Chapter 5

Steady Flow of Blood through a Stenosed Artery: A Non-Newtonian Fluid Model

5.1 Introduction

Heart disease or acute problems in heart, is usually called a coronary artery disease which arises due to some disorders in blood circulation. This kind of disorders happens due to some blockage in arteries, supplying blood to heart and organs. The cause behind the formation of such blockage in arteries, may be included as the accumulation of fat or oily substances at vessel wall. Other odd activities like, consuming alcohol, smoking cigarette, tension, excitement etc. may increase blood viscosity and lead to heart disease. Due to this blockage in main arteries, patients feel discomfort in regular routine work and finally complete cutoff in blood supply to organs and tissues, may be caused. However, to restore normal blood supply, blockage at an artery wall should be completely removed.

It is already known that atherosclerosis or stenosis is a dangerous cardiovascular disease that may be caused due to an abnormal growth in the lumen of an arterial wall (Young and Tsai, 1973). If the stenosis is present in an artery, normal blood flow is disturbed. In order to understand the effect of stenosis on blood flow through and beyond the narrowed segment of the artery, many studies have been undertaken experimentally and theoretically, by taking blood to act as a Newtonian fluid (Verma and Parihar, 2010; Biswas and Chakraborty, 2010; Srivastava et al., 2010; Biswas and Chakraborty, 2009). It has been observed that whole blood, being predominantly a suspension of erythrocytes in plasma; behave as a non-Newtonian fluid at low shear rates and in small diameter arteries (Cokelet et al., 1963; Goldsmith and Skalak, 1975). Existing literature in this area also reveals that the shear rate of blood is low in the stenosed region. A number of researchers have studied the flow of non-Newtonian fluids (Misra and Shit, 2006; Singh and Shah, 2010; Varshney et al., 2010; Sankar and Hemalatha, 2006; Sankar, 2009) with various perspectives. It is worthwhile to note that the Herschel-Bulkley model is of
general type and the results obtained by this model enables one to derive the corresponding Newtonian fluid, Bingham fluid and also power law fluid, by putting specific measures for the parameters (Nubar, 1971).

Many investigators have theoretically studied the flow of blood through uniform artery and stenosed tubes and, analyzed the influence of slip velocity on the flow variables such as velocity, flow rate, wall shear stress etc. (Haymen, 1973; Chaturani and Biswas, 1983; Majhi and Usha, 1984; Biswas, 2000; Ponalgusamy, 2007; Nubar, 1967; Chaturani and Biswas, 1984). Blood flow experiments (Bennet, 1967; Bugliarello and Hayden, 1962) indicates the existence of slip at the tube wall. Recently, Misra and Shit (2007) and Ponalgusamy (2007) have developed mathematical models for blood flow through stenosed arterial segment, by taking a velocity slip condition at the constricted wall.

In the class of non-Newtonian fluids, an important visco-inelastic fluid that is characterized by the flow behavior index \( n (<, =, > 1) \) and yield stress \( \tau_H \) \( (\geq 0) \), familiar as Herschel–Bulkley fluid (Fung, 1981). A linear profile in shear stress versus strain rate relationship, is observed for \( n=1, \tau_H = 0 \) which case stands for a Newtonian fluid (Schlichting, 1979), whereas for \( n<1 \), fluid behaves as Pseudoplastic (Fung, 1981) and for \( n>1 \), fluid behavior is dialatant (Fung, 1981). Herschel–Bulkley (H-B) fluid models have been investigated by many authors for various perspectives. Siddiqui et al., have studied pulsatile blood flow in a constricted artery, in considering blood as a Herschel–Bulkley fluid (Siddiqui et al., 2010). The Herschel–Bulkley behavior of blood from various considerations, has been dealt in the theoretical model of Misra and Shit (2006).

In view of the above, the present investigation has been devoted to the problem of blood flow through a stenosed segment of an artery with employing a velocity slip at the stenosed wall where the rheology of blood is described by a Herschel–Bulkley model. The variations in the velocity, wall shear stress, the volumetric flow rate, apparent viscosity, pressure gradient and resistance to flow at different stages of the growth of the stenosis, have also been reported.
5.2 Formulation of the problem

We have assumed an axially symmetric, laminar, steady and fully developed flow of blood through a stenosed artery with the formation of a stenosis, as shown in Figure 5.1. Blood is assumed to act like a non-Newtonian (H-B) fluid and the artery is assumed to be a rigid circular tube.

\[ R(z) = R_0 - \frac{\delta}{2} \left[ 1 + \cos \frac{2\pi}{L} (z - d - L_0) \right]; d \leq z \leq d + L_0, \quad (5.2.1) \]

where \( R_0 \) and \( R(z) \) represents the radii of the uniform and constricted regions, \( L_0, d \) and \( \delta \) are the length, location and maximum height of the stenosis and \( L \), the length of the artery, \( r \) and \( z \) are the radial and axial coordinates.

