Chapter 5
Prime Ternary Subsemimodules

5.1 Introduction
Prime ideals of ternary semirings are studied by T. K. Dutta and S. Kar [29]. In [4], R. E. Atani, has extended this work for semimodules over semirings. Prime avoidance theorem for ideals in ternary semirings is given by J. N. Chaudhari and K. J. Ingale [25]. In this chapter, we introduce the concept of prime ternary subsemimodule and hence we extend some basic results of ternary semirings [29], semimodules over semirings [4] and partial semimodules [49] to ternary semimodules over ternary semirings. Also we prove the prime avoidance theorem for ternary semimodules over ternary semirings. D. D. Anderson and E. Smith [3] introduced the notion of weakly prime ideals in commutative ring with non-zero identity in 2003. Later on, this concept has been studied in modules and semirings by many authors [7, 11, 39]. Further it is extended for semimodule by J. N. Chaudhari and D. R. Bonde [20]. We introduce the concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules.
5.2 Prime ternary subsemimodules

In this section, we introduce the notion of prime ternary subsemimodule of a ternary semimodule over a ternary semiring which is a generalization of prime subsemimodule introduced by R. E. Atani [4]. Moreover, we extend the results of semimodules, ternary semirings and partial semimodules to ternary semimodules over ternary semirings.

**Definition 5.1.** A proper ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ is called prime ternary subsemimodule if $r, s \in R, n \in M$ and $rsn \in N$, then $r \in (N : M)$ or $s \in (N : M)$ or $n \in N$.

**Lemma 5.2.** Let $N$ be a proper ternary subsemimodule of a ternary $R$-semimodule $M$. Then the following statements are equivalent:

i) $N$ is a prime ternary subsemimodule;

ii) If $IJD \subseteq N$ where $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$, then $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$.

**Proof.** (i)$\Rightarrow$(ii) Let $IJD \subseteq N$ where $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$. Suppose that $J \not\subseteq (N : M)$ and $D \not\subseteq N$. Choose $r_2 \in J$ and $x \in D$ such that $r_2 \not\in (N : M)$ and $x \not\in N$. Let $r_1 \in I$. Now $r_1r_2x \in IJD \subseteq N$. Since $N$ is a prime ternary subsemimodule, $r_1 \in (N : M)$. Hence $I \subseteq (N : M)$.

(ii)$\Rightarrow$(i) Let $r_1r_2m \in N$ where $r_1, r_2 \in R$ and $m \in M$. Take $I = \langle r_1 \rangle = RRr_1$, $J = \langle r_2 \rangle = RRr_2$ and $D = \langle m \rangle = RRm$. Then $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$ such that $IJD \subseteq N$. By assumption either $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$. So either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Hence $N$ is a prime ternary subsemimodule of $M$. $\square$
**Theorem 5.3.** If \( N \) is a prime ternary subsemimodule of a ternary \( R \)-semimodule \( M \), then \( (N : M) \) is a prime ideal of \( R \).

**Proof.** Suppose that \( N \) is a prime ternary subsemimodule of \( M \). Let \( a, b, c \in R, abc \in (N : M) \) and \( a \notin (N : M), b \notin (N : M) \). Now \( ab(crM) = (abc)rM \subseteq N \) for all \( r \in R \). Since \( N \) is a prime ternary subsemimodule, \( crM \subseteq N \) for all \( r \in R \). Hence \( c \in (N : M) \). Thus, \( (N : M) \) is a prime ideal of \( R \). \( \square \)

The following example shows that the converse of the Theorem 5.3 is not true.

**Example 5.4.** Consider the ternary \( \mathbb{Z}_0^- (= R) \)-semimodule \( \mathbb{Z}_0^- \times \mathbb{Z}_0^- (= M) \) under the ternary scalar multiplication \( * : (r, s, (m, n)) \mapsto (rsm, rsn) \). Clearly, \( N = \{0\} \times (-8)\mathbb{Z}_0^- \mathbb{Z}_0^- \) is a ternary subsemimodule of \( M \) but it is not a prime ternary subsemimodule of \( M \) because \( (-2) * (-2) * (0, -2) = ((-2)(-2)0, (-2)(-2)(-2)) = (0, -8) \in N \) but \( -2 \notin (N : M) \) and \( (0, -2) \notin N \). Clearly, \( (N : M) = \{0\} \) is a prime ideal of \( \mathbb{Z}_0^- \).

**Definition 5.5.** A ternary \( R \)-semimodule \( M \) is said to be multiplication ternary semimodule if for each ternary subsemimodule \( N \) of \( M \), there exists an ideal \( I \) of \( R \) such that \( N = I RM \).

**Lemma 5.6.** If \( M \) is a multiplication ternary \( R \)-semimodule and \( N \) is a ternary subsemimodule of \( M \), then \( N = (N : M)RM \).

**Proof.** Since \( M \) is a multiplication ternary semimodule, there exists an ideal \( I \) of \( R \) such that \( N = I RM \). Then \( I RM \subseteq N \Rightarrow I \subseteq (N : M) \). So \( N = I RM \subseteq (N : M)RM \subseteq N \). Hence \( N = (N : M)RM \). \( \square \)

Now we show that the converse of the Theorem 5.3 is true for the multiplication ternary semimodules.
Theorem 5.7. Let $N$ be a ternary subsemimodule of a multiplication ternary $R$-semimodule $M$. Then $N$ is a prime ternary subsemimodule of $M$ if and only if $(N : M)$ is a prime ideal of $R$.

Proof. Proof of the direct part follows from Theorem 5.3. Conversely, suppose that $(N : M)$ is a prime ideal of $R$. Let $I, J$ be ideals of $R$ and $K$ be a ternary subsemimodule of $M$ such that $IJK \subseteq N$. Since $M$ is a multiplication ternary semimodule, there exists an ideal $L$ of $R$ such that $K = LRM$. Now $N \supseteq IJK = IJ(LRM) = (IJL)RM$. Hence $IJL \subseteq (N : M)$. Since $(N : M)$ is prime ideal of $R$, $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $L \subseteq (N : M)$. Thus $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $K = LRM \subseteq N$. Hence by Lemma 5.2, $N$ is a prime ternary subsemimodule of $M$.

