In some of the earlier chapters, we have used the following results on the sums of martingales. These results are not directly available in the literature and as such are presented here for the ease of reference.

**Lemma A.1**

A sum of martingales is a martingale

**Proof:** For each $l$, let $\{Y_{l,k}; k \geq 1\}$ be a sequence of real r.vs, on the complete probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_k, k \geq 1\}$ be a sequence of increasing sub-$\sigma$-fields of $\mathcal{F}$ and $Y_{l,k}$ be $\mathcal{F}_k$-measurable ($l = 1,2,\ldots$) Let

$$
\mathcal{E}_{l,k} = Y_{l,k} - E(Y_{l,k} | \mathcal{F}_{k-1}) \quad \text{then}
$$

(1) \quad \mathbb{E}(\mathcal{E}_{l,k} | \mathcal{F}_{k-1}) = 0,

Also define $X_{l,k} = \sum_{v=1}^{k} \mathcal{E}_{l,v}$, then

$$
\mathbb{E}(X_{l,k} | \mathcal{F}_{k-1}) = \mathbb{E}(\mathcal{E}_{l,k} | \mathcal{F}_{k-1}) + \mathbb{E}(X_{l,k-1} | \mathcal{F}_{k-1}) = X_{l,k-1}
$$
on using (1). Thus for given \( l, \{ X_{l,k}, \mathcal{F}_k \mid k \geq 1 \} \)

is a martingale sequence \( (l = 1, 2, \ldots) \).

For some positive integer value \( m \), define the sum of martingales

\[
M_{m,k} = \sum_{l=1}^{m} X_{l,k}.
\]

We note that

\[
E \left( M_{m,k} \mid \mathcal{F}_{k-1} \right) = \sum_{l=1}^{m} E(X_{l,k} \mid \mathcal{F}_{k-1}) = \sum_{l=1}^{m} X_{l,k-1} = M_{m,k-1},
\]

and as such for each \( m \), \( \{ M_{m,k}, \mathcal{F}_k \mid k \geq 1 \} \) is a martingale sequence and the lemma is proved.

In general, using the linearity property of conditional expectations, we may prove the following

**Corollary A.1.** Any linear function of martingales is a martingale.

For fixed \( k \), let \( \nu(k) \) be a \( \mathcal{F}_{k-1} \)-measurable r.v. which takes values on the integer set \( \{ 1, 2, \ldots \} \) and whose distribution is independent of \( X \)'s.

Let \( \nu(k) \geq \nu(k-1) \) for all \( k \). We define
a random sum of zero-mean martingales by

\[ M \nu(k), k = \sum_{l=1}^{\omega} X_{l,k} = \sum_{m=1}^{\omega} M_{m,k} I \{ \nu(k) = m \}. \]

Then,

\[ E \left\{ M \nu(k), k \mid \mathcal{F}_{k-1} \right\} = E \left\{ \sum_{l=1}^{\nu(k)} X_{l,k} \mid \mathcal{F}_{k-1} \right\} \]

\[ = \sum_{l=1}^{\nu(k-1)} X_{l,k-1} \]

\[ = M \nu(k-1), k-1, \]

since \( \nu(k) \) is \( \mathcal{F}_{k-1} \)-measurable, and \( \nu(k) - \nu(k-1) \) martingales start from zero at the \( (k-1) \)th instant.

Thus as \( k \) varies, \( \{ M \nu(k), k, \mathcal{F}_k \mid k \geq 1 \} \) is a zero mean martingale and we have the following

**Lemma 4.2** A sum of (random number) of zero-mean martingales is a zero-mean martingale.