CHAPTER IV
ASYMPTOTIC INFERENCE FOR A CLASS OF BIRTH AND DEATH PROCESS

4.1 INTRODUCTION

In the previous chapter, we have derived a class of likelihood ratio test statistics for transition dependence of parameters of birth and death process based on small samples. In section 3.3 we have determined the percentile points or the null distribution by using simulation technique. In this chapter, a general large sample approximation technique is discussed instead of simulation technique. Though Wolff (1965) has studied the large sample inference in queueing models of birth and death type, he has considered the likelihood observing the process for fixed period of time. Also he has considered some known function of the state j. Here we observe the process for fixed number of transitions and consider the birth and death rates (parameters) to be a linear function of the parameter θ, which is unknown. The
hypothesis testing is made for the parameter $\theta$ which is more general than the form of the hypotheses discussed in chapter-III. Maximum likelihood estimators (m.l.e.) of parameters are obtained in section 4.2. Some test criteria have been derived in section 4.3.

### 4.2 MAXIMUM LIKELIHOOD ESTIMATORS OF PARAMETERS OF A CLASS OF BIRTH AND DEATH PROCESS.

Suppose $\{X(t), t \geq 0\}$ is a birth and death process with parameters $\lambda_i$ and $\mu_i$ where $\lambda_i$ the birth rate and $\mu_i$ is the death rate. The likelihood function of the sample path of the process till $n$ transitions as given by 3.2.1 is

$$L(\lambda, \mu) = \prod_{i=1}^{n} \lambda_i^{u_i} \prod_{i=1}^{n} \mu_i^{d_i} \exp \left\{ -(\lambda_i + \mu_i) T_i \right\}$$

...(4.2.1)

We reparametrise $\lambda_i$ and $\mu_i$ as follows:

$\lambda_i = \lambda [1 + (i-1)\theta]$ for all $i$

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...(4.2.2)
where $\lambda$ and $\mu$ are fixed unknown birth and death rates. $\theta \geq 0$ is the unknown parameter, $i$ stands for the $i^{th}$ transition.

It may be noted that if $\theta = 1$, then $\lambda_i = i\lambda$ and $\mu_i = i\mu$ viz., $\lambda_i$ and $\mu_i$ are linear functions of $i$. This situation corresponds to linear birth and death process, and if $\theta = 0$ then $\lambda_i = \lambda$ and $\mu_i = \mu$ viz.; $\lambda_i$ and $\mu_i$ are independent of $i$. This corresponds to the immigration and emigration process or $M/M/1$ queueing process.

**THEOREM 4.2.1.** For a birth and death process $\{X(t), t \geq 0\}$ with parameters $\lambda_i = \lambda [1 + (i-1) \theta ]$ and $\mu_i = \mu [1 + (i-1) \theta ]$, $\lambda$, $\mu$, and $\theta$ the m.l.e of $\lambda$, $\mu$ and $\theta$ are given by

\[
\hat{\lambda} = \frac{\sum u_i [n \sum T_i - K \sum (i-1) T_i]}{n [\sum T_i]^2} \tag{4.2.3}
\]

\[
\hat{\mu} = \frac{\sum d_i [n \sum T_i - K \sum (i-1) T_i]}{n [\sum T_i]^2} \tag{4.2.4}
\]
\[ \hat{\theta} = \frac{k \sum T_i}{n \sum T_i - k \sum (i-1) T_i} \]  
\[ \ldots (4.2.5) \]

where,

\[ k = \sum_{i=2}^{n} \left[ \frac{1}{(i-1)} \right] \]
\[ \ldots (4.2.6) \]

Proof: The likelihood (4.2.1) after substituting the value of \( \lambda_i \) and \( \mu_i \) becomes

\[ L(\lambda, \mu) = \lambda \sum u_i \mu \sum d_i e^{-(\lambda+\mu) \sum [1+(i-1)\theta] T_i} \sum_{i=1}^{n} [1+(i-1)\theta] \]
\[ \ldots (4.2.7) \]

The m.l.e of \( \lambda, \mu \) and \( \theta \) are the solutions of the likelihood equations.