Figure 5.1: Flow geometry and coordinate system.
5.3 Method of Solution

Steady laminar flow of blood—a red cell suspension through a straight rigid tube of circular cross-section, is considered. Flowing fluid blood which is an incompressible one, has been assumed to behave like a Herschel-Bulkley fluid.

The equation of motion for laminar and incompressible, steady, fully-developed, one-dimensional flow of blood is given by

\[ \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{xz} \right) = 0, \quad (5.3.1) \]

\[ \frac{\partial p}{\partial r} = 0, \quad (5.3.2) \]

in which \( \tau_{xz} \) represents the shear stress of blood behaving as a Herschel-Bulkley fluid and \( p \) the pressure at any point.

The constitutive equation for H-B fluid may be written as

\[ -\frac{\partial u}{\partial r} = f(\tau) = \frac{1}{k} \left( \tau - \tau_{H} \right)^n, \tau \geq \tau_{H}, \]

\[ = 0, \quad \tau \leq \tau_{H}, \quad (5.3.3) \]

where \( u \) stands for the axial velocity of blood and \( \tau_{xz} \) is the yield shear stress and \( k \), the fluid viscosity and \( n (\prec, =, \succ 1) \), the fluid behavior index.

The boundary conditions applied for integrating the governing equations (5.3.1) and (5.3.2) are the following:

\[ u = u_s \quad \text{at} \quad r = R(z), \quad (5.3.5) \]

\[ \tau \quad \text{is finite at} \quad r = 0, \quad (5.3.6) \]

where \( u_s \) the axial slip velocity.

Using analytical methods and employing the conditions (5.3.5-5.3.6) in equation (5.3.1), we get
\[ \tau_{rr} = -\frac{r \, dp}{2 \, dz} \]  \hspace{1cm} (5.3.7)

The wall shear stress \( \tau_R \) is given by

\[ \tau_R = -\frac{R \, dp}{2 \, dz} \]  \hspace{1cm} (5.3.8)

where \( R = R(z) \)

Now integrating equation (5.3.3) and using equations (5.3.5-5.3.8) thereof, the velocity function becomes

\[ u(r) = u_s + \frac{R(z) \, \tau_R}{(n+1)k} \left[ (1 - \beta)^{n+1} - \left( \frac{r}{R(z)} - \beta \right)^{n+1} \right], \text{ for } r_p \leq r \leq R(z) \]  \hspace{1cm} (5.3.9)

where \( \beta = \frac{\tau_H}{\tau_R} \).

And the expression for plug flow velocity is

\[ u_p = u_s + \frac{R(z) \, \tau_R}{(n+1)k} \left[ (1 - \beta)^{n+1} \right], \text{ for } 0 \leq r \leq r_p \]  \hspace{1cm} (5.3.10)

The volumetric flow rate \( Q \) is defined as

\[ Q = \int_0^{R(z)} 2\pi ru(r) dr, \]  \hspace{1cm} (5.3.11)

and this after using equations (5.3.9)-(5.3.10), becomes

\[ Q = \pi R(z)^2 u_s + \frac{\pi R(z)^3 \, \tau_R}{(n+3)k} \left\{ 1 + \left( \frac{2}{n+2} \right) \beta + \left( \frac{2}{(n+1)(n+2)} \right) \beta^2 \right\} (1 - \beta)^{n+1}. \]  \hspace{1cm} (5.3.12)

When \( \frac{\tau_H}{\tau_R} \ll 1 \), equation (5.3.12) reduces to
Using equation (5.3.8) in equation (5.3.13), the pressure gradient becomes,

\[
\frac{dp}{dz} = \left\{ \frac{2^n k(n+3)(Q - \pi R(z)^2 u_s)}{\pi R(z)^{n+2}} \right\}^{1/n} + \frac{2(n+3)}{n+2} \frac{\tau_H}{R(z)}. \tag{5.3.14}
\]

