Chapter 4

Factor Ternary Semimodules

4.1 Introduction

In this chapter, we study factor ternary semimodules over ternary semirings, fundamental theorem of ternary semimodule homomorphism, second isomorphism theorem. We obtain the relation between factor and quotient ternary semimodules.

4.2 Factor ternary semimodules

In this section, we introduce the concept of factor ternary semimodule. Also we characterize ternary subsemimodule, subtractive ternary subsemimodule for factor ternary semimodule.

Definition 4.1. Let $N$ be a ternary subsemimodule of a ternary $R$-semimodule $M$. For $x, y \in M$, we define $x \equiv_N y$ if and only if there exist $n_1, n_2 \in N$ such that $x + n_1 = y + n_2$. Then the relation $\equiv_N$ is an equivalence relation (called Bourne relation) on $M$. The equivalence class of $x \in M$ is denoted by $[x]_N^M = \{y \in M : y \equiv_N x\}$.

The set $\frac{M}{N} = \{[x]_N^M : x \in M\}$ forms a ternary $R$-semimodule under the addition
and ternary scalar multiplication defined by $[x]_N^M + [y]_N^M = [x + y]_N^M$ and $r_1r_2[x]_N^M = [r_1r_2x]_N^M$ for all $x, y \in M$ and $r_1, r_2 \in R$. This ternary $R$-semimodule $\frac{M}{N}$ is called the factor ternary semimodule of $M$ modulo $N$.

The following example shows that “if $N, K$ are ternary subsemimodules of a ternary $R$-semimodule $M$ with $N \subseteq K$ and $x \in K$, then $[x]_{N}^{M}$ and $[x]_{K}^{M}$ may be different”. So we denote the symbol $[x]_{N}^{M}$ instead of $x + N$ or $x/N$.

**Example 4.2.** Let $M = (\mathbb{Z}_0, +)$ and $R = (\mathbb{Z}_0, +, \cdot)$. Then $N = \{-4\} = \{0, -4, -8, -12, -16, ...\}$ and $K = \{0, -4, -8, -9, -10, -11, ...\}$ are ternary subsemimodules of a ternary $R$-semimodule $M$ with $N \subseteq K$. Here $-9 \in K \subseteq M$ and $[-9]_N^K = \{x \in K : x \equiv_N 9\} = \{-9, -13, -17, ...\}$ and $[-9]_N^M = \{x \in M : x \equiv_N 9\} = \{-1, -5, -9, -13, -17, ...\}$. Hence $[-9]_N^K \neq [-9]_N^M$.

If $N$ is a ternary subsemimodule of a ternary $R$-semimodule $M$, then clearly $N \subseteq [0]_N^M$ but $[0]_N^M \neq N$.

**Example 4.3.** Let $M$ and $R$ be defined as in Example 4.2. Then $N = \{0, -6, -9, -12, -15, ...\}$ is a ternary subsemimodule of $M$. Clearly, $[0]_N^M = \{0, -3, -6, -9, -12, ...\} \neq N$.

**Lemma 4.4.** If $N$ is a ternary subsemimodule of a ternary $R$-semimodule $M$, then

1) $[0]_N^M$ is the smallest subtractive ternary subsemimodule of $M$ containing $N$.

2) $x \in N \Rightarrow N \subseteq [x]_N^M$.

**Corollary 4.5.** If $N$ is a ternary subsemimodule of a ternary $R$-semimodule $M$, then $[0]_N^M$ is the subtractive closure of $N$. 
The following theorem gives equivalent conditions for a ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ to be subtractive ternary subsemimodule.

**Theorem 4.6.** Let $N$ be a ternary subsemimodule of a ternary $R$-semimodule $M$. Then the following statements are equivalent:

1) $N$ is a subtractive ternary subsemimodule of $M$;

2) $x \in N \iff [x]_N^M = N$;

3) $[0]_N^M = N$.

