussed. The MLE and UMVUE of the parameters are obtained. Lastly the estimators of reliability are obtained by substituting the estimators of parameters in reliability function.
3.1 Reliability function

Consider a system of two identical and independent components. One component serves as basic component and the other one functions as standby component, which is lightly loaded. The times to failure of basic component and standby component are exponentially distributed with parameters $\lambda$ and $\lambda_1$ respectively, $\lambda, \lambda_1 > 0$ and $\lambda_1 < \lambda$. When a component fails, it undergoes repair and the time to repair of a component is also a exponential r.v. with parameter $\mu$, $\mu > 0$. When a basic component fails, the lightly loaded standby will start working as basic component automatically. If a component fails, irrespective of either basic or standby, the failed component will be under repair. After the repair is over, it is put into the system as standby component. During the repair time of failed component, the one which is operating as basic component fails, then the system fails.

Let $X(t)$ denote the number of non operating components at time $t$. Then the reliability of this system at mission time $\tau$ is

$$R(\tau) = P\left\{X(u) \leq 2, \quad 0 \leq u \leq \tau \mid X(0) = 0\right\}$$

$$= q_{00}(\tau) + q_{01}(\tau),$$

where $q_{ij}$ is the probability of $i$ basic and $j$ standby components being non operating at time $t$. [Note: The expression for $q_{00}(\tau)$ and $q_{01}(\tau)$ would need to be provided or derived based on the specific system model and parameter values.]
where

$$q_{0j}(T) = P\{X(u)=j, X(u) \notin 2, 0 \leq u \leq T \mid X(0)=0\}.$$ 

Let

$$q_{ij}(r) = P\{X(u)=j, X(u) \notin 2, 0 \leq u \leq r \mid X(0)=i\}, i, j = 0, 1.$$ 

Then, we have

$$q_{00}(T) = e^{-\lambda \tau} + \int_0^\tau (\lambda + \lambda) e^{-\lambda u} q_{10}(T-u) du,$$ 

where

$$q_{10}(t) = P\{X(u)=0, X(u) \notin 2, 0 \leq u \leq t \mid X(0)=1\}$$ 

$$= \int_0^t e^{-\mu u} e^{-\lambda u} q_{00}(t-u) du.$$ 

Similarly, we have

$$q_{01}(T) = \int_0^\tau (\lambda + \lambda) e^{-\lambda u} q_{11}(T-u) du,$$ 

with

$$q_{11}(t) = P\{X(u)=1, X(u) \notin 2, 0 \leq u \leq t \mid X(0)=1\}$$ 

$$= e^{-(\lambda + \mu) t} + \int_0^t e^{-(\lambda + \mu) u} q_{01}(t-u) du.$$ 

Taking the Laplace transform of (6.22) to (6.25), and simplifying we have
\[ q_{00}(s) = \frac{s + \lambda + \mu}{s^2 + s(2\lambda + \lambda_1 + \mu) + (\lambda + \lambda_1)} . \] (6.26)

Letting \( s^2 + s(2\lambda + \lambda_1 + \mu) + (\lambda + \lambda_1) = (s + \alpha)(s + \beta) \), and solving for \( \alpha \) and \( \beta \), we have

\[ 2\alpha = (2\lambda + \lambda_1 + \mu) + \sqrt{(2\lambda + \lambda_1 + \mu)^2 - 4\lambda(\lambda + \lambda_1)} \] (6.27)

and

\[ 2\beta = (2\lambda + \lambda_1 + \mu) - \sqrt{(2\lambda + \lambda_1 + \mu)^2 - 4\lambda(\lambda + \lambda_1)} . \] (6.28)

Considering the partial fractions, we have

\[ \frac{s + \lambda + \mu}{(s + \alpha)(s + \beta)} = \frac{A_o}{s + \alpha} + \frac{B_o}{s + \beta} , \] (6.29)

and solving for \( A_o \) and \( B_o \), we have

\[ A_o = \frac{\lambda + \mu - \alpha}{\beta - \alpha} \] (6.30)

\[ B_o = \frac{\lambda + \mu - \beta}{\alpha - \beta} . \] (6.31)