Again using equation (5.3.15) in equation (5.3.8) yields, wall shear stress as

\[
\tau_R = \left\{ \frac{k(n+3)(Q - \pi R(z)^2 u_s)}{\pi R(z)^3} \right\}^{1/n} + \frac{n+3}{n+2} \frac{\tau_H}{R(z)}, \tag{5.3.15}
\]

Expression for apparent viscosity is defined as

\[
\mu_a = \frac{\pi CR^4(z)}{8Q}, \tag{5.3.16}
\]

this has the following expression after using the equation (5.3.14)

\[
\mu_a = CR^2(z) \left[ 8u_s + \frac{R(z)}{(n+3)k} \left( \frac{\tau_R}{\tau_0} - \frac{n+3}{n+2} \frac{\tau_H}{\tau_0} \right) \right]^{1/n}, \tag{5.3.17}
\]

Resistance to flow

\[
\lambda = \frac{1}{Q_0} \int_0^l \left( -\frac{dp}{dz} \right) dz, \tag{5.3.18}
\]

where Q and (-dp/dz) are expressed in equations (5.3.13-5.3.14) above.

A second representation of the above flow variables can be obtained as follows:

\[
\frac{-u(r)}{U_0} = u(r), \quad \frac{Q}{Q_0} = \frac{Q}{Q_0}, \quad \tau_{R(z)} = \frac{\tau_{R(z)}}{\tau_0}, \quad \frac{\mu_a}{\mu_0} = \mu_a, \quad \frac{u_s}{U_0} = \frac{u_s}{U_0}, \quad \frac{r}{R_0} = \frac{r}{R_0}, \quad \frac{R(z)}{R_0} = \frac{R(z)}{R_0},
\]

\[
U_0 = \left( \frac{CR_0^{n+1}}{2k} \right)^{1/n}, \quad Q_0 = \pi \left( \frac{CR_0^{n+1}}{2k} \right)^{1/n}, \quad \tau_0 = \left( \frac{1}{k} \right)^{1/n} \left( \frac{CR_0}{2} \right)^{1/n}, \tag{5.3.19}
\]

\[
Q = \pi R(z)^2 u_s + \frac{\pi R(z)^3}{(n+3)k} \left[ \tau_R - \frac{n+3}{n+2} \tau_H \right]^{n}. \tag{5.3.13}
\]
Velocity function:

\[ u(r) = u_s + \frac{R}{(n+1)k} \left[ \frac{1 - \frac{\tau_H}{R}}{R} \right]^{n+1} - \left( \frac{\tau_H}{R} \right)^{n+1} \]  

(5.3.20)

Flow rate:

\[ \bar{Q} = R^2 u_s + \frac{R^3}{(n+3)k} \left[ \frac{2(n+3)}{n+2} \frac{\tau_H}{R} \right]^{n} \]  

(5.3.21)

Wall shear stress:

\[ \frac{\bar{\tau}_{R(z)}}{k(n+3)}Q - \pi R_0^2 \left( \frac{R(z)}{R_0} \right)^2 u_s \right]^{\frac{1}{n}} + \frac{n+3}{n+2} \left( \frac{\pi R_0}{R} \right)^{\frac{1}{n}} \frac{\tau_H}{R} \]  

(5.3.22)

Pressure gradient:

\[ \frac{d\bar{p}}{dz} = \left\{ \frac{2(n+3)k}{R} \left( \frac{\bar{Q}}{R} - \frac{2^{-n+1} u_s}{R} \right) \right\}^{\frac{1}{n}} + \frac{2(n+3)}{n+2} \tau_H. \]  

(5.3.23)

Apparent viscosity:

\[ \frac{\bar{\mu}_o}{R(z)} = \frac{\bar{Q}}{Q}, \]  

(5.3.24)

where non-dimensional \( \bar{Q} \) is defined in equation (5.3.21)
Resistance to flow:

\[ \lambda = \frac{1}{Q} \int_{0}^{L} \left( \frac{dp}{dz} \right) dz. \]  

(5.3.25)

5.4 Results and Discussions

In this theoretical analysis for blood flow, three gradual stages of a stenosis developed at the inner most artery wall for blood flow in a stenosed vessel, subject to a velocity slip or no-slip condition, is taken for investigation (Figure 5.1). In the present study, H-B behavior of blood, is considered. Analytical expressions for axial velocity, flow rate, wall shear stress, apparent viscosity, pressure gradient and resistance to flow are obtained and their variations with different flow parameters are shown graphically.