Theorem 5.8. A ternary $R$-semimodule $M$ is a multiplication ternary semimodule if and only if for each $m \in M$, there exists an ideal $I$ of $R$ such that $RRm =IRM$.

Proof. Suppose that $M$ is a multiplication ternary $R$-semimodule. Let $m \in M$. Then $RRm$ is a ternary subsemimodule of $M$. Hence there exists an ideal $I$ of $R$ such that $RRm =IRM$. Conversely, suppose that for each $m \in M$, there exists an ideal $I$ of $R$ such that $RRm =IRM$. Let $N$ be a ternary subsemimodule of $M$. Then for $n \in N$, there exists an ideal $I_n$ of $R$ such that $RRn = I_nRM$. Denote $I = \sum_{n \in N} I_n$. Then $I$ is an ideal of $R$. Now $N = \sum_{n \in N} RRn = \sum_{n \in N} I_nRM = \left( \sum_{n \in N} I_n \right)RM = IRM$. Hence $M$ is a multiplication ternary $R$-semimodule.

Definition 5.9. Let $N_1, N_2, N_3$ be ternary subsemimodules of a multiplication ternary $R$-semimodule $M$ such that $N_1 =IRM, N_2 = JRM$ and $N_3 = KRM$ for some ideals $I, J, K$ of $R$. Then the ternary multiplication of $N_1, N_2$ and $N_3$ is defined as
\[ N_1N_2N_3 = (IRM)(JRM)(KRM) = (IJK)RM. \]

**Definition 5.10.** Let \( M \) be a multiplication ternary \( R \) semimodule and \( m_1, m_2, m_3 \in M \) be such that \( RRm_1 = I_1RM, RRm_2 = I_2RM \) and \( RRm_3 = I_3RM \) for some ideals \( I_1, I_2 \) and \( I_3 \) of \( R \). Then the ternary multiplication of \( m_1, m_2 \) and \( m_3 \) is defined as \( m_1m_2m_3 = (I_1RM)(I_2RM)(I_3RM) = (I_1I_2I_3)RM. \)

Now the following theorem give characterizations of prime ternary subsemimodules of multiplication ternary \( R \)-semimodule \( M \).

**Theorem 5.11.** Let \( N \) be a proper ternary subsemimodule of a multiplication ternary \( R \)-semimodule \( M \). Then the following statements are equivalent:

1) \( N \) is a prime ternary subsemimodule.

2) For any ternary subsemimodules \( U, V \) and \( W \) of \( M, UVW \subseteq N \) implies \( U \subseteq N \) or \( V \subseteq N \) or \( W \subseteq N \).

3) For any \( m_1, m_2, m_3 \in M, m_1m_2m_3 \subseteq N \) implies \( m_1 \in N \) or \( m_2 \in N \) or \( m_3 \in N \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( U, V, W \) be ternary subsemimodules of \( M \) such that \( UVW \subseteq N \). Since \( M \) is a multiplication ternary semimodule, there exist ideals \( I, J \) and \( K \) of \( R \) such that \( U = IRM, V = JRM \) and \( W = KRM \). Now \((IJK)RM = UVW \subseteq N \Rightarrow IJK \subseteq (N : M)\). By Theorem 5.3, \((N : M)\) is a prime ideal of \( R \). So \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( K \subseteq (N : M) \). Hence \( U = IRM \subseteq N \) or \( V = JRM \subseteq N \) or \( W = KRM \subseteq N \).

(2) \( \Rightarrow \) (3) Let \( m_1, m_2, m_3 \in M \) such that \( m_1m_2m_3 \subseteq N \). Since \( M \) is a multiplication ternary semimodule, there exist ideals \( I, J \) and \( K \) of \( R \) such that \( RRm_1 = IRM, RRm_2 = JRM \) and \( RRm_3 = KRM \). Now \((RRm_1)(RRm_2)(RRm_3) = m_1m_2m_3\).
\( \subseteq N \Rightarrow RRM_1 \subseteq N \) or \( RRM_2 \subseteq N \) or \( RRM_3 \subseteq N \). Hence \( m_1 \in N \) or \( m_2 \in N \) or \( m_3 \in N \).

(3) \( \Rightarrow \) (1) Let \( I, J \) and \( K \) be ideals of \( R \) such that \( IJK \subseteq (N : M) \). Then \((IJK)RM \subseteq N\). Suppose that \( I \not\subseteq (N : M) \), \( J \not\subseteq (N : M) \) and \( K \not\subseteq (N : M) \). Therefore \( IRM \not\subseteq N \), \( JRM \not\subseteq N \) and \( KRM \not\subseteq N \). Choose \( i \in I, j \in J, k \in K, r_1, r_2, r_3 \in R \) and \( m_1, m_2, m_3 \in M \) such that \( ir_1m_1 \in IRM \setminus N, jr_2m_2 \in JRM \setminus N \) and \( kr_3m_3 \in KRM \setminus N \). Now \(((ir_1m_1)(jr_2m_2)(kr_3m_3)) \subseteq (IRM)(JRM)(KRM) = (IJK)RM \subseteq N \Rightarrow ir_1m_1 \in N \) or \( jr_2m_2 \in N \) or \( kr_3m_3 \in N \), a contradiction to \((*)\). Hence \((N : M)\) is a prime ideal of \( R \). Therefore by Theorem 5.7, \( N \) is a prime ternary subsemimodule of \( M \).

Definition 5.12. A subset \( S \) of a multiplication ternary \( R \)-semimodule \( M \) is said to be multiplicatively closed subset (in short closed subset) i.e. \( m \)-system if \( n_1, n_2, n_3 \in S \), then \( n_1n_2n_3 \cap S \neq \phi \).

