CHAPTER III
A CLASS OF TEST STATISTICS FOR BIRTH AND DEATH PROCESSES

3.1 INTRODUCTION

In chapter I, we have sufficiently elaborated the role of birth and death process for the study of storage models in general and for the study of queueing and inventory models in particular. In chapter II, we have studied the UMPU test for transition dependence of birth and death process by reparameterising the birth rates and death rates. In this chapter we have derived a class of likelihood ratio test criteria for transition dependence of the parameters of birth and death process, based on small samples. The large sample inference in queueing models of birth and death process type has been studied by Wolff (1965). Muddapur and Hegde (1991) have studied the "Asymptotic inference for a class of birth and death process", which we discuss in chapter IV. The simulation study of the distribution of the standardised test statistics
has been done in some cases. A table of percentage points of 1% and 5% have been given in section 3.4 by using a simulation technique.

3.2 LIKELIHOOD RATIO TEST CRITERIA

Suppose \( \{X(t), t \geq 0\} \) is a birth and death process with \( \lambda_i \) and \( \mu_i \) as birth and death rates (parameters) respectively. Let the event birth or death be called a transition. Suppose the process is observed for a fixed number of transitions. Let \( u_i \) be the upward (birth) transition and \( d_i \) be the downward (death) transition. Thus for the \( i \)th transition \( u_i \) is either zero or one and correspondingly \( d_i \) is either one or zero. So that \( u_i + d_i = 1 \) for all \( i \). Also \( \Sigma u_i \) is the total upward transition and \( \Sigma d_i \) is the total downward transition. Since \( u_i \) and \( d_i \) assume either of the two values zero or one the total number of transitions is \( n = \Sigma u_i + \Sigma d_i \). Let \( t_0 < t_1 < \ldots < t_n \) be the epochs at which the transitions occur. \( T_i = t_i - t_{i-1} \) be the inter transition time which is known to have a negative exponential distribution with mean \( (\lambda_i + \mu_i)^{-1} \). Since the events occur independently the likelihood function for the process \( \{X(t), t \geq 0\} \) is given by
L(λ, μ) = \prod_{i=1}^{n} \lambda_i^u_i \prod_{i=1}^{n} \mu_i^d_i \exp \left\{ -(\lambda_1 + μ_1) T_1 \right\} \ldots (3.2.1)

Theorem 3.2.1. For a birth and death process \{X(t), t \geq 0\} with parameters \(λ_i\) and \(μ_i\) a level \(α\) likelihood ratio test for testing \(H_0: \lambda_i = λ\) and \(μ_i = μ\) for all \(i\), against \(H_1: \lambda_i \neq λ\) and \(μ_i \neq μ\) for all \(i\), \(λ\) and \(μ\) unknown, is given by the critical function,

\[ \phi(T) = \begin{cases} 1 & \text{if } T \leq T_0 \\ 0 & \text{if } T > T_0 \end{cases} \ldots (3.2.2) \]

where,

\[ T = \prod (T_i \div \Sigma T_i) \quad \text{and} \quad \ldots (3.2.3) \]

\[ P\left[ T \leq T_0 \mid H_0 \right] = α \ldots (3.2.4) \]

Proof: The likelihood ratio test criteria is based on the ratio
\[ \Lambda = \frac{\text{Max } L_0 (\lambda, \mu)}{\text{Max } L_1 (\lambda, \mu)} \quad (3.3.5) \]

\[ \Lambda = \frac{L_0 (\hat{\lambda}, \hat{\mu})}{L_1 (\hat{\lambda}_1, \hat{\mu}_1)} \]

where \( \hat{\lambda} \) and \( \hat{\mu} \) are maximum likelihood estimators (M.L.Es) of \( \lambda \) and \( \mu \) under \( H_0 \) which are obtained by using (3.2.1) as

\[ \hat{\lambda} = \frac{\Sigma u_i}{\Sigma T_i} \quad (3.3.6) \]

\[ \hat{\mu} = \frac{\Sigma d_i}{\Sigma T_i} \quad (3.3.7) \]

and \( \hat{\lambda}_i \) and \( \hat{\mu}_i \) are maximum likelihood estimators of \( \lambda_i \) and \( \mu_i \) under \( H_1 \) which are obtained by using (3.2.1) as

\[ \hat{\lambda}_i = \frac{u_i}{T_i} , \quad i = 1, 2, \ldots, n \quad (3.3.8) \]

\[ \hat{\mu}_i = \frac{d_i}{T_i} , \quad i = 1, 2, \ldots, n \quad (3.3.9) \]
Therefore, using (3.3.6) - (3.3.9) in (3.3.5) with (3.2.1) we have

\[ A = \frac{(\sum u_i)^{\Sigma u_i} (\sum d_i)^{\Sigma d_i}}{\prod T_i} \]

\[ \times \left( \frac{u_i + d_i}{u_i^u d_i^d} \right)^{n(T_i)} \]

