CHAPTER VII

INVERSE SAMPLING WITHOUT REPLACEMENT

FOR POST-CLUSTER SAMPLING

7.1 Introduction

We studied in Chapter VI the method of post-cluster sampling which can be used when the information relating to the composition of the clusters is lacking. One of the defects of this sampling method is that the number of clusters formed on the basis of the initial sample would not be a fixed number and consequently we cannot select a predetermined number of distinct post-clusters in the sample. If we wish to have a fixed number, say \( n \), of post-clusters in the initial sample, we can use the method of inverse sampling without replacement, considered in Chapter V, for this purpose. In this chapter we formulate the theory of inverse sampling without replacement for post-cluster sampling. First we consider the case where all the post-clusters, formed with the initial sample, are selected for estimating the correlation mean. Then we consider the case where a sample of post-clusters is drawn to estimate the population mean.
7.2. Selection of all the γ post-clusters.

Sampling Method:
Consider a population of $K$ clusters with the $i$-th cluster having $N_i$ units such that $\sum_{i=1}^{K} N_i = N$. As in Chapter 6, to have a predetermined number $\gamma$ of post-clusters in the initial sample, the units are selected one by one without replacement with equal probabilities until $(\gamma+1)$ distinct clusters are represented in the sample. The selection of the units is discontinued soon after the first unit from the $(\gamma+1)$-th cluster appears in the sample, say, at the $(n+1)$-th draw. This unit appearing at the last draw is rejected from the sample and hence we are left with $n$ units coming from $\gamma$ distinct clusters. In this section we consider the estimation of the population mean $\bar{Y} = \frac{\sum_{i=1}^{K} \bar{Y}_i}{\gamma}$ on the basis of all the $\gamma$ post-clusters.

Suppose that out of $n$ units $n_i (>0)$ come from the $i$-th cluster such that $\sum_{i=1}^{\gamma} n_i = n$. Let $s_i$ be the specified set of $\gamma$ clusters from which the $n$ units are selected. Then the probability that the $n$ draws will yield a specified set, $s_i$ of $\gamma$ post-clusters of sizes $n_1, n_2, \ldots, n_\gamma$, is given by

$$
P \left[ n_1, n_2, \ldots, n_\gamma ; n, s_i \right] = \frac{N - \sum_{i=1}^{\gamma} N_i}{N} \binom{N}{n} \frac{1}{n!} \left( \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_\gamma} \right)$$

as in Chapter 7. In this chapter also we make use of
Theorem 5.1 and Corollaries 5.1, 5.2 and 5.3.

Unbiased estimator:

An unbiased estimator of the population mean $\bar{Y}$ is given by

$$\hat{\bar{Y}} = \frac{1}{n} \sum_{i=1}^{n} n_i \bar{Y}_i$$

where $\bar{Y}_i$ is the mean of $n_i$ units coming from the $i$-th cluster.

Here we take the expectation over two stages; firstly when the vector $(n_1, \ldots, n_s, n_Y)$, the number $n$ and the set $s_Y$ are kept fixed and secondly when all the above factors are allowed to vary. Thus

$$E(\hat{\bar{Y}}) = E \left[ E \left( \frac{1}{n} \sum_{i=1}^{n} n_i \bar{Y}_i \right) \right]$$

$$= E \left( \frac{1}{n} \sum_{i=1}^{n} n_i \bar{Y}_i \right)$$

$$= \bar{Y}$$

which follows by proceeding along the lines followed to prove $E(\hat{\bar{Y}}) = \bar{Y}$ in Chapter V. It should be noted that $\bar{Y}$ is the same as $\bar{Y}$ of Chapter V (eqn. 9) with $x_i$ replaced by $x_i$, $N$ by $N$ and $y_i$ by $n_i \bar{Y}_i$.
Variance of the estimator:

\[
\text{Var}(\widehat{Y}) = \text{Var}(\widehat{Y} \text{ in } t_{1}, \ldots, t_{n}; n, s_{t})
\]

\[
= \sum_{t=1}^{T} \text{Var}(\widehat{Y} \text{ in } t_{1}, \ldots, t_{n}; n, s_{t})
\]

The first term on the right-hand side of (4) can be written as

\[
\text{Var}(\widehat{Y} \text{ in } t_{1}, \ldots, t_{n}; n, s_{t}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} t_{i} \right)
\]