Proposition 5.13. A proper ternary subsemimodule \( N \) of a multiplication ternary \( R \)-semimodule \( M \) is prime if and only if \( M \setminus N \) is closed subset of \( M \).

Proof. Proof is straightforward.

Theorem 5.14. Let \( A \) be a ternary subsemimodule of a multiplication ternary \( R \)-semimodule \( M \) and \( S \) be a closed subset of \( M \) such that \( A \cap S = \phi \). Then there is a ternary subsemimodule \( N \) of \( M \) maximal with respect to the property that \( A \subseteq N \) and \( N \cap S = \phi \). Further \( N \) is a prime ternary subsemimodule of \( M \).

Proof. Take \( C = \{ B : B \) is a ternary subsemimodule of \( M, A \subseteq B \) and \( B \cap S = \phi \} \). Then \((C, \subseteq)\) is a non-empty partially ordered set in which every simply ordered family has an upper bound. By Zorn’s Lemma, \( C \) has a maximal element. Let it be \( N \). i.e.
$N$ is a ternary subsemimodule of $M$ which is maximal with respect to the property that $A \subseteq N$ and $N \cap S = \phi$. Let if possible $N$ is not prime ternary subsemimodule of $M$. Therefore there exist $a, b, c \in M$ such that $abc \subseteq N$ and $a \notin N, b \notin N$ and $c \notin N$. Now $N \subseteq N + RRa, N \subseteq N + RRb$ and $N \subseteq N + RRc \Rightarrow (N + RRa) \cap S \neq \phi, (N + RRb) \cap S \neq \phi$ and $(N + RRc) \cap S \neq \phi$. Choose $s_1, s_2, s_3 \in S$ such that $s_1 \in N + RRa, s_2 \in N + RRb$ and $s_3 \in N + RRc$. Since $S$ is a closed subset of $M$, $s_1 s_2 s_3 \cap S \neq \phi$. Moreover $s_1 s_2 s_3 \subseteq (N + RRa)(N + RRb)(N + RRc) \subseteq N$. Thus $N \cap S \neq \phi$, a contradiction. Hence $N$ is a prime ternary subsemimodule of $M$.

**Theorem 5.15.** Every prime ternary subsemimodule of a multiplication ternary $R$-semimodule $M$ contains a minimal prime ternary subsemimodule.

**Proof.** Let $N$ be a prime ternary subsemimodule of $M$. Let $A = \{H : H$ is a prime ternary subsemimodule of $M, H \subseteq N\}$. Since $N \in A, (A, \subseteq)$ is a non-empty partially ordered set. Let $\{H_i : i \in \mathbb{N}\}$ be a descending chain of prime ternary subsemimodules of $M$ such that $H_i \subseteq N$ for all $i \in \mathbb{N}$ and let $H' = \bigcap_{i \in \mathbb{N}} H_i$. Then $H'$ is a ternary subsemimodule of $M$ and $H' \subseteq N$. Now we prove $H'$ is a prime ternary subsemimodule of $M$. Let $m_1, m_2, m_3 \in M$ be such that $m_1 m_2 m_3 \subseteq H'$ and $m_1 \notin H', m_2 \notin H'$. Then $m_1 \notin H_k$ for some $k \in \mathbb{N}$ and $m_2 \notin H_l$ for some $l \in \mathbb{N}$. Take $n = \max\{k, l\}$. Now $m_1, m_2 \notin H_n \Rightarrow m_3 \in H_n$, since $m_1 m_2 m_3 \subseteq H' \subseteq H_n, H_n$ is a prime ternary subsemimodule and by Theorem 5.11. For any $i \leq n, H_i \supseteq H_n$ and hence $m_3 \in H_i,...(1)$. For any $i > n, H_i \subseteq H_n \Rightarrow m_1, m_2 \notin H_i$ and hence $m_3 \in H_i$ for all $i > n...(2)$. From (1) and (2), we get $m_3 \in H'$. Hence $H' \in A$. Then by Zorn’s Lemma, $A$ has a minimal element. Hence the theorem.\]
5.3 Prime avoidance theorem

In this section, we prove prime avoidance theorem for ternary semimodules over ternary semirings.

**Definition 5.16.** Let $L, L_1, L_2, L_3, ..., L_n$ be ternary subsemimodules of a ternary $R$-semimodule $M$ and $L \subseteq L_1 \cup L_2 \cup L_3 \cup ... \cup L_n$. Then $L_1 \cup L_2 \cup L_3 \cup ... \cup L_n$ is said to be efficient covering of $L$ if $L \not\subseteq \bigcup_{i=1}^{n} L_i$ for any $j \in \{1, 2, 3, ..., n\}$.

**Definition 5.17.** Let $L, L_1, L_2, L_3, ..., L_n$ be ternary subsemimodules of a ternary $R$-semimodule $M$ and $L = L_1 \cup L_2 \cup L_3 \cup ... \cup L_n$. Then $L_1 \cup L_2 \cup L_3 \cup ... \cup L_n$ is said to be efficient union of $L$ if $L \not\subseteq \bigcup_{i=1}^{n} L_i$ for any $j \in \{1, 2, 3, ..., n\}$.

**Lemma 5.18.** Let $L$ be a ternary subsemimodule of a ternary $R$-semimodule $M$ and $L_1, L_2, ..., L_n$ be subtractive ternary subsemimodules of $M$. If $L = L_1 \cup L_2 \cup ... \cup L_n$ is an efficient union, then $\bigcap_{i=1}^{n} L_i = \bigcap_{i=1, \ i \neq j}^{n} L_i$ for all $1 \leq j \leq n$.