...(3.3.10)

where \( u_i > 0 \), \( d_i > 0 \)

Here for a given \( i^{th} \) transition \( (u_i + d_i) = 1 \) for all \( i \) and \( \sum u_i + \sum d_i = n \). Thus taking logs, we can get from (3.3.10),

\[ \ln A = (\sum u_i) \ln (\sum u_i) + (\sum d_i) \ln (\sum d_i) \]

\[ - n \ln (\sum T_i) + \sum \ln T_i \]

...(3.3.11)

Since the process is observed for a fixed number of transitions, one cannot expect the number of downward transitions (deaths) to exceed the number of upward transitions. Obviously \( \sum d_i \leq n/2 \) and \( \sum u_i \geq n/2 \) and \( (\sum d_i / \sum u_i) \) is bounded.

Thus (3.2.11) reduces to
\[ \ln \Lambda = \ln C + \ln \left[ \prod \left( \frac{T_i}{\sum T_i} \right) \right] \]  
...(3.3.12)

where \( C \) is some known constant which is independent of \( T_i \)s.

Thus,

\[ \Lambda \propto T, \]

where,

\[ T = \prod_{i=1}^{n} \left[ \frac{T_i}{\sum T_i} \right] \]

Reject \( H_0 \) if \( T \leq T_0 \)

where \( T_0 \) is found such that

\[ P \left[ T \leq T_0 \mid H_0 \right] = \alpha \]

Hence the theorem.

**Corollary: 3.2.1.** (Pure birth process) If the death rate \( \mu_i \) is zero for all \( i \), then we get pure birth process. Further if the hypotheses are

\[ H_0 : \lambda_i = \lambda \quad \text{for all } i \quad \text{against} \quad H_1 : \lambda_i \neq \lambda \quad \text{for all } i, \]

\( \lambda \) unknown, then the likelihood ratio test remains the
same as (3.2.2) with change in the nature of the
distribution of \( T_i \). In this case \( T_i \)s are distributed
as exponential with scale parameter \( \lambda \) under \( H_0 \)

Corollary. 3.2.2. (Pure death process) If the
death rate \( \lambda_i \) is zero for all \( i \), then we get pure
death process. Further if the hypotheses are

\[
H_0 : \mu_i = \mu \text{ for all } i \text{ against } H_1 : \mu_i \neq \mu \text{ for all } i,
\]

\( \mu \) unknown, then the likelihood remains the same as
(3.2.2) with change in the nature of the distribution
of \( T_i \). In this case \( T_i \) are distributed as exponential
with parameter \( \mu \) under \( H_0 \).

Theorem 3.2.2. For a birth and death process
\( \{X(t), t \geq 0\} \) with parameters \( \lambda_i \) and \( \mu_i \) a level \( \alpha \)
likelihood ratio test for testing

\[
H_0 : \lambda_i = i\lambda \text{ and } \mu_i = i\mu \text{ for all } i, \text{ against } \\
H_1 : \lambda_i \neq i\lambda \text{ and } \mu_i \neq i\mu \text{ for all } i,
\]

\( \lambda \) and \( \mu \) unknown, is given by the critical function.
\( \phi(T) = \begin{cases} 1 & \text{if } T \leq T_0 \\ 0 & \text{if } T > T_0 \end{cases} \) ...(3.2.14)

where,

\[ T = \prod (i_T / \Sigma i_T) \] ...(3.2.15)

and

\[ P [ T \leq T_0 \mid H_0 ] = \alpha \] ...(3.2.16)

Proof: The likelihood ratio test criteria is based on the ratio

\[ \Lambda = \frac{\text{Max } L_0(\lambda, \mu)}{\text{Max } L_1(\lambda, \mu)} \]

\[ = \frac{L_0(\hat{\lambda}, \hat{\mu})}{L_1(\hat{\lambda}, \hat{\mu})} \] ...(3.3.17)

where \( \hat{\lambda} \) and \( \hat{\mu} \) are maximum likelihood estimators (M.L.Es) of \( \lambda \) and \( \mu \) under \( H_0 \) which are obtained by
using (3.2.1) as

\[ \hat{\lambda} = \frac{\sum u_i}{\sum i T_i} \quad \text{...(3.3.18)} \]

\[ \hat{\mu} = \frac{\sum d_i}{\sum i T_i} \quad \text{...(3.3.19)} \]

where \( \hat{\lambda} \) and \( \hat{\mu} \) are maximum likelihood estimators of \( \lambda \) and \( \mu \) under \( H_1 \) which are obtained by using (3.2.1) as

\[ \hat{\lambda}_i = \frac{u_i}{T_i}, \quad i = 1, 2, \ldots, n \quad \text{...(3.3.20)} \]

\[ \hat{\mu}_i = \frac{d_i}{T_i}, \quad i = 1, 2, \ldots, n \quad \text{...(3.3.21)} \]