The variance on the right-hand side of (5) can be obtained from Var(\widehat{Y}), given in (22) of Chapter V, by replacing \(N\) by \(N_{1}\), \(N_{2}\) by \(N_{2}\), \(x\) by \(\gamma\) and \(y_{1}\) by \(N_{1} \widehat{Y}_{1}\) and then dividing by \(n^{2}\). Thus

\[
\text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} \widehat{Y}_{i} \right)
\]

\[
= \sigma_{b}^{2} - \frac{1}{N} \sum_{r=1}^{N} (\gamma - r) \left(\frac{N_{r}}{N_{r+1}} - 1\right) \left(\frac{\sum_{i=1}^{N_{1}} t_{i} \widehat{Y}_{i} \widehat{Y}_{1}}{\sum_{i=1}^{N_{1}} t_{i}} \right)^{2}
\]

where

\[
\sigma_{b}^{2}(\gamma) = \frac{1}{2} \sum_{i=1}^{n} \left(\gamma - \frac{\sum_{i=1}^{N_{1}} t_{i} \widehat{Y}_{i} \widehat{Y}_{1}}{\sum_{i=1}^{N_{1}} t_{i}} \right)^{2}
\]

\[
\sigma_{b}(n) = \sigma_{b}^{2}
\]
The second term on the right-hand of (4) can be written as

$$E \left[ \mathbf{V}(\mathbf{Y}_{1}, \ldots, n, \mathbf{y}, s, k) \right] = E \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right) \right]$$

$$= E \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right) \right]$$

where $\sigma_{i}^{2}$ is the variance of $Y_{i}$ units in the $i$-th cluster.

From Corollary 5.1 and eqns. (4) and (11) of Chapter V, we have

$$E \left[ \mathbf{V}(\mathbf{Y}_{1}, \ldots, n, \mathbf{y}, s, k) \right] = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \left( 1 - \frac{1}{\sum_{i=1}^{n} x_{i}^{2}} \right)$$
Similarly from Corollary 5.2 and eqns. (4), (5) and (11)
of Chapter V, we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\sigma_i}{\gamma - r} \right) \leq 1
\]

Substituting from (10) and (11) in (9), we obtain
\[ E \left[ \hat{Y}(x) \right] = \frac{1}{n} \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \left( \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \right) \left[ \frac{1}{\sum_{r=1}^{N} \sum_{r=1}^{N} \sigma_r^2} \right] \]

\[ = \frac{1}{n} \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \left( \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \right) \left[ \frac{1}{\sum_{r=1}^{N} \sum_{r=1}^{N} \sigma_r^2} \right] \]

\[ = \frac{1}{n} \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \left( \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \right) \left[ \frac{1}{\sum_{r=1}^{N} \sum_{r=1}^{N} \sigma_r^2} \right] \]

which follows by adding and subtracting the quantity

\[ \frac{1}{n} \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \left( \sum_{r=1}^{N} \frac{1}{\sum_{r=1}^{N} \sigma_r^2} \right) \left[ \frac{1}{\sum_{r=1}^{N} \sum_{r=1}^{N} \sigma_r^2} \right] \]

Adding (6) and (12) and making use of the relations

\[ \text{[Ref. eqn. (7) of Chapter V]} \]

\[ \text{relations} \]
\[ \sigma^2(r) = \frac{1}{\sum_{i=1}^{N} \Sigma_{j=1}^{n_i}} \left( \frac{\sum_{i=1}^{N} \Sigma_{j=1}^{n_i} \frac{v_{ij}}{r} \frac{y_{ij}}{r} \frac{x_{ij}}{r}}{r} \right)^2 \]

and for \( r=1 \),

\[ \sigma^2(1) = \sigma^2 = \frac{1}{n} \Sigma \Sigma \Sigma \Sigma (-1)^{n} \frac{(n-2)}{n-1} \left( \frac{1}{n-2} \right)^{2} \]

we obtain

\[ \hat{V}(Y) = \sigma^2 - \frac{1}{N} \Sigma \Sigma \Sigma (-1)^{n} \frac{(n-2)}{n-1} \left( \frac{1}{n-2} \right)^{2} \frac{(r=1)}{r} \left( \frac{1-r}{r} \right) \sigma^2 \]

Estimator of the variance:

An unbiased estimator of the second component of the variance (4) can be obtained from (9) as

\[ \hat{\text{Var}}[\hat{V}(\hat{Y} \mid n_1, \ldots, n_{\gamma} ; n, \Sigma_{\gamma})] = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n_i} - \frac{1}{\Sigma_{i}} \right) s_i^2 \]

where \( s_i^2 \) is the sample mean square of \( n_i (\gamma i) \) units from the \( i \)-th cluster. When \( n_i = 1 \), \( s_i^2 \) may be taken to be zero.