**Proof.** Let $1 \leq j \leq n$ and let $x \in \bigcap_{i=1}^{n} L_i$. Since $L = L_1 \cup L_2 \cup ... \cup L_n$ is an efficient union, $L \not\subseteq \bigcup_{i=1}^{n} L_i$. So there exists $y \in L$ such that $y \notin \bigcup_{i=1, \ i \neq j}^{n} L_i$...(1). Hence $y \in L_j$. Now $x + y \in L = L_1 \cup L_2 \cup ... \cup L_n$. Suppose that $x + y \in L_i$ for some $i \neq j$. Since $x \in L_i$ and $L_i$ is a subtractive ternary subsemimodule, $y \in L_i$, a contradiction to (1). Hence $x + y \in L_j$. Since $y \in L_j$ and $L_j$ is a subtractive ternary subsemimodule, $x \in L_j$. Hence $x \in \bigcap_{i=1}^{n} L_i$. Now $\bigcap_{i=1, \ i \neq j}^{n} L_i \subseteq \bigcap_{i=1}^{n} L_i$. Other inclusion is trivial. \[\square\]

The following theorem is essential to prove the prime avoidance theorem.
**Theorem 5.19.** Let \( L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n \) be an efficient covering of subtractive ternary subsemimodules of a ternary \( R \)-semimodule \( M \) where \( n > 2 \). If \((L_j : M) \nsubseteq (L_k : M)\) for any \( k \neq j \), then \( L_k \) is not a prime ternary subsemimodule of \( M \).

**Proof.** Since \( L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n \) is an efficient covering, \( L = (L \cap L_1) \cup (L \cap L_2) \cup \ldots \cup (L \cap L_n) \) is an efficient union. Then by Lemma 5.18, \( \bigcap_{j=1, j \neq k}^{n} (L \cap L_j) \subseteq L \cap L_k \). Since \( L \nsubseteq L_k \), there exists \( l_k \in L \setminus L_k \). Suppose \( L_k \) is a prime ternary subsemimodule of \( M \). Then \((L_k : M)\) is a prime ideal of \( R \). Since \((L_j : M) \nsubseteq (L_k : M)\), there exists \( s_j \in (L_j : M) \setminus (L_k : M) \). Then \( s = s_1^{n+1} s_2 \ldots s_k s_{k+1} \ldots s_n \in (L_j : M) \setminus (L_k : M) \). Hence \( s \ell_k \in L \cap L_j \) for all \( j \neq k \), for all \( r \in R \) and \( s \ell' \ell_k \notin L \cap L_k \) for some \( r' \in R \). So \( \bigcap_{j=1, j \neq k}^{n} (L \cap L_j) \nsubseteq L \cap L_k \), a contradiction to (1). Thus, \( L_k \) is not a prime ternary subsemimodule. \( \square \)

Now we prove the prime avoidance theorem for ternary semimodules over ternary semirings.

**Theorem 5.20.** (Prime Avoidance Theorem) Let \( L \) be a ternary subsemimodule of a ternary \( R \)-semimodule \( M \) and \( L_1, L_2, \ldots, L_n \) be subtractive ternary subsemimodules of \( M \) such that atmost two of \( L_i^s \) are not prime and for any \( j \neq k \), \((L_j : M) \nsubseteq (L_k : M)\). If \( L \subseteq L_1 \cup L_2 \cup \ldots \cup L_n \), then \( L \subseteq L_k \) for some \( k \).

**Proof.** Let \( L \subseteq L_{i_1} \cup L_{i_2} \cup \ldots \cup L_{i_m} \) be an efficient covering where \( i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\} \) and \( 1 \leq m \leq n \). Suppose \( m = 2 \). Then \( L \subseteq L_{i_1} \cup L_{i_2} \) is an efficient covering\( \ldots(1) \). By Lemma 3.12, \( L \subseteq L_{i_{i_1}} \) or \( L \subseteq L_{i_{i_2}} \), a contradiction to (1). Suppose that \( m \geq 3 \). Then by assumption, there exists at least one prime ternary subsemimodule
\( L_{i_j} \) for some \( i_j \) which is impossible by Theorem 5.19. Hence \( m = 1 \). Now \( L \subseteq L_k \) for some \( k \in \{1, 2, ..., n\} \).

**Theorem 5.21.** Let \( P \) be a prime ternary subsemimodule and \( N_1, N_2, ..., N_n \) be subtractive ternary subsemimodules of a multiplication ternary \( R \)-semimodule \( M \). Then \( \bigcap_{i=1}^n N_i \subseteq P \) if and only if \( N_j \subseteq P \) for some \( j \in \{1, 2, ..., n\} \).

**Proof.** Suppose that \( \bigcap_{i=1}^n N_i \subseteq P \). Then \( \left( \bigcap_{i=1}^n N_i : M \right) \subseteq (P : M) \Rightarrow \bigcap_{i=1}^n (N_i : M) \subseteq (P : M) \). Claim: \( (N_j : M) \subseteq (P : M) \) for some \( j \). Suppose that \( (N_j : M) \notin (P : M) \) for all \( j \). So there exists \( s_j \in (N_j : M) \) such that \( s_j \notin (P : M) \) for all \( j \)...(1). Then \( s = s_1^n s_2^n ... s_n \in (N_j : M) \) for all \( j \Rightarrow s_1^n s_2^n ... s_n \in \bigcap_{j=1}^n (N_j : M) \subseteq (P : M) \)...(2).

Since \( P \) is prime ternary subsemimodule, by Theorem 5.3, \( (P : M) \) is a prime ideal of \( R \). Hence by (2), \( s_j \in (P : M) \) for some \( j \), a contradiction to (1). Now \( (N_j : M) \subseteq (P : M) \) for some \( j \). Hence \( (N_j : M)RM \subseteq (P : M)RM \Rightarrow N_j \subseteq P \), since by Lemma 5.6. Converse is trivial. \( \square \)

**Corollary 5.22.** Let \( N \) be a ternary subsemimodule of a multiplication ternary \( R \)-semimodule \( M \) and \( P_1, P_2, ..., P_n \) be subtractive ternary subsemimodules of \( M \) such that \( N \subseteq P_1 \cup P_2 \cup ... \cup P_n \) and at most two of \( P_i \)'s are not prime, then \( N \subseteq P_k \) for some \( k \).