\[ \sum [1+(i-1)\theta] T_i - (\sum u_i) / \hat{\lambda} = 0 \]
\[ \ldots (4.2.8) \]

\[ \sum [1+(i-1)\theta] T_i - (\sum d_i) / \hat{\mu} = 0 \]
\[ \ldots (4.2.9) \]
\[ \sum \frac{(i-1)}{[1+(i-1)\theta]} - \left( \hat{\lambda} + \hat{\mu} \right) \Sigma (i-1) T_i = 0 \]  \hspace{1cm} \ldots(4.2.10)

(4.2.8) yields,

\[ \hat{\lambda} = \frac{\Sigma u_i}{\Sigma [1+(i-1)\theta]T_i} \]  \hspace{1cm} \ldots(4.2.11)

(4.2.9) yields,

\[ \hat{\mu} = \frac{\Sigma d_i}{\Sigma [1+(i-1)\theta]T_i} \]  \hspace{1cm} \ldots(4.2.12)

The expression for \( \hat{\lambda} \) and \( \hat{\mu} \) consists of \( \theta \). Thus as a first approximation, \( \theta \) can be obtained by using expressions (4.2.10), (4.2.11) and (4.2.12). Adding (4.2.11) and (4.2.12) and simplifying we get,

\[ n - (\hat{\lambda} + \hat{\mu}) \Sigma T_i - \theta (\hat{\lambda} + \hat{\mu}) \Sigma (i-1) T_i = 0 \]  \hspace{1cm} \ldots(4.2.13)

From (4.2.10) and (4.2.13) we get,
\[ [n - (\lambda + \mu) \sum T_i] / \theta \]

\[ = (\hat{\lambda} + \hat{\mu}) \sum (i-1) T_i \]

\[ = \sum \frac{(i-1)}{1 + (i-1) \theta} \]

\[ \text{...(4.2.14)} \]

Implies,

\[ [n - (\lambda + \mu) \sum T_i] = \theta \sum (i-1) / [1 + (i-1) \theta] \]

\[ = \sum \left[ 1 + 1 / (i-1) \theta \right]^{-1} \]

Considering only first two terms of the expansion of the expression in right hand side one can get the m.l.e. \( \hat{\theta} \) of \( \theta \) as

\[ \hat{\theta} = k / (\hat{\lambda} + \hat{\mu}) \sum T_i \]

where,

\[ k = \sum_{i=2}^{n} \left[ 1 / (i-1) \right] \]

Thus one can obtain
\[ \hat{\theta} = \frac{k \sum T_i}{n \sum T_i - k \sum (i-1)T_i} \] (after simplification)

Which is (4.2.5).

Substituting this estimator of \( \theta \) in (4.2.11) and (4.2.12) gives the m.l.e of \( \lambda \) and \( \mu \) as in (4.2.2) and (4.2.3).

Hence the theorem.

4.3 ASYMPTOTIC TEST FOR BIRTH AND DEATH PROCESS

In this section we derive the likelihood ratio test for testing the parameter \( \theta \).

THEOREM 4.3.1. For a birth and death process \((X(t), t \geq 0)\) with parameters \( \lambda_i = \lambda [1 + (i-1) \theta] \) and \( \mu_i = \mu [1 + (i-1) \theta] \) and a level \( \alpha \) likelihood ratio test for testing \( H_0: \theta = \theta_0 \) Vs : \( \theta \neq \theta_0 \) is given by the critical function
\[ \phi(W) = \begin{cases} 
1 & \text{if } W \leq C \\
0 & \text{if } W > C 
\end{cases} \quad \text{...(4.3.1)} \]

where,

\[ W = [1 + K_1 S + K_2 S^2]^{-n} \]

\[ K_1 = \theta_0 - K / n, \]

\[ k = \sum_{i=2}^{n} \left[ \frac{1}{(i-1)} \right], \]

\[ K_2 = K \theta_0 / n, \]

\[ S = \sum_{i=2}^{n} (i-1) T_i / \sum_{i=1}^{n} T_i \text{ and} \]

\[ P[W \leq C \mid H_0] = \alpha \]