In order to obtain an unbiased estimator of the first component of the variance (4), we consider the quantity
defined by
\[
\frac{N-n}{n} s_b^2, \quad \ldots \quad (17)
\]
with
\[
s_b^2 = \frac{1}{n-1} \sum_{i=1}^{\gamma} n_i \left( \bar{Y}_i - \frac{\gamma}{n} \right)^2. \quad \ldots \quad (18)
\]

Now, \( s_b^2 \) can be written as
\[
s_b^2 = \frac{1}{n-1} \left[ \sum_{i=1}^{\gamma} n_i \left( 1 - \frac{n_i}{n} \right) \bar{Y}_i^2 - \frac{1}{n} \sum_{i=1}^{\gamma} n_i \sum_{j=1}^{\gamma} n_j \bar{Y}_i \bar{Y}_j \right]. \quad (19)
\]

Hence
\[
\mathbb{E} \left( s_b^2 \mid \eta_1, \ldots, \eta_\gamma; n, s_\gamma \right)
\]
\[
= \frac{1}{n-1} \left[ \sum_{i=1}^{\gamma} n_i \left( 1 - \frac{n_i}{n} \right) \frac{1}{n} \left( 1 - \frac{n_i}{n} \right) s_i^2 + \frac{1}{n} \sum_{i=1}^{\gamma} n_i \left( \bar{Y}_i - \frac{\gamma}{n} \right) \bar{Y}_i \right] - \frac{1}{n} \sum_{i=1}^{\gamma} n_i \bar{Y}_i \bar{Y}_j \quad (20)
\]
\[
= \frac{1}{n-1} \sum_{i=1}^{\gamma} n_i \left( 1 - \frac{n_i}{n} \right) \left( \frac{1}{n} - \frac{1}{n} \right) s_i^2 + \frac{1}{n-1} \sum_{i=1}^{\gamma} n_i \left( \bar{Y}_i - \frac{\gamma}{n} \right) \bar{Y}_i \bar{Y}_j. 
\]

By making the substitutions as in obtaining (\ref{eq:5}), it can be seen from eqn. (87) of Chapter V that
\[
\mathbb{E} \left[ \frac{N-n}{n(n-1)} \sum_{i=1}^{\gamma} n_i \left( \bar{Y}_i - \frac{\gamma}{n} \right) \left( \bar{Y}_i - \frac{\gamma}{n} \right) \right] = \mathbb{V} \left[ \mathbb{E} \left( \bar{Y}_i \mid \eta_1, \ldots, \eta_\gamma; n, s_\gamma \right) \right]. \quad (21)
\]
From (20) and (21) we have

\[
E\left( \frac{N-n}{n} \frac{S_b}{s_b} \right) = E \left[ \frac{N-n}{n} \hat{\beta}(s_b^2, n_1, \ldots, n_\gamma; n, s_\gamma) \right] = \frac{N-n}{n} \sum_{j=1}^{\gamma} n_j \left( 1 - \frac{n_j}{n} \right) \left( \frac{1}{n_j} - \frac{1}{N_j} \right) \frac{S_b^2}{s_b^2} + \frac{N-n}{n} \sum_{j=1}^{\gamma} n_j \left( 1 - \frac{n_j}{n} \right) \left( \frac{1}{n_j} - \frac{1}{N_j} \right) \frac{s_\gamma^2}{s_b^2}.
\]

Thus, we obtain

\[
E \left( \hat{\beta} \left( \frac{\bar{Y}}{n}, n_1, \ldots, n_\gamma; n, s_\gamma \right) \right) = \frac{N-n}{n} \hat{S}_b - \frac{1}{n-1} \sum_{j=1}^{\gamma} n_j \left( 1 - \frac{n_j}{n} \right) \left( \frac{1}{n_j} - \frac{1}{N_j} \right) s_\gamma^2.
\]  