**Proof.** Let \( N \subseteq P_{i_1} \cup P_{i_2} \cup ... \cup P_{i_m} \) be an efficient covering where \( i_1, i_2, ..., i_m \in \{1, 2, ..., n\} \) and \( 1 \leq m \leq n \). Suppose \( m = 2 \). Then \( N \subseteq P_{i_1} \cup P_{i_2} \) is an efficient covering...(1). By Lemma 3.12, \( N \subseteq P_{i_1} \) or \( N \subseteq P_{i_2} \), a contradiction to (1). Suppose that \( m \geq 3 \). Then by assumption, there exists at least one prime ternary subsemimodule \( P_{i_j} \) for some \( i_j \). Now if \( (P_{i_k} : M) \subseteq (P_{i_l} : M) \) for some \( i_k \neq i_l \). Then
\[(P_{i_k} : M)RM \subseteq (P_{i_t} : M)RM \Rightarrow P_{i_k} \subseteq P_{i_t} \text{ for some } i_k \neq i_t, \text{ since by Lemma 5.6.} \]
Which is impossible. Hence for any \(i_k \neq i_t, (P_{i_k} : M) \nsubseteq (P_{i_t} : M). \) Which is impossible, by Theorem 5.19. Hence \(m = 1. \) Thus, \(N \subseteq P_k \) for some \(k \in \{1, 2, \ldots, n\}. \)

\section{5.4 Weakly prime ternary subsemimodules}

V. Gupta and J. N. Chaudhari [39], extended some results of weakly prime ideals of rings to weakly prime subtractive ideals in semirings and characterized them. J. N. Chaudhari and D. R. Bonde [20], introduced and characterized weakly prime and weakly primary subsemimodules of semimodules. This work has been extended for prime subsemimodules and weakly primary subsemimodules of partial semimodules in [49, 50]. In this section, we introduce the concept of weakly prime ternary subsemimodules of ternary semimodules over ternary semirings which is analogous to weakly prime subsemimodules of semimodules. Also we characterize the weakly prime ternary subsemimodules of ternary semimodules.

\textbf{Definition 5.23.} A proper ternary subsemimodule \(N\) of a ternary \(R\)-semimodule \(M\) is said to be weakly prime if \(0 \neq rsn \in N, r, s \in R, n \in M, \) then either \(r \in (N : M)\) or \(s \in (N : M)\) or \(n \in N. \)

Clearly, every prime ternary subsemimodule of a ternary semimodule is weakly prime. Following example shows that the converse implication is not true.

\textbf{Example 5.24.} Consider the ternary semiring \(R = (\mathbb{Z}_{0}, +, \cdot). \) Then \(\{0\}\) is a weakly prime ternary subsemimodule of a ternary \(R\)-semimodule \(M = (\mathbb{Z}_{-6}, +, \cdot)\), which is not a prime ternary subsemimodule.
Theorem 5.25. Let $N$ be a weakly prime ternary subsemimodule of an entire ternary $R$-semimodule $M$. Then $(N : M)$ is a weakly prime ideal of $R$.

Proof. Let $I, J, K$ be ideals of $R$ such that $0 \neq IJK \subseteq (N : M)$ and $I \not\subseteq (N : M), J \not\subseteq (N : M)$. So there exist $0 \neq a \in I \setminus (N : M)$ and $0 \neq b \in J \setminus (N : M)$. Let $0 \neq c \in K$. Then $0 \neq abc \in IJK \subseteq (N : M)$. So for any $0 \neq x \in M, 0 \neq r \in R$, we have $0 \neq (abc)rx = a(brx) = ab(cx) \in N$, since $M$ is entire. Therefore $crx \in N$, since $N$ is a weakly prime ternary subsemimodule. Now $crx \in N$ for all $0 \neq r \in R$ and for all $0 \neq x \in M$. So $c \in (N : M)$ for all $c \in K$. So $K \subseteq (N : M)$. Thus, $(N : M)$ is a weakly prime ideal of $R$. 

In Theorem 5.25 the condition that, $M$ is an entire, is essential.

Example 5.26. Consider the ternary $R$-semimodule $M$ as stated in Example 5.24. Then $\{0\}$ is a weakly prime ternary subsemimodule of $M$, but $(\{0\} : M) = (−6)Z_0^-Z_0^−$ is not a weakly prime ideal because $0 \neq (−2) \cdot (−3) \cdot (−1) \in (−6)Z_0^-Z_0^−$, but $−2 \notin (−6)Z_0^-Z_0^−, −3 \notin (−6)Z_0^-Z_0^−, −1 \notin (−6)Z_0^-Z_0^−$.

The following example shows that the converse of the Theorem 5.25 is not true.

Example 5.27. Let $M, N$ be defined as in Example 5.4. Then $(N : M) = \{0\}$ is a weakly prime ideal of $Z_0^-$. But $N = \{0\} \times (−8)Z_0^-Z_0^−$ is not a weakly prime ternary subsemimodule of $M$ because $0 = (0, 0) \neq (−2)∗(−2)∗(0, −2) = ((−2)(−2)0, (−2)(−2)(−2)) = (0, −8) \in N$ but $−2 \notin (N : M)$ and $0, −2 \notin N$.