5.4.1 Axial Velocity

Axial velocity is computed from equation (5.3.20) in dimensionless form and its variation with different parameters is presented graphically.

Variation of axial velocity with radial distance in the constricted region, for different values of slip velocities at the stenotic wall, different stenosis size viz. mild, moderate and severe forms, yield stress and fluid behaviour index, have been represented in Figures 5.2-5.4 respectively.
**Figure 5.2:** Variation of axial velocity with radial distance for different values of yield stress and slip velocity for $\delta=1/2$.

**Figure 5.3:** Variation of axial velocity with radial distance for different values of fluid behaviour index $n$ and slip velocity for $\delta=1/2$. 
Figure 5.4: Variation of axial velocity with radial distance for different values of stenosis size and slip velocity for $n=1$.

From the figures, it is observed that in all cases of stenosis formations, and for variations of fluid index $n$, axial velocity increases for the employment of a slip condition at the stenotic wall. Its magnitude, calculated with an axial slip, is greater than that obtained with zero slip. As stenosis size increases from mild to severe stage, axial velocity decreases for both slip and no-slip cases at the stenotic wall. Axial velocity is maximum at the initiation of the stenosis and minimum at the throat of the stenosis. However, as fluid behavior index $n$ increases in magnitude for the cases $n<1$, $n=1$, $n>1$, velocity decreases throughout the stenotic region. Axial velocity decreases with the increase of yield stress.
5.4.2 Flow Rate

Variation of flow rate can be obtained from equation (5.3.21) against axial distance for the cases of slip and no-slip at the stenotic region, different stenosis size, yield stress and fluid viscosity are drawn in Figures 5.5-5.7.

![Diagram showing variation of flow rate with axial distance for different values of slip velocity and stenosis size.](image)

**Figure 5.5:** Variation of flow rate with axial distance for different values of slip velocity and stenosis size.
Figure 5.6: Variation of flow rate with axial distance for different values of yield stress for $\delta=1-\sqrt{3}/2$.

Figure 5.7: Variation of flow rate with axial distance for different values of fluid viscosity and slip velocity for $\delta=1-\sqrt{3}/2$. 
The figures show that the flow rate is minimum at the throat of the stenosis and maximum at its either end (i.e., at the initiation and at the termination of the stenosis). As slip increases, magnitude of $\bar{Q}$ increases rapidly. However, values of $\bar{Q}$ obtained with velocity slip are greater than those with no-slip at the interface. As height of the stenosis increases from mild to severe form, flow rate decreases significantly. As fluid viscosity and yield stress increase, flow rate reduces from higher magnitude at the constricted region of an artery.

5.4.3 Wall Shear stress

Wall shear stress is computed from the equation (5.3.22) and its variations with axial distance for slip and no-slip cases, different fluid behavior index $n$, yield stresses and different stenosis size, are shown in Figures 5.8-5.10.

Figure 5.8: Variation of wall shear stress with axial distance for different values of fluid behavior index and slip velocity for $\delta=1-\sqrt{3}/2$. 
**Figure 5.9:** Variation of wall shear stress with axial distance for different values of yield stress and for $\delta = 1 - \sqrt{3}/2$.

**Figure 5.10:** Variation of wall shear stress with axial distance for different values of stenosis size and slip velocity for $\tau_H = 0.01$. 
From the figures it has been noticed that wall shear stress is the minimum at the either end of attains stenosis and the maximum magnitude at the throat of the stenosis. As fluid behavior index \( n \) increases from \( n<1 \) to \( n>1 \), wall shear stress decreases significantly, which is similar to (Misra and Shit, 2006). It is also important to observe that the effect of yield stress (figure 5.9) on wall shear stress is marginal, which is also consistent with the result found by the authors (Misra and Shit, 2006). Also, it is observed that increase in the stenosis size up to a certain level, does not have any significant effect on wall shear stress. In all the cases, wall shear stress is the minimum for a velocity slip at the stenotic wall than that of the the same with no-slip cases.

5.4.4 Pressure Gradient

Pressure gradient is computed from the equation (5.3.23) in dimensionless form and its variations with different flow parameters are presented graphically, in Figures (5.11-5.12).