Therefore, using (3.2.18) - (3.2.21) in (3.3.17) with (3.2.1) we have

\[ \Lambda = \frac{(\sum u_i)^{\sum u_i} (\sum d_i)^{\sum d_i}}{\prod T_i^T(u_i + d_i)^\prod (u_i + d_i)} \frac{\prod u_i^{u_i} \prod d_i^{d_i}}{\prod T_i^{T_i} (\sum T_i)^n} \quad \text{...(3.3.22)} \]

where \( u_i > 0, \quad d_i > 0 \)
Here for a given $i^{th}$ transition $(u_i + d_i) = 1$ for all $i$ and $\sum u_i + \sum d_i = n$. Thus taking logs we can get from (3.3.22),

$$\ln \Lambda = (\sum u_i) \ln (\sum u_i) + (\sum d_i) \ln (\sum d_i)$$

$$- n \ln (\Sigma t_i) + \sum \ln t_i - \sum u_i \ln u_i$$

$$- \sum d_i \ln d_i + \sum \ln T_i$$

...(3.3.23)

Since the process is observed for a fixed number of transitions, one cannot expect the number of downward transitions (deaths) to exceed the number of upward transitions. Obviously $\sum d_i \leq n/2$ and $\sum u_i \geq n/2$ and $(\sum d_i / \sum u_i)$ is bounded.

Thus (3.2.22) reduces to

$$\ln \Lambda \approx \ln C + \ln \left[ \prod \left( \frac{iT_i}{\sum iT_i} \right) \right]$$

...(3.3.24)

where $C$ is some known constant which is independent of $T_i$'s.

Thus,

$$\Lambda \propto T,$$

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where,

\[ T = \prod_{i=1}^{n} \left( iT_i / \sum iT_i \right) \]

Reject \( H_0 \) if \( T \leq T_0 \)

where \( T_0 \) is found such that

\[ P \left[ T \leq T_0 \mid H_0 \right] = \alpha \]

Hence the theorem.

**Corollary : 3.2.3.** (Pure birth process) If the death rate \( \mu_i \) is zero for all \( i \), then we get pure birth process. Further if the hypotheses are

\[ H_0 : \lambda_i = i\lambda \quad \text{for all } i \quad \text{against} \]
\[ H_1 : \lambda_i \neq i\lambda \quad \text{for all } i, \]

\( \lambda \) unknown, then the likelihood ratio test remains the same as (3.2.14) with change in the nature of the distribution of \( iT_i \). In this case \( iT_i \) are distributed as exponential with scale parameter \( \lambda \) under \( H_0 \).
Corollary 3.2.4. (Pure death process) If the death rate $\lambda_i$ is zero for all $i$, then we get pure death process. Further if the hypotheses are

$H_0 : \mu_i = i\mu$ for all $i$, against

$H_1 : \mu_i \neq i\mu$ for all $i$,

$\mu$ unknown, then the likelihood ratio test remains the same as (3.2.14) with change in the nature of the distribution of $iT_i$. In this case $iT_i^S$ are distributed as exponential with scale parameter $\mu$ under $H_0$.

Theorem 3.2.3. For a birth and death process with parameters $\lambda_i$ and $\mu_i$ a level $\alpha$ likelihood ratio test for testing

$H_0 : \lambda_i = \lambda$ and $\mu_i = \mu$ for all $i$, against

$H_1 : \lambda_i = i\lambda$ and $\mu_i = i\mu$ for all $i$,

$\lambda$ and $\mu$ unknown, is given by the critical function

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\[
\phi(T) = \begin{cases} 
1 & \text{if } T \leq T_0 \\
0 & \text{if } T > T_0
\end{cases}
\] 
\[\text{...(3.2.25)}\]

where,
\[
T = \left[ \sum_i T_i / \sum T_i \right]^n \left[ \Pi_i \right]^{-1}
\] 
\[\text{...(3.2.26)}\]

\[
P \left[ T \leq T_0 \mid H_0 \right] = \alpha
\] 
\[\text{...(3.2.27)}\]

