CHAPTER II

UMPU TEST FOR TRANSITION DEPENDENCE OF PARAMETERS IN A LINEAR BIRTH AND DEATH PROCESS

2.1 INTRODUCTION

In chapter I, the role of birth and death process for the study of storage models in general and for the study of queueing and inventory models in particular was discussed. We have also pointed out that the need for developing a class of linear birth and death processes model is from biological applications. Muddapur (1993) has studied the state (transition) dependence of a birth process. In this chapter UMPU test for transition dependence of parameters of a linear birth and death process is obtained by reparametering the birth and death rates by introducing a parameter \( \theta \). Some approximate percentage points of the test statistics have been given using simulation technique.

2.2 THE UNIFORMLY MOST POWERFUL TEST

Suppose \( \{X(t), t \geq 0\} \) is a birth and death process
with parameters $\lambda_i$ and $\mu_i$, where $\lambda_i$ is the birth rate and $\mu_i$ is the death rate. To construct the likelihood we proceed as follows. We observe the sample path of the process $\{X(t), t \geq 0\}$ till $n$ transitions or events (after the first event at the epoch $t_0$).

Let $t_0 < t_1 < \ldots < t_n$ be the epochs at which the transition occurs and $T_i = t_i - t_{i-1}$ for $i = 1, 2, \ldots, n$, are inter transition times. It is well known that the inter transition time $T_i$ has an exponential distribution with mean $(\lambda_i + \mu_i)^{-1}$. Let $u_i$ be the upward transition (birth) and $d_i$ be the downward transition (death). Thus for the $i^{th}$ transition $u_i$ is either zero or one and correspondingly $d_i$ is either one or zero. So, $u_i + d_i = 1$, for all $i$, and $\sum u_i + \sum d_i = n$. Thus $u_i$ can be treated as a Bernoulli random variable with probability of birth as: $p = \lambda_i / (\lambda_i + \mu_i)$ and that of death as $1 - p = \mu_i / (\lambda_i + \mu_i)$. Thus the likelihood for the process, based on the above sample path is

$$L(\lambda, \mu) = \prod_{i=1}^{n} u_i \lambda \prod_{i=1}^{n} d_i \mu \prod_{i=1}^{n} e^{-(\lambda_i + \mu_i) T_i}$$

...(2.2.1)
We reparameterise $\lambda_i$ and $\mu_i$ as follows:

$$\lambda_i = \lambda [1 + (i-1)\theta] \text{ for all } i \geq 1$$

$$\mu_i = \mu [1 + (i-1)\theta] \text{ for all } i \geq 1$$

where $\lambda$ and $\mu$ are fixed unknown birth and death rates. $\theta$, $0 \leq \theta \leq 1$ is the unknown parameter, $i$ stands for $i^{th}$ transition.

It may be noted that if $\theta = 1$, then $\lambda_i = i\lambda$ and $\mu_i = i\mu$, viz., $\lambda_i$ and $\mu_i$ are linear functions of $i$. This situation corresponds to linear birth and death process. If $\theta = 0$, then $\lambda_i = \lambda$ and $\mu_i = \mu$ viz., $\lambda_i$ and $\mu_i$ are independent of $i$. This corresponds to the immigration and emigration process or M/M/1 queueing. Now it is of some interest to check whether the birth and death rates are affected by the transition or not. That is, to check whether $\lambda_i$ and $\mu_i$ are linear functions of $i$ or not. This is same as testing

$$H_0 : \theta = 0 \text{ Vs. } H_1 : \theta = 1$$

... (2.2.3)

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Theorem 2.2.1. For a birth and death process 
\( \{X(t), t \geq 0\} \) with parameters \( \lambda_i = \lambda [1 + (i-1) \theta] \) and 
\( \mu_i = \mu [1=(i-1) \theta] \) the UMPU test of size \( \alpha \) for testing
\( H_0 : \theta = 0 \) Vs. \( H_1 : \theta = 1 \) is given by

\[
\phi(T) = \begin{cases} 
1 & \text{if } T \geq C_1 \text{ or } T \leq C_2 \\
0 & \text{otherwise}
\end{cases}
\]

\[\ldots (2.2.4)\]

where,

\[ T = \sum i T_i / \sum T_i \]

\[ E_{H_0} \phi(T) = \alpha \text{ and } \]

\[ E_{H_0} \left[ \sum T_i \phi(T) \right] = \alpha E_{H_0} \left( \sum T_i \right), \]

provided the number of upward transitions is approximately equal to the number of downward transitions. i.e. \( \sum u_i \approx \sum d_i \approx n/2. \)