Substituting value from (6.29) in (6.26) and taking the inverse of Laplace transformation we obtain

\[ q_{00}(\tau) = A_o e^{-\alpha \tau} + B_o e^{-\beta \tau} , \] (6.32)

where \( A_o \) and \( B_o \) are given by (6.30) and (6.31).
Following on the similar lines, $q_{01}(\tau)$ is obtained as

$$q_{01}(\tau) = A_1 e^{-\alpha \tau} + B_1 e^{-\beta \tau}, \quad (6.33)$$

where

$$A_1 = \frac{\lambda + \lambda_1}{\beta - \alpha}, \quad (6.34)$$
$$B_1 = \frac{\lambda + \lambda_1}{\alpha - \beta}. \quad (6.35)$$

Substituting $q_{00}(\tau)$ and $q_{01}(\tau)$ from (6.32) and (6.33) in (6.21), we get the system reliability $R(\tau)$, and the distribution function of this system is given by

$$F(t) = 1 - R(t), \quad 0 < t < \infty. \quad (6.36)$$

$R(\tau)$ is also obtained by Ravichandran (1990) using birth and death process.

For this model, the transition probability matrix is

$$
\begin{array}{cccc}
0 & 1 & 2 \\
0 & 0 & 1 & 0 \\
1 & \frac{\mu}{\lambda + \mu} & 0 & \frac{\lambda}{\lambda + \mu} \\
2 & 0 & 0 & 1 \\
\end{array}
\quad (6.37)
$$

3.2 Life testing experiment

Suppose, $r$ systems with life distribution (6.36) are put on life testing experiment, and the experiment is conducted until all the systems fail.
Let

\[ n_j = \text{total number of transitions observed during the operating period of j-th system} \]

\[ t_{ij}^0 = \text{time spent in state 0 at i-th transition for j-th replication} \]

\[ t_{ij}^1 = \text{time spent in state 1 at i-th transition for j-th replication} \]

and \( t_{ij}^0 \sim e(\lambda + \lambda_1), \quad i=1,3,\ldots,n_j-1; \ j=1,\ldots,r \)

\[ t_{ij}^1 \sim e(\lambda + \mu), \quad i=2,4,\ldots,n_j; \ j=1,2,\ldots,r \]

Letting \( 2m_j = n_j \), the joint pdf of the r.v.'s \( T_{2j-1}, T_{2j}, \ldots, T_{2m_j} \) for j-th replication is

\[
\sum_{i=1}^{m_j} \frac{e^{-(\lambda + \lambda_1) t_{ij}^0} (\lambda + \lambda_1)^{t_{ij}^0}}{t_{ij}^0!} \frac{e^{-(\lambda + \mu) t_{ij}^1} (\lambda + \mu)^{t_{ij}^1}}{t_{ij}^1!} \left( \frac{\mu}{\lambda + \mu} \right)^{m_j-1} \left( \frac{\lambda}{\lambda + \mu} \right)^{m_j-1}.
\]

\[
= \frac{e^{-(\lambda + \lambda_1) \sum_{i=1}^{m_j} t_{ij}^0} (\lambda + \lambda_1)^{\sum_{i=1}^{m_j} t_{ij}^0}}{(\sum_{i=1}^{m_j} t_{ij}^0)!} \frac{e^{-(\lambda + \mu) \sum_{i=1}^{m_j} t_{ij}^1} (\lambda + \mu)^{\sum_{i=1}^{m_j} t_{ij}^1}}{(\sum_{i=1}^{m_j} t_{ij}^1)!} \left( \frac{\mu}{\lambda + \mu} \right)^{m_j-1} \left( \frac{\lambda}{\lambda + \mu} \right)^{m_j-1}.
\]
The pdf of \( N_j, T_{ij}, T_{2j}, T_{3j}, \ldots, T_{2m_j}, j \) for all the \( r \) systems is

\[
\begin{align*}
\lambda^r \mu^m - r \left( \lambda + \lambda_1 \right)^m \cdot e^{-(X+X^+)^2} \cdot \left( \lambda + \lambda_1 \right)^{t^0} \cdot \left( \lambda + \mu \right)^{t^1}
\end{align*}
\]

(6.39)

where

\[
\begin{align*}
t^0 &= \sum_{j=1}^{r} \sum_{i=1}^{m_j} t^0 \cdot (2i-1), j, \\
t^1 &= \sum_{j=1}^{r} \sum_{i=1}^{m_j} t^1, j
\end{align*}
\]

and

\[
m = \sum_{j=1}^{r} m_j = \frac{1}{2} \sum_{j=1}^{r} n_j
\]

(see Basava and Prakasa Rao (1980)).