\text{Proof: The likelihood ratio test criteria is based on the ratio}

\[
\Lambda = \frac{\text{Max } L_0 \left( \lambda, \mu \right)}{\text{Max } L_1 \left( \lambda, \mu \right)}
\]

\[
= \frac{L_0 \left( \hat{\lambda}, \hat{\mu} \right)}{L_1 \left( \hat{\lambda}, \hat{\mu} \right)}
\] 
\[\text{...(3.2.28)}\]

where \(\hat{\lambda}\) and \(\hat{\mu}\) are maximum likelihood estimators (M.L.E s) of \(\lambda\) and \(\mu\) under \(H_0\) which are obtained by using (3.2.1) as

\[
\hat{\lambda} = \frac{\sum u_i}{\sum T_i}
\] 
\[\text{...(3.2.29)}\]
\[ \hat{\mu} = \frac{\sum d_i}{\sum T_i} \] ...(3.2.30)

and \( \hat{\lambda}_i \) and \( \hat{\mu}_i \) are maximum likelihood estimators of \( \lambda_i \) and \( \mu_i \) under \( H_1 \), which are obtained by using (3.2.1) as

\[ \hat{\lambda}_i = \frac{\sum u_i}{\sum iT_i} \] ...(3.3.31)

\[ \hat{\mu}_i = \frac{\sum d_i}{\sum iT_i} \] ...(3.3.32)

Therefore, using (3.2.29) - (3.3.32) in (3.3.28) with (3.2.1) we have,

\[ \Lambda = \left[ \frac{\sum iT_i}{\sum T_i} \right]^n \left( \prod i \right)^{-1} \] ...(3.2.33)

Reject \( H_0 \) if \( T \leq T_0 \)

where,

\[ T = \left[ \frac{\sum iT_i}{\sum T_i} \right]^n \left( \prod i \right)^{-1} \]
Where $T_0$ is found such that

$$P \left[ T \leq T_0 \mid H_0 \right] = \alpha$$

Hence the theorem.

**Corollary. 3.2.5. (Pure birth process)** If the death rate $\mu_i$ is zero for all $i$, then we get pure birth process. Further if the hypotheses are

- $H_0 : \lambda_i = \lambda$ for all $i$ against
- $H_1 : \lambda_i = i\lambda$ for all $i$,

$\lambda$ unknown, then the likelihood ratio test remains the same as (3.2.17) with change in the nature of the distribution of $T_i$. In this case $T_i^g$ are distributed as exponential with scale parameter $\mu$ under $H_0$.

**Corollary. 3.2.6. (Pure death process)** If the death rate $\lambda_i$ is zero for all $i$, then we get pure death process. Further if the hypotheses are
$H_0 : \mu_i = \mu$ for all $i$ against $H_1 : \mu_i = i\mu$ for all $i$.

$\mu$ unknown, then the likelihood ratio test remains the same as (3.2.17) with change in the nature of the distribution of $T_i$. In this case $T_i$ are distributed as exponential with scale parameter $\mu$ under $H_0$.

A summary of test statistics for different hypotheses has been given in Table (3.3) at the end of the chapter.

3.3 NULL DISTRIBUTION OF THE TEST STATISTICS

In the previous section we found the likelihood ratio test statistics to test the different hypotheses. But it is essential to find the distribution of the test statistics $T$ under $H_0$ (Null distribution) to determine the value of $T_0$ for given $\alpha$. We have determined the distribution of $T$ as follows:

In Theorem (3.2.1) we have the test statistics $T = \prod_{i=1}^{n}(T_i / \sum T_i)$. In this case $T_i$'s are i.i.d. exponential. Since $T_i$'s are i.i.d. exponential, one can
view them as i.i.d. chi-square variates with 2 df. Then \( \frac{T_i}{\Sigma T_i} \) has Dirichlet's distribution (See Johnson and Kotz (1972, p. 231)). Hence it is essential to find the distribution of the product of these Dirichlet's distribution to find \( T_0 \). This is a difficult task. One can obtain the approximate distribution function or approximate percentile points, by using either

(i) large sample approximation technique or
(ii) the simulation technique.

In this chapter we have discussed simulation technique to obtain the percentile points for the statistics. However, the general large sample approximation technique will be discussed in chapter IV.