Theorem 5.28. If $N$ is a weakly prime subtractive ternary subsemimodule of a ternary $R$-semimodule $M$, then either $N$ is prime or $(N : M)(N : M)N = 0$. 
Proof. Suppose that \((N : M)(N : M)N \neq 0\). Let \(r_1r_2m \in N\) with \(r_1, r_2 \in R\) and \(m \in M\). If \(r_1r_2m \neq 0\), then we are through. Suppose \(r_1r_2m = 0\). If \(r_1r_2N \neq 0\), then there exists \(n \in N\) such that \(r_1r_2n \neq 0\). Now \(0 \neq r_1r_2(m + n) = r_1r_2n \in N \Rightarrow \) either \(r_1 \in (N : M)\) or \(r_2 \in (N : M)\) or \(m \in N\), as \(N\) is a weakly prime subtractive ternary subsemimodule. Now suppose that \(r_1r_2N = 0\). If \((N : M)r_2m \neq 0\), then there exists \(r_1' \in (N : M)\) such that \(r_1'r_2m \neq 0\). Now \(0 \neq (r_1 + r_1')(r_2 + r_2)m = r_1'r_2m \in N \Rightarrow \) either \(r_1 + r_1' \in (N : M)\) or \(r_2 \in (N : M)\) or \(m \in N\). By Theorem 3.6, \((N : M)\) is a subtractive ideal, and hence either \(r_1 \in (N : M)\) or \(r_2 \in (N : M)\) or \(m \in N\). So suppose that \((N : M)r_2m = 0\). On the similar lines we can assume that \(r_1(N : M)m = 0\). If \((N : M)(N : M)m \neq 0\), then there exist \(r''_1, r''_2 \in (N : M)\) such that \(r''_1r''_2m \neq 0\). Now \(0 \neq (r_1 + r''_1)(r_2 + r''_2)m = r''_1r''_2m \in N \Rightarrow \) either \(r_1 + r''_1 \in (N : M)\) or \(r_2 + r''_2 \in (N : M)\) or \(m \in N\). Again by using Theorem 3.6, either \(r_1 \in (N : M)\) or \(r_2 \in (N : M)\) or \(m \in N\). So suppose that \((N : M)(N : M)m = 0\). Again on the similar lines we can assume that \((N : M)r_2N = 0\) and \(r_1(N : M)N = 0\). Since \((N : M)(N : M)N \neq 0\), there exist \(r''_1, r''_2 \in (N : M)\) and \(n^* \in N\) such that \(r''_1r''_2n^* \neq 0\). Now \(0 \neq (r_1 + r''_1)(r_2 + r''_2)(m + n^*) = r''_1r''_2n^* \in N \Rightarrow \) either \(r_1 + r''_1 \in (N : M)\) or \(r_2 + r''_2 \in (N : M)\) or \(m + n^* \in N\). Since \(N\) is a subtractive ternary subsemimodule and by using Theorem 3.6, either \(r_1 \in (N : M)\) or \(r_2 \in (N : M)\) or \(m \in N\). Hence \(N\) is a prime ternary subsemimodule of \(M\).

\[\square\]

**Theorem 5.29.** If \(N\) is a proper subtractive ternary subsemimodule of a ternary \(R\)-semimodule \(M\), then the following statements are equivalent:

1) \(N\) is a weakly prime ternary subsemimodule of \(M\);

2) If \(0 \neq IJD \subseteq N\), with \(I, J\) are ideals of \(R\) and \(D\) is a ternary subsemimodule of \(M\), then either \(I \subseteq (N : M)\) or \(J \subseteq (N : M)\) or \(D \subseteq N\).
Proof. (1)⇒(2) If \( N \) is prime, then the result is clear by using Lemma 5.2. So we can assume that \( N \) is not prime. Let \( 0 \neq IJD \subseteq N \) where \( I, J \) are ideals of \( R \) and \( D \) is a ternary subsemimodule of \( M \). To show \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \). Suppose that \( I \not\subseteq (N : M), J \not\subseteq (N : M) \) and \( D \not\subseteq N \). Choose \( r_1 \in I, r_2 \in J \) and \( x \in D \) such that \( r_1, r_2 \not\in (N : M) \) and \( x \notin N \). If \( 0 \neq r_1r_2x \in IJD \subseteq N \), then \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( x \in N \), as \( N \) is a weakly prime ternary subsemimodule. It is impossible. Hence assume that \( r_1r_2x = 0 \). If \( r_1r_2D \neq 0 \), then choose \( d \in D \) such that \( r_1r_2d \neq 0 \). Now \( 0 \neq r_1r_2d \in IJD \subseteq N \Rightarrow d \in N \), since \( N \) is weakly prime ternary subsemimodule. Now \( 0 \neq r_1r_2(d + x) = r_1r_2d \in N \Rightarrow d + x \in N \). Since \( N \) is a subtractive ternary subsemimodule and \( d \in N \), so \( x \in N \), a contradiction. Hence assume that \( r_1r_2D = 0 \). If \( Ir_2x \neq 0 \), then there exists \( r_1' \in I \) such that \( 0 \neq r_1'r_2x \in IJD \subseteq N \). Since \( N \) is a weakly prime ternary subsemimodule, \( r_1' \in (N : M) \). Now \( 0 \neq (r_1 + r_1')r_2x = r_1'r_2x \in N \Rightarrow r_1 + r_1' \in (N : M) \), as \( N \) is a weakly prime ternary subsemimodule. By Theorem 3.6, \( r_1 \in (N : M) \), a contradiction. Hence assume that \( Ir_2x = 0 \). On the similar lines we can assume that \( r_1Jx = 0 \). If \( IJx \neq 0 \), then there exist \( r_1'' \in I \) and \( r_2'' \in J \) such that \( 0 \neq r_1''r_2''x \in IJD \subseteq N \). Since \( N \) is a weakly prime ternary subsemimodule, \( r_1'' \in (N : M) \) or \( r_2'' \in (N : M) \).

Case (i) \( r_1'' \in (N : M) \) and \( r_2'' \notin (N : M) \). Now \( 0 \neq (r_1 + r_1'')r_2''x = r_1'r_2''x \in N \Rightarrow r_1 + r_1'' \in (N : M) \). Now by Theorem 3.6, \( r_1 \in (N : M) \), a contradiction.

Similarly, Case (ii) \( r_1'' \notin (N : M) \) and \( r_2'' \in (N : M) \) is impossible.