Variation of pressure gradient with axial distance for different stenosis size, slip and no-slip cases and fluid viscosity are represented, in Figures 5.14-5.15 respectively.

![Figure 5.11](image.png)

**Figure 5.11**: Variation of pressure gradient with axial distance for different values of stenosis size and slip velocity for \( n=1 \).
Figure 5.12: Variation of pressure gradient with axial distance for different values of fluid viscosity and slip velocity for $\delta = 1 - \sqrt{3}/2$.

It is clearly indicated in the figures that pressure gradient is lower in magnitude in all the cases for a velocity slip than that of no-slip at the constricted wall. However, as stenosis size increases from mild to severe, pressure gradient significantly increases to a higher magnitude. As fluid viscosity $k$ increases, pressure gradient is also increases prominently.

5.4.5 Apparent viscosity

Apparent viscosity $\mu_a$ is computed with the help of the equation (5.3.24) and its variation with axial distance in the stenotic region, for three cases of stenosis size, different values of yield stress, fluid viscosity and for slip and no-slip cases, are shown in Figures 5.13-5.15.
Figure 5.13: Variation of apparent viscosity with axial distance for different values of yield stress and slip velocity for $\delta=1-\sqrt{3}/2$.

Figure 5.14: Variation of apparent viscosity with axial distance for different values of stenosis size and slip velocity for $n=1$. 
In the figures, it has been clearly indicated that apparent viscosity decreases with a slip at the boundary and as the slip increases in magnitude, it decreases accordingly. Further, apparent viscosity increases as yield stress increases. Also, as height of the stenosis rises from mild to severe one, apparent viscosity increases from lower to greater magnitude. Magnitude of apparent viscosity is found to be higher as fluid viscosity increases from a lower to higher value.

**5.4.6 Resistance to Flow**

Resistance to flow is obtained from the equation (5.3.25) and its variation with other flow parameters are shown graphically, in (Figures 5.16-5.17).

Variation of resistance to flow with different stenosis size, flow behavior index n, different yield stress and for slip and no-slip cases are presented in, Figures 5.16-5.17, respectively.
Figure 5.16: Variation of resistance to flow with stenosis size for different values of fluid behavior index $n$ and slip velocity.

Figure 5.17: Variation of resistance to flow with yield stress for different values of slip velocity for $\delta=0.1$. 
The results presented in these figures indicate that flow resistance increases with the increasing value of stenosis size. Further, as the fluid behavior index $n$ decreases, the flow resistance increases significantly, which is also similar to (Misra and Shit, 2006). Also, it is clear that as the yield stress increases from 0.00 to 0.20, flow resistance increases considerably. In all the cases, flow resistance decreases with the employment of a velocity slip than that with no-slip at the vessel wall.

5.5 Conclusion

In this investigation, the steady flow of blood in a uniform artery with stenosis, has been analyzed. A condition for an axial velocity slip, is applied at the flow boundaries and three different stages of stenosis viz., mild, moderate and severe, have been considered. Analytical expressions for axial velocity, flow rate, wall shear stress, apparent viscosity, pressure gradient, resistance to flow, are obtained and their variations with different flow parameters have been depicted in figures.

It is observed that both axial velocity and the flow rate, will increase with an increase of a velocity slip but decreases with an increase in height of the stenosis. Shear stress attains the maximum magnitude at the throat of the stenosis and lower magnitudes at either end of the stenosis. Further, the wall shear stress is found to be lower, due to the employment of a slip at the wall. An increase in the size of the stenosis increases resistance to blood flow through arteries in the brain, heart and other organs of the body. This may lead to stroke, heart attack and various cardiovascular diseases.

The present study reveals that by the introduction of a velocity slip at the vessel wall, speed and flow rate can be accelerated on one hand, and shear stress, apparent viscosity, pressure gradient and resistance to flow, can be retarded on the other. As a result, this will help in avoiding rupture at the constricted wall as well as it will help in improving the diseased artery wall which in turn, will help in making the flow normal and regular.

It is already known that due to the intermitted pumping of heart muscles, there produces a pressure difference (called pressure pulse) arising due to the difference of diastolic and systolic states, the double pump heart which in turn leads to the fact that blood flow is
pulsatile in human circulatory system. Further, due to vibration motion of any kind, a body acceleration may be caused intentionally or unintentionally. In order to include both the characteristics, a pulsatile blood flow model with body acceleration has been proposed in the next chapter.