3.4 SIMULATION TECHNIQUE

Case (i): Consider the test statistic obtained in theorem (3.2.1)
\[ T = \prod_{i=1}^{n} \left( \frac{T_i}{\Sigma T_i} \right). \]

In the previous section we have noted that determination of the distribution of the test statistic is a difficult task. However, the expression for the mean and variance of the test statistics
\[ T = \prod_{i=1}^{n} \left( \frac{T_i}{\Sigma T_i} \right) \]
are (see Johnson and Kotz (1972, p.233)).

\[ E(T) = \frac{n!}{(2n+1)!} \]
\[ V(T) = \left\{ \frac{n! 2^{n+1}}{(3n+2)!} - \left[ \frac{n!}{(2n+1)!}\right]^2 \right\} \]

The simulation study of the distribution of the standardised test statistics has been done and percentage points are given in Table (3.1). The simulation algorithm is outlined as follows:

ALGORITHM

1. 1000 random samples of sizes varying from 2 to 13 were taken from exponential distribution using MINITAB.
2. Using the expression for mean and variance find the standardised value of the test statistics.

3. Obtain 1% and 5% cut off points and present in Table (3.1).

Case (ii): In theorem (3.2.2) we have the test statistics \( T \) is

\[
T = \prod_{i=1}^{n} \left( \frac{V_i}{\Sigma V_i} \right)
\]

where \( V_i = iT_i \) are i.i.d exponential. Hence the ratio \((V_i / \Sigma V_i)\) follows Dirichlet's distribution as in case (i). Hence the same cut-off points in Table (3.1) can be used to test the hypotheses in theorem (3.2.2)

Case (iii): In theorem (3.2.3) the test statistics is

\[
T = \left[ \Sigma i T_i / \Sigma T_i \right]^n \left[ \Pi i \right]^{-1}.
\]

Though the \( T_i'^6 \) are negative exponential, neither the exact distribution nor moments can be obtained easily. However simulation technique is used to
determine the 1% and 5% cut-off points. For standardisation we have substituted the sample estimates of mean and standard deviation. The cut-off points are given in Table (3.2)

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<th>5% value</th>
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<td>3.012782E-03</td>
</tr>
<tr>
<td>3</td>
<td>7.541620E-04</td>
<td>9.069191E-04</td>
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<td>1.922247E-07</td>
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<td>6.306778E-10</td>
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TABLE (3.2)

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<td>----------------</td>
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<td>I</td>
<td>$H_0 : \lambda_i = \lambda$ and $\mu_i = \mu$ for all $i$</td>
<td>$\prod T_i / [ \sum T_i ]^n$</td>
</tr>
<tr>
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<td>$H_1 : \lambda_i \neq \lambda$ and $\mu_i \neq \mu$ for all $i$</td>
<td></td>
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<tr>
<td>II</td>
<td>$H_0 : \lambda_i = i\lambda$ and $\mu_i = i\mu$ for all $i$</td>
<td>$\prod iT_i / [ \sum iT_i ]^n$</td>
</tr>
<tr>
<td></td>
<td>$H_1 : \lambda_i \neq i\lambda$ and $\mu_i \neq i\mu$ for all $i$</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$H_0 : \lambda_i = \lambda$ and $\mu_i = \mu$ for all $i$</td>
<td>$(\prod i)^{-1}[ \sum iT_i / \sum T_i ]^n$</td>
</tr>
<tr>
<td></td>
<td>$H_1 : \lambda_i = i\lambda$ and $\mu_i = i\mu$ for all $i$</td>
<td></td>
</tr>
</tbody>
</table>

For pure birth process

| V     | $H_0 : \lambda_i = \lambda$ for all $i$ | $\prod T_i / [ \sum T_i ]^n$ |
|       | $H_1 : \lambda_i \neq \lambda$ for all $i$ |                |
| VI    | $H_0 : \lambda_i = \lambda$ for all $i$ | $(\prod i)^{-1}[ \sum iT_i / \sum T_i ]^n$ |
|       | $H_1 : \lambda_i = i\lambda$ for all $i$ |                |
| VII   | $H_0 : \lambda_i = \lambda$ for all $i$ | $\prod iT_i / [ \sum iT_i ]^n$ |
|       | $H_1 : \lambda_i \neq \lambda$ for all $i$ |                |

For pure death process

| VIII  | $H_0 : \mu_i = \mu$ for all $i$ | $\prod T_i / [ \sum T_i ]^n$ |
|       | $H_1 : \mu_i \neq \mu$ for all $i$ |                |
| IX    | $H_0 : \mu_i = \mu$ for all $i$ | $(\prod i)^{-1}[ \sum iT_i / \sum T_i ]^n$ |
|       | $H_1 : \mu_i = i\mu$ for all $i$ |                |
| X     | $H_0 : \mu_i = i\mu$ for all $i$ | $\prod iT_i / [ \sum iT_i ]^n$ |
|       | $H_1 : \mu_i \neq i\mu$ for all $i$ |                |