Proof: The likelihood ratio test criteria is based on the ratio

\[ W = \frac{\text{Max. } L_0 (\lambda, \mu, \theta)}{\text{Max } L_1 (\lambda, \mu, \theta)} \quad \text{...(4.3.3)} \]
\[ L_0 = \text{Max } L_0 (\hat{\lambda}, \hat{\mu}, \theta) = L_0 (\hat{\lambda}, \hat{\mu}, \theta) \] where \( \hat{\lambda} \) and \( \hat{\mu} \) are m.l.e of \( \lambda \) and \( \mu \) under \( H_0 \) given respectively by

\[
\hat{\lambda} = \frac{\sum u_i}{\sum [1 + (i-1) \theta_0] T_i}
\]

\[
\hat{\mu} = \frac{\sum d_i}{\sum [1 + (i-1) \theta_0] T_i}
\]

\[ L_1 = \text{Max } L_1 (\hat{\lambda}, \hat{\mu}, \theta) = L_1 (\hat{\lambda}, \hat{\mu}, \hat{\theta}) \] where \( \hat{\lambda}, \hat{\mu} \) and \( \hat{\theta} \) are m.l.e of \( \lambda, \mu \) and \( \theta \) under \( H_1 \) given respectively by

\[
\hat{\lambda} = \frac{\sum u_i [n \sum T_i - K \sum (i-1) T_i]}{n [\sum T_i]^2}
\]

\[
\hat{\mu} = \frac{\sum d_i [n \sum T_i - K \sum (i-1) T_i]}{n [\sum T_i]^2}
\]

\[
\hat{\theta} = \frac{k \sum T_i}{n \sum T_i - k \sum (i-1) T_i}
\]

where,

\[ k = \sum_{1 \leq i} ^ {n} \left[ \frac{1}{(i-1)} \right] \]

Therefore,
\[
W = \frac{L_0}{L_1}
\]

\[
= \left[ \frac{n \left[ \sum T_i \right]^2}{\left\{ n \sum T_i - k \sum (i-1) T_i \right\} \left\{ n \sum T_i + \theta_0 \sum (i-1) T_i \right\}} \right]^n
\]

\[
= \prod_{i=1}^{n} \left[ \frac{n \sum T_i - k \sum (i-1) T_i \left[ 1 + (i-1) \theta_0 \right]}{n \sum T_i - k \sum (i-1) T_i + (i-1) k \sum T_i} \right]
\]

\[
= \left[ 1 + \left( \theta_0 - k/n \right) S - k\theta_0 S^2 \right]^{-n} \prod_{i=1}^{n} \left( \frac{(n/k) - S}{(n/k) - S + (i-1)} \right)
\]

\[
= \left[ 1 + k_1 S + k_2 S^2 \right]^{-n} \prod_{i=1}^{n} \left( \frac{(n/k) - S}{(n/k) - S + (i-1)} \right)
\]

where,

\[
k_1 = (\theta_0 - k/n),
\]

\[
k_2 = -K \theta_0 / n \quad \text{and}
\]

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\[
S = \left\{ \frac{\sum (i - 1) T_i}{\sum T_i} \right\}
\]

Ignoring the second term which tends to one as \( n \) becomes sufficiently large, we get

\[
W \approx \left[ 1 + k_1 S + k_2 S \right]^{-n}
\]

Thus the null hypotheses \( H_0 \) is rejected if

\[
W \leq C
\]

such that,

\[
P [ W \leq C | H_0 ] = \alpha
\]

\((4.3.12)\)

\((4.3.13)\)

Hence the theorem.

Remark: The critical region in (4.3.13) can be written equivalently, as (See Anderson (1974, p.74) )

\[
S > C_1, \quad S < C_2
\]

\((4.3.14)\)
Where $C_1$ and $C_2$ are given by

\[ C_1 = \frac{-K_1 + \sqrt{K_2^2 - 4K_2(1-c)}}{2K_2} \quad \ldots (4.3.15) \]

\[ C_2 = \frac{-K_1 - \sqrt{K_2^2 - 4K_2(1-c)}}{2K_2} \quad \ldots (4.3.16) \]

Thus the likelihood ratio test of $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \neq \theta_0$ has a rejection region of the form, (4.3.14) but $C_1$ and $C_2$ are not chosen so that the probability of each inequality is $\alpha / 2$ when $H_0$ is true, but they are to be of the form given in (4.3.15) and (4.3.16) where $C$ is chosen so that the probability of the two inequalities is $\alpha$.

It is not simple to obtain the exact distribution of $W$. However (see Wolff, (1965)) an approximate distribution of $\frac{2}{n} \ln(W)$ is chi-square with 3 degrees of freedom; which can be used to obtain $C$. 

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