**Proof.** (1) $\Rightarrow$ (2) Let $N$ be a subtractive ternary subsemimodule of $M$ and $x \in N$. Clearly, $N \subseteq [x]_N^M$. Let $z \in [x]_N^M$. Therefore $z + n_1 = x + n_2$ for some $n_1, n_2 \in N$. Since $N$ is a subtractive ternary subsemimodule of $M$, $z \in N$. So $[x]_N^M \subseteq N$. Now $[x]_N^M = N$. Conversely, if $[x]_N^M = N$, then $x \in [x]_N^M = N$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Follows from Lemma 4.4.

Example 4.3 shows that if $N$ is not a subtractive ternary subsemimodule then Theorem 4.6 is not true.

**Lemma 4.7.** Let $N, K$ be ternary subsemimodules of a ternary $R$-semimodule $M$ with $N \subseteq K$. Then the following statements are equivalent:

1) $\frac{K}{N} \subseteq \frac{M}{N}$;

2) $x \in K \iff [x]_N^M = [x]_K^M$;

3) $x \in N, x + y \in K, y \in M \Rightarrow y \in K$. 
Proof. (1) $\Rightarrow$ (2) Let $x \in K$. Then $[x]_N^K \subseteq \frac{K}{N} \subseteq \frac{M}{N}$. So there exists $z \in M$ such that $[x]_N^K = [z]_N^M$. Now $x \in [x]_N^K = [z]_N^M$ and $x \in [x]_N^M \Rightarrow [z]_N^M = [x]_N^M \Rightarrow [x]_N^K = [x]_N^M$. On the other hand, $x \in [x]_N^M = [x]_N^K \subseteq K$.

(2) $\Rightarrow$ (3) Let $x \in N, x + y \in K$ and $y \in M$. Now $y + x = x + y + 0 \Rightarrow y \in [x + y]_N^K = [x + y]_N^K \subseteq K$.

(3) $\Rightarrow$ (1)Let $[x]_N^K \subseteq \frac{K}{N}$. Then $x \in K$. If $y \in [x]_N^K$, then $y + n_1 = x + n_2 \in K$ for some $n_1, n_2 \in N$. By using given condition, $y \in K$ and so $y \in [x]_N^K$. Hence $[x]_N^K \subseteq [x]_N^K$. But $[x]_N^K \subseteq [x]_N^K$. Thus $[x]_N^K = [x]_N^K \subseteq \frac{M}{N}$. 

\[ \square \]

**Theorem 4.8.** Let $N, K$ be ternary subsemimodules of a ternary $R$-semimodule $M$ with $N \subseteq K$. Then

1) $K$ is a subtractive ternary subsemimodule of $M$ $\Rightarrow$ $\frac{K}{N} \subseteq \frac{M}{N}$.

2) $\frac{K}{N} \subseteq \frac{M}{N} \iff \frac{K}{N}$ is a ternary subsemimodule of $\frac{M}{N}$.

3) $K$ is a subtractive ternary subsemimodule of $M$ $\iff \frac{K}{N}$ is a subtractive ternary subsemimodule of $\frac{M}{N}$.

Proof. (1) If $K$ is a subtractive ternary subsemimodule of $M$ and $x \in N, x + y \in K, y \in M$, then $y \in K$. By Lemma 4.7, $\frac{K}{N} \subseteq \frac{M}{N}$.

(2) Let $\frac{K}{N} \subseteq \frac{M}{N}$. Since $K$ is a ternary subsemimodule of $M$, $\frac{K}{N}$ is a ternary subsemimodule of $\frac{M}{N}$. Converse is trivial.

(3) Suppose that $K$ is a subtractive ternary subsemimodule of $M$. Then by (1) and (2), $\frac{K}{N}$ is a ternary subsemimodule of $\frac{M}{N}$. Let $[x]_N^K, [y]_N^K \in \frac{K}{N}$ where $[y]_N^M \in \frac{M}{N}$. Then by Lemma 4.7(2), $[x]_N^K = [x]_N^K$. Hence $[x + y]_N^K = [x]_N^K + [y]_N^K \subseteq \frac{K}{N}$. So $x + y \in K$. Since $K$ is a subtractive ternary subsemimodule of $M, y \in K$. Again...
Example 4.9. Consider a ternary \((Z_0^-,+,\cdot)\)-semimodule \(M = (Z_0^-,+)\) and ternary subsemimodules \(N = \{(-5)r_1r_2 \in M : r_1, r_2 \leq (-3)\} \cup \{0\}, \; K = \{r \in M : r \leq (-7)\} \cup \{0\}\) of \(M\) with \(N \subseteq K\). Here \(-45 \in N, (-45) + (-2) \in K, -2 \in M\) but \(-2 \notin K\). Hence by Lemma 4.7, \(\frac{K}{N} \nsubseteq \frac{M}{N}\). Now by Theorem 4.8, \(\frac{K}{N}\) is not a ternary subsemimodule of \(\frac{M}{N}\).