Case (iii) \( r_1'' \in (N : M) \) and \( r_2'' \in (N : M) \). Now \( 0 \neq (r_1 + r_1'')(r_2 + r_2'')x = r_1'r_2''x \in N \Rightarrow \) either \( r_1 + r_1'' \in (N : M) \) or \( r_2 + r_2'' \in (N : M) \). By Theorem 3.6, either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \), a contradiction.
Hence assume that $IJx = 0$.

On the similar lines we can assume that $Ir_2D = 0$ and $r_1JD = 0$. Since $IJD \neq 0$, there exist $r_1^* \in I$, $r_2^* \in J$ and $d^* \in D$ such that $0 \neq r_1^*r_2^*d^* \in IJD \subseteq N$. Since $N$ is a weakly prime ternary subsemimodule, either $r_1^* \in (N : M)$ or $r_2^* \in (N : M)$ or $d^* \in N$.

Case $(\alpha_1)$ $r_1^* \in (N : M)$, $r_2^* \notin (N : M)$ and $d^* \notin N$. Now $0 \neq (r_1^* + r_1^*)r_2^*d^* = r_1^*r_2^*d^* \in N \Rightarrow r_1 + r_1^* \in (N : M)$. By Theorem 3.6, $r_1 \in (N : M)$, a contradiction.

On the similar lines Case $(\alpha_2)$ $r_1^* \notin (N : M)$, $r_2^* \in (N : M)$, $d^* \notin N$ and Case $(\alpha_3)$ $r_1^* \notin (N : M)$, $r_2^* \notin (N : M)$ and $d^* \in N$ are impossible.

Case $(\alpha_4)$ $r_1^*, r_2^* \in (N : M)$ and $d^* \notin N$. Now $0 \neq (r_1^* + r_1^*)(r_2^* + r_2^*)d^* = r_1^*r_2^*d^* \in N \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$. By Theorem 3.6, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, a contradiction.

Again on the similar lines Case $(\alpha_5)$ $r_1^* \notin (N : M)$, $r_2^* \in (N : M)$, $d^* \in N$ and Case $(\alpha_6)$ $r_1^* \in (N : M)$, $r_2^* \notin (N : M)$ and $d^* \in N$ are impossible.

Case $(\alpha_7)$ $r_1^*, r_2^* \in (N : M)$ and $d^* \in N$. Now $0 \neq (r_1^* + r_1^*)(r_2^* + r_2^*)(x + d^*) = r_1^*r_2^*d^* \in N \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$ or $(x + d^*) \in N$. By Theorem 3.6 and $N$ is subtractive, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $x \in N$, a contradiction.

Now $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$.

(2)$\Rightarrow$(1) Suppose that $0 \neq r_1r_2m \in N$ where $r_1, r_2 \in R$ and $m \in M$. Take $I = \langle r_1 \rangle = RRr_1$, $J = \langle r_2 \rangle = RRr_2$ and $D = \langle m \rangle = RRm$. Then $0 \neq IJD \subseteq N$. So either $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$ and hence either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Thus $N$ is a weakly prime ternary subsemimodule of $M$. □

**Theorem 5.30.** Let $N$ be a weakly prime subtractive ternary subsemimodule of a ternary $R$-semimodule $M$. Then the following statements hold:
1) For \( m \in M \setminus N \), \((N : m) = (N : M) \cup (0 : m)\);

2) For \( m \in M \setminus N \), \((N : m) = (N : M)\) or \((N : m) = (0 : m)\).

Proof. (1) Let \( m \in M \setminus N \). Clearly, \((N : M) \cup (0 : m) \subseteq (N : m)\). Now let \( a \in (N : m) \). Then \( arm \in N \) for all \( r \in R \). If \( 0 \neq a1m \in N \), then \( a \in (N : M) \) or \( 1 \in (N : M) \) as \( N \) is a weakly prime ternary subsemimodule. Hence \( a \in (N : M) \).

Suppose that \( a1m = 0 \). Then \( arm = 1r(a1m) = 0 \) for all \( r \in R \). So \( a \in (0 : m) \).

Thus \( a \in (N : M) \cup (0 : m) \). Now \((N : m) \subseteq (N : M) \cup (0 : m)\).

(2) It follows by Theorem 3.6, Lemma 3.7 and Lemma 3.12. \( \square \)

5.5 Weakly prime ternary subsemimodules in quotient ternary semimodules

In this section, we extend some results of [7] and [20] to ternary semimodules over ternary semirings and give a relation between weakly prime ternary subsemimodules of a ternary \( R \)-semimodule \( M \) and weakly prime ternary subsemimodules of the quotient ternary \( R \)-semimodule \( M/N(Q) \) where \( N \) is a \( Q \)-ternary subsemimodule of \( M \).

Lemma 5.31. Let \( N \) be a \( Q \)-ternary subsemimodule of a ternary \( R \)-semimodule \( M \). If \( r, s \in R \) and \( m \in M \), then there exists a unique \( q \in Q \) such that \( rsm \in r \odot s \odot (q+N) \).

Proof. Let \( r, s \in R \) and \( m \in M \). Since \( N \) is a \( Q \)-ternary subsemimodule of \( M \) and \( m, rsm \in M \), there exist unique \( q, q' \in Q \) such that \( m + N \subseteq q + N \) and \( rsm + N \subseteq q' + N \). Also \( r \odot s \odot (q + N) = q'' + N \) where \( q'' \in Q \) is a unique element such that \( rsq + N \subseteq q'' + N \). By Lemma 1.17, \( rsm + N \subseteq rsq + N \subseteq q'' + N \). Now
\( rsm \in (q' + N) \cap (q'' + N) \). Hence \((q' + N) \cap (q'' + N) \neq \emptyset \). So \( q' = q'' \). Thus \( rsm \in q' + N = q'' + N = r \odot s \odot (q + N) \).