Adding (16) and (23) and then collecting like terms, we obtain the estimator of the variance as

\[
\hat{\sigma}^2(\bar{Y}) = \frac{N-n}{n} s_b^2 + \frac{1}{n(n-1)} \sum_{j=1}^{\gamma} n_j \left( 1 - \frac{n_j}{n} \right) \left( \frac{1}{n_j} - \frac{1}{N_j} \right) s_\gamma^2.
\]

7.2 Selection of \( m \) out of \( \gamma \) post-clusters.

Now we consider the case where out of the \( \gamma \) post-clusters formed on the basis of the initial sample, \( m \) post-clusters are selected with equal probabilities without replacement.
Unbiased estimator

Here an unbiased estimator of the population mean $\bar{Y}$ is given by

$$\hat{\bar{Y}} = \frac{1}{m} \sum_{i=1}^{m} Y_i$$

($\bar{Y} = n/\gamma$), which is structurally the same as the estimator $\hat{\bar{Y}}_1$ given by (8) of Chapter VI.

Taking the expectation of $\hat{\bar{Y}}'$ firstly over all the possible samples of $m$ post-clusters that can be drawn from the set, $s_v$, of $\gamma$ post-clusters, we obtain

$$E(\hat{\bar{Y}}' | u_1, \ldots, u_m; n_1, \ldots, n_\gamma; n, s_v) = \frac{1}{n} \sum_{i=1}^{n} n_i \hat{Y}_i$$

$$= \hat{\bar{Y}}$$

Since $\hat{\bar{Y}}$ is found to be an unbiased estimator of $\bar{Y}$, $\hat{\bar{Y}}'$ is also an unbiased estimator of $\bar{Y}$.

Variance of $\hat{\bar{Y}}'$:

$$V(\hat{\bar{Y}}') = V \left[ E (\hat{\bar{Y}}' | u_1, \ldots, u_m; n_1, \ldots, n_\gamma; n, s_v) \right]$$

$$+ E \left[ V (\hat{\bar{Y}}' | u_1, \ldots, u_m; n_1, \ldots, n_\gamma; n, s_v) \right]$$

The first component of this variance is
which is given by (15). The second component can be written as

$$\mathbb{E} \left[ \gamma \sum_{i=1}^{\gamma} \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right] = \mathbb{E} \left( \bar{Y} \right)$$

(29)

$$= \frac{\gamma \left( \gamma - m \right)}{m \left( \gamma - 1 \right)} \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{\gamma \gamma} \left( \bar{Y}_i - \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right)^2 \right]$$

(30)

$$= \frac{\gamma \left( \gamma - m \right)}{m \left( \gamma - 1 \right)} \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{\gamma \gamma} \left( \bar{Y}_i - \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right)^2 \right]$$

The expectation of the first term inside the square brackets is given by (13). And the expectation of the second term inside the square brackets can be written as

$$m \left( \begin{array}{c} \frac{1}{n^2} \sum_{i=1}^{\gamma \gamma} \bar{Y}_i^2 \\ \frac{1}{n^2} \sum_{i=1}^{\gamma \gamma} \left( \bar{Y}_i - \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right)^2 \end{array} \right)$$

$$= \frac{1}{m} \sum_{i=1}^{\gamma \gamma} \bar{Y}_i \left( \bar{Y}_i - \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right)^2$$

$$+ \frac{\sum_{i=1}^{\gamma \gamma} \left( \bar{Y}_i - \frac{1}{\gamma} \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right)^2}{\gamma - m}$$

(30)
This follows by substituting \( \frac{Y_i^2}{\sigma_i^2} \) for \( \sigma_i^2 \) in (10) and
\( \frac{Y_i^2}{\sigma_i^2} \) for \( \left[ \sigma_i^2 / (T-1) \right] \) in (11). Adding and subtracting the quantity
\[
\frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) (-1)^{\gamma-r} \left[ \sigma_i^2 / (T-1) \right] \sum_{r=1}^{T} \frac{1}{\sigma_i^2}
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) (-1)^{\gamma-r} \left[ \sigma_i^2 / (T-1) \right] \sum_{r=1}^{T} \frac{1}{\sigma_i^2}
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right),
\]

we obtain from (30)

\[
\begin{align*}
\frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) &= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) &= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) &= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) &= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right) \\
&= \frac{1}{N} \sum_{i=1}^{N} \sum_{i=1}^{N} \left( \frac{Y_i^2}{\sigma_i^2} \right)
\end{align*}
\]
where
\[ \nu_c(r) = \frac{1}{r} \sum_{i=1}^{r} \left( \frac{r \nu_1}{i} \right) \]  \hspace{1cm} (25)
and \( \nu_1 = \frac{r}{\lambda_1} \).