Proof: From (2.2.1), the likelihood function with reparameterised birth and death rates as given in the theorem is

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Here $\Sigma u_i$ and $\Sigma d_i$ may be treated as total number of births and total number of deaths in $n$ events or transitions viz., $n = \Sigma (u_i + d_i)$. For fairly large $n$ one may approximate $\Sigma u_i = \Sigma d_i = n/2$. (see Wolff (1965)). Thus the likelihood in (2.2.5) may be written as

$$L(\lambda, \mu, \theta) = \lambda \sum u_i \mu \sum d_i \sum_{i=1}^{n} \left[ 1 + (i-1) \theta \right]$$

$$e^{-(\lambda+\mu)} \sum_{i=1}^{n} \left[ 1 + (i-1) \theta \right] T_i$$

...(2.2.5)

From the above equation it follows from Lehman (1976) that $(\Sigma T_i, \Sigma iT_i)$ is a minimal sufficient statistics for $(\lambda + \mu, \theta)$, since it belongs to two parameter exponential family. Further when $\theta = 0$, $\Sigma T_i$ is minimal sufficient statistic for family indexed by the parameter $(\lambda+\mu)$ and the distribution of the statistic $T = \Sigma iT_i / \Sigma T_i$ is independent of $(\lambda+\mu)$. 25
Hence by Basu's theorem \( T \) and \( \Sigma T_i \) are independent under \( H_0 \). Since the family of distributions defined by \( \Sigma T_i \) is complete. Further since \( T \) is an increasing function of \( \Sigma iT_i \) for each \( \Sigma T_i \), there exists a UMPU size \( \alpha \) test based on \( T \) for testing the hypotheses \( \theta = 0 \) given by

\[
\phi(T) = \begin{cases} 
1 & \text{if } T \geq C_1 \text{ or } T \leq C_2 \\
0 & \text{otherwise}
\end{cases}
\]

...(2.2.7)

where \( C_1 \) and \( C_2 \) are chosen so that following conditions hold:

\[
E_{H_0} \phi(T) = \alpha \text{ and } \quad E_{H_0} \left[ \Sigma T_i \phi(T) \right] = \alpha \quad E_{H_0} (\Sigma T_i).
\]

Hence the theorem.

2.3. The Distribution of the Test Statistic \( T \)

Using the fact that \( T_i \) has a negative exponential distribution and \( T \) and \( \Sigma T_i \) are independently distributed under \( H_0 \), we can easily obtain the mean and
variance of $T$ as

$$E_{H_0}(T) = \frac{(n + 1)}{2},$$

$$\text{Var}(T) = \frac{(n - 1)}{12}.$$

Let,

$$V = T - \frac{(n + 1)}{2}$$

$$= \sum [i - \frac{(n + 1)}{2}] T_i / \Sigma T_i$$

$$= \sum W_i T_i / \Sigma T_i$$

where $W_i = [\sqrt{i - n + 1}/2]$. Under $H_0$, the distribution of $V$ is symmetric around the mean 0. Further, from the simulation study, we have found that the percentage points of $Z = \frac{V}{\sqrt{\text{Var}(V)}}$ lie well within -3 to +3 values. Consequently, the UMPU test based on $Z$ can be carried out in practice as follows:

The UMPU test is given by

$$\phi(Z) = \begin{cases} 
1 & \text{if } |Z| \geq k, \\
0 & \text{otherwise}.
\end{cases}$$

Where $k$ is determined from the size condition. That is,
\[ E[\Phi(Z) \mid H_0] = \alpha = \Pr(\mid Z \mid \geq k \mid H_0) \]

where the approximate values of \( k \) can be obtained from Table given below.

2.4. Percentage points of \( Z \)

Based on 1026 samples of sizes \( n = 3, 4, 5, \ldots, 10 \) each for \( Z \) we have obtained the 95%, 97.5%, 99%, and 99.5% points approximately. These are given in Table.

**TABLE 2.1**

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<tr>
<th>Sample size</th>
<th>95%</th>
<th>97.5%</th>
<th>99%</th>
<th>99.5%</th>
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<td>1.9042</td>
<td>2.0890</td>
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