The pdf (6.39) belongs to exponential family of distributions with three parameters, \( \lambda, \lambda_1 \) and \( \mu \). By Theorem 1 of Basheen (1959, p.132), it follows that \( m, t^0 \) and \( t^1 \) are jointly complete sufficient statistics for the family of distributions (6.39).

3.3 Estimators of reliability

Using (6.39), the MLEs of \( \lambda, \lambda_1 \) and \( \mu \) are obtained as:

\[
\hat{\lambda} = \frac{r}{t^1} \tag{6.40}
\]
Substituting these MLEs of parameters in (6.32) and (6.33), we obtain the MLEs of \( q_{00}(\tau) \) and \( q_{01}(\tau) \), in turn using these in (6.21), we obtain the MLE \( R(\tau) \) of \( R(\tau) \).

In order to obtain the estimator of \( R(\tau) \), using UMVUE of parameters, we proceed as follows:

Letting \( \theta = \frac{1}{\lambda + \lambda_1} \), \( \eta = \frac{1}{\lambda + \mu} \) and \( p = \frac{\lambda}{\lambda + \mu} \), the reliability of the system \( R(\tau) \) given by (6.21), can be expressed as

\[
R(\tau) = A_0 e^{-\alpha \tau} + B_0 e^{-\beta \tau} + A_1 e^{-\alpha \tau} + B_1 e^{-\beta \tau},
\]

where

\[
A_0 = \frac{1}{\eta} - \alpha / (\beta - \alpha) \quad B_0 = \frac{1}{\eta} - \beta / (\alpha - \beta) \\
A_1 = \frac{1}{\eta (\beta - \alpha)} \quad B_1 = \frac{1}{\eta (\alpha - \beta)}
\]

with

\[
2\alpha = \theta^{-1} + \eta^{-1} + \sqrt{(\theta^{-1} + \eta^{-1})^2 - 4p(\theta \eta)}^{-1}
\]

and

\[
2\beta = \theta^{-1} + \eta^{-1} - \sqrt{(\theta^{-1} + \eta^{-1})^2 - 4p(\theta \eta)}^{-1}
\]
The UMVUE of $p$, $\Theta$, and $\eta$ are obtained as follows:

Since $m_1 = \frac{1}{2}$ has geometric distribution with parameter $p$, we have

$$P(m_1 = 1) = p$$

Using Rao-Blackwell and Lehmann-Scheffe theorems, the UMVUE of $p$ is given by

$$p^* = P(m_1 = 1 | m_2).$$  \hspace{1cm} (6.46)

Following on the lines of Basu (1964), the conditional distribution, (6.46) is obtained. Letting $m' = \sum_{j=2}^{r} m_j$. Now $m'$ and $m_1$ are independent and their joint pmf is

$$P_1(m_1, m') = \binom{m' - 1}{r - 2} p^{m'} q^{m_1 - r}.$$  \hspace{1cm} (6.47)

Making the transformations $m = m' + m_1$ and $m_1 = m_1$, the joint pmf of $m_1$ and $m$ is

$$P_2(m_1, m) = \binom{m - m_1 - 1}{r - 2} p^{m_1} q^{m_1 - r}.$$  \hspace{1cm} (6.48)

The marginal pmf of $m_1$ is

$$P_3(m_1) = \binom{m_1 - 1}{r - 1} p^{r - 1} q^{m_1 - r}.$$  \hspace{1cm} (6.49)

Dividing (6.48) by (6.49), the conditional distribution of $m_1$ given $m$ is obtained as
Using this conditional distribution in (6.46), the UMVUE of \( p \) is given by

\[
P^* = \frac{\left(\frac{m-2}{r-2}\right)}{\left(\frac{m-1}{r-1}\right)}, \quad m = 1, 2, \ldots, m-r+1
\]

(6.50)

In order to obtain UMVUE of \( \theta \), consider

\[
E \left( \frac{t_{m-2}}{m} \right) = \sum_{m=r}^{\infty} E \left( \frac{t_{m-2}}{m} \right) P_3(m) = \theta
\]

(6.53)

implying that the UMVUE of \( \theta \) is

\[
\theta^* = \frac{t_{m-2}}{m}.
\]

(6.53)

On the same line, the UMVUE of \( \eta \) is given by

\[
\eta^* = \frac{t_{m-1}}{m}.
\]

(6.54)

The estimator of \( R(\tau) \) with UMVUEs of parameters is obtained by substituting \((p^*, \theta^*, \eta^*)\) in (6.43).