Also,
\[ \hat{\nu}^2 = \hat{\nu}(\hat{y}) + \nu^2 \]  \hspace{1cm} (26)
where \( \hat{\nu}(\hat{y}) \) is given by (25).

Substituting from (12), (25) and (26) into (39), we obtain
\[ E\left[ \nu \left( \hat{y} \right| u_1, \ldots, u_n ; n, \ldots, n, \nu, \nu_1 \right) \nu \]
\[ = \frac{\gamma - \alpha}{m(\gamma - 1)} \left[ \sigma^2 = \frac{1}{\gamma} \sum_{i=1}^{\gamma} \frac{1}{\gamma - r} \left( \frac{1 - F^2(x)}{\gamma - r} \right) \right] \]
\[ = \frac{\gamma - m}{m(\gamma - 1)} \hat{\nu}(\hat{y}) + \frac{\gamma - m}{m} \nu^2 \]  \hspace{1cm} (27)

Adding (23) and (27) and simplifying we obtain the variance
of \( \hat{\nu} \) as
\[ \nu(\hat{y}) = \frac{2}{m} \sum_{i=1}^{N} \left( \frac{1}{\gamma - r} \right) \left( \frac{1}{\gamma - r} \right) \frac{1 - F^2(x)}{\gamma - r} \]
\[ = \frac{\gamma - m}{m(\gamma - 1)} \nu_c(r) \]
Estimator of the variance:

An unbiased estimator of the second component of the variance \( (27) \) can be easily seen to be

\[
\text{Est} \left[ \hat{V}(\hat{Y}^2, \ldots, \hat{Y}_n; \hat{Y}_1, \ldots, \hat{Y}_n) \right] = \frac{\gamma(n-m)}{mn} \sum_{i=1}^{m} \left( \frac{\hat{y}_i - \hat{Y}}{n} \right)^2,
\]

where \( \hat{Y} = \frac{1}{n} \sum_{i=1}^{n} y_i \).

An unbiased estimator of the second component \( (28) \) of the variance \( (27) \) is obtained by finding the estimator of \( \hat{V}(\hat{Y}) \), given by \( (24) \), on the basis of the sample of \( m \) post-clusters. Thus we have

\[
\text{Est} \left[ \hat{V}(\hat{Y}) \right] = \frac{(n-m)}{mn} \sum_{i=1}^{m} \left( \frac{\hat{y}_i - \hat{Y}}{n} \right)^2 + \frac{\gamma}{mn} \sum_{i=1}^{m} \left( \frac{1 - \frac{n_i}{n}}{n} \right) \left( n_i - (n-1) \right) \hat{y}_i^2.
\]

Adding \( (39) \) and \( (44) \), we obtain an unbiased estimator.
\[
\hat{\text{V}}(\hat{\bar{y}}) = \frac{\gamma (\gamma - m)}{m n^2} \left[ \frac{1}{n-1} \sum_{i=1}^{\frac{m}{\gamma}} \left( \frac{1}{n} \sum_{j=1}^{\frac{m}{\gamma}} \left( \bar{Y}_i - \frac{1}{\gamma} \bar{Y}_j \right) \right)^2 \right]
\]

\[
+ \frac{\gamma}{\Gamma n} \sum_{i=1}^{\frac{m}{\gamma}} \frac{1}{n - 1} \sum_{j=1}^{\frac{m}{\gamma}} n_i (\bar{Y}_i \hat{-} \hat{\bar{y}})^2
\]

\[
+ \frac{\gamma}{\Gamma n (n - 1)} \sum_{i=1}^{\frac{m}{\gamma}} \left( 1 - \frac{n_i}{\gamma} \right) \left\{ n_i (n-1) - (n-\bar{n}) \right\}\sigma_i^2
\]

\[
\text{... (41)}
\]