**Theorem 5.32.** Let \( N \) be a \( Q \)-ternary subsemimodule of a ternary \( R \)-semimodule \( M \) and \( P \) be a subtractive ternary subsemimodule of \( M \) with \( N \subseteq P \). Then

1) If \( P \) is a weakly prime ternary subsemimodule of \( M \), then \( P/N(Q\cap P) \) is a weakly prime ternary subsemimodule of \( M/N(Q) \).

2) If \( N, P/N(Q\cap P) \) are weakly prime ternary subsemimodules of \( M, M/N(Q) \) respectively, then \( P \) is a weakly prime ternary subsemimodule of \( M \).

**Proof.** Let \( q_0 \) be the unique element of \( Q \) such that \( q_0 + N \) is the zero element of \( M/N(Q) \), since Lemma 2.3.

(1) Let \( P \) be a weakly prime ternary subsemimodule of \( M \). Let \( r, s \in R \) and \( q_1 + N \in M/N(Q) \) be such that \( q_0 + N \neq r \odot s \odot (q_1 + N) \in P/N(Q\cap P) \). By Lemma 3.19, \( N \) is a \( Q \cap P \)-ternary subsemimodule of \( P \). Hence there exists a unique \( q_2 \in Q \cap P \) such that \( r \odot s \odot (q_1 + N) = q_2 + N \) where \( rsq_1 + N \subseteq q_2 + N \). Since \( N \subseteq P \), \( rsq_1 \in P \).

If \( rsq_1 = 0 \), then \( rsq_1 \in (q_0 + N) \cap (q_2 + N) \), since by Lemma 2.3, \( 0 \in N = q_0 + N \). So \( q_0 = q_2 \) and hence \( q_0 + N = q_2 + N \), a contradiction. Thus \( rsq_1 \neq 0 \). As \( P \) is a weakly prime ternary subsemimodule, either \( r \in (P : M) \) or \( s \in (P : M) \) or \( q_1 \in P \). If \( q_1 \in P \), then \( q_1 \in Q \cap P \) and hence \( q_1 + N \in P/N(Q\cap P) \). Without loss of generality suppose that \( r \in (P : M) \). For \( q + N \in M/N(Q) \) and \( s' \in R \), let \( r \odot s' \odot (q + N) = q_3 + N \) where \( q_3 \) is a unique element of \( Q \) such that \( rs'q + N \subseteq q_3 + N \). Therefore \( rs'q = q_3 + n \) for some \( n \in N \). Now \( r \in (P : M) \Rightarrow rs'q \in P \Rightarrow q_3 + n \in P \Rightarrow q_3 \in P \), as \( P \) is a subtractive ternary subsemimodule and \( n \in N \subseteq P \). Hence \( q_3 \in Q \cap P \). Now \( r \odot s' \odot (q + N) = q_3 + N \in P/N(Q\cap P) \) for all \( s' \in R \) and \( q + N \in M/N(Q) \). Therefore
r ∈ (P/N_{Q∩P} : M/N_{Q}). Thus P/N_{Q∩P} is a weakly prime ternary subsemimodule of M/N_{Q}.

(2) Suppose that N, P/N_{Q∩P} are weakly prime ternary subsemimodules of M, M/N_{Q} respectively. Let 0 ≠ rsm ∈ P where r, s ∈ R, m ∈ M. If rsm ∈ N, then we are through, since N is a weakly prime ternary subsemimodule of M. So suppose that rsm ∈ P \ N. By using Lemma 5.31, there exists a unique q₁ ∈ Q such that m ∈ q₁ + N and rsm ∈ r ⊙ s ⊙ (q₁ + N) = q₂ + N where q₂ is a unique element of Q such that rsq₁ + N ⊆ q₂ + N. Now rsm ∈ P, rsm ∈ q₂ + N implies q₂ ∈ P, as P is a subtractive ternary subsemimodule and N ⊆ P. Hence q₀ + N ≠ r ⊙ s ⊙ (q₁ + N) = q₂ + N ∈ P/N_{Q∩P}. As P/N_{Q∩P} is a weakly prime ternary subsemimodule, r ∈ (P/N_{Q∩P} : M/N_{Q}) or s ∈ (P/N_{Q∩P} : M/N_{Q}) or q₁ + N ∈ P/N_{Q∩P}. If q₁ + N ∈ P/N_{Q∩P}, then q₁ ∈ P. Hence m ∈ q₁ + N ⊆ P. Now without loss of generality assume that r ∈ (P/N_{Q∩P} : M/N_{Q}). Let x ∈ M and s' ∈ R. By using Lemma 5.31, there exists a unique q₃ ∈ Q such that x ∈ q₃ + N and rs'x ∈ r ⊙ s' ⊙ (q₃ + N) = q₄ + N where q₄ is a unique element of Q such that rs'q₃ + N ⊆ q₄ + N. Now q₄ + N = r ⊙ s' ⊙ (q₃ + N) ∈ P/N_{Q∩P} and hence q₄ ∈ P. As rs'x ∈ q₄ + N and N ⊆ P, rs'x ∈ P for all s' ∈ R and for all x ∈ M. So r ∈ (P : M).

**Theorem 5.33.** Let N be a Q-ternary subsemimodule of a ternary R-semimodule M and P be a subtractive ternary subsemimodule of M with N ⊆ P. Then P is a prime ternary subsemimodule of M if and only if P/N_{Q∩P} is a prime ternary subsemimodule of M/N_{Q}.

**Proof.** The proof is similar as in the proof of Theorem 5.32.

Every ternary semiring R is a ternary semimodule over itself and hence every ideal
$I$ of a ternary semiring $R$ is a ternary subsemimodule of a ternary $R$-semimodule $R$. So we have:

**Corollary 5.34.** Let $I$ be a $Q$-ideal and $P$ be a subtractive ideal of a ternary semiring $R$ with $I \subseteq P$. Then $P$ is a prime ideal of $R$ if and only if $P/I_{(Q \cap P)}$ is a prime ideal of quotient ternary semiring $R/I_{(Q)}$. 