The following example shows that the converse of Theorem 4.8(1) is not true.

Example 4.10. Consider a ternary \((Z_0^-,+,\cdot)\)-semimodule \(M = (Z_0^-,+)\) and ternary subsemimodules \(N = \{0\}, K = \{0, -2, -3, -4, \ldots\}\) of \(M\) with \(N \subseteq K\). Clearly, \(x \in N, x + y \in K\) and \(y \in M \Rightarrow y \in K\). By Lemma 4.7, \(\frac{K}{N} \subseteq \frac{M}{N}\). But by Lemma 3.2, \(K\) is not a subtractive ternary subsemimodule of \(M\).

Example 4.11. Consider a ternary \((Z_0^-,+,\cdot)\)-semimodule \(M = (Z_0^-,+)\) and ternary subsemimodules \(N = \langle -3 \rangle = \{0, -3, -6, -9, \ldots\}, K = \langle -2, -3 \rangle = \{0, -2, -3, -4, \ldots\}\) of \(M\) with \(N \subseteq K\). Clearly, \(\frac{K}{N} = \{[x]_N^K : x \in K\} = \{[0]_N^K, [-2]_N^K, [-4]_N^K\}\).
[0, -3, -6, ...], [-2]_{K}^{N} = \{ -2, -5, -8, ... \} \text{ and } [-4]_{K}^{N} = \{ -4, -7, -10, ... \}. \quad \text{Also } \frac{M}{N} = \{ [0]_{N}^{M}, [-2]_{N}^{M}, [-1]_{N}^{M} \} \text{ where } [0]_{N}^{M} = \{ 0, -3, -6, ... \}, [-2]_{N}^{M} = \{ -2, -5, -8, ... \} \text{ and } [-1]_{N}^{M} = \{ -1, -4, -7, -10, ... \}. \text{ So } [-4]_{N}^{M} \neq [-4]_{K}^{N}. \text{ Here } \frac{K}{N} \not\subseteq \frac{M}{N} \text{ and hence cannot be a ternary subsemimodule of } \frac{M}{N}.

**Theorem 4.12.** Let \( N \) be a ternary subsemimodule of a ternary \( R \)-semimodule \( M \). Then a subset \( \mathcal{L} \) of \( \frac{M}{N} \) is a ternary subsemimodule of \( \frac{M}{N} \) if and only if there exists a ternary subsemimodule \( K \) of \( M \) with \( N \subseteq K \) and \( \frac{K}{N} = \mathcal{L} \).

**Proof.** If \( \mathcal{L} \) is a ternary subsemimodule of \( \frac{M}{N} \), then \( K = \{ x \in M : [x]_{N}^{M} \in \mathcal{L} \} \) is a ternary subsemimodule of \( M \) with \( N \subseteq K \) and \( \frac{K}{N} = \mathcal{L} \). Conversely, if \( \mathcal{L} \) is a subset of \( \frac{M}{N} \), \( \frac{K}{N} = \mathcal{L} \subseteq \frac{M}{N} \) and \( K \) is a ternary subsemimodule of \( M \) with \( N \subseteq K \), then by Theorem 4.8(2), \( \mathcal{L} \) is a ternary subsemimodule of \( \frac{M}{N} \).

**Theorem 4.13.** Let \( N \) be a ternary subsemimodule of a ternary \( R \)-semimodule \( M \). Then a subset \( \mathcal{L} \) of \( \frac{M}{N} \) is a subtractive ternary subsemimodule of \( \frac{M}{N} \) if and only if there exists a subtractive ternary subsemimodule \( K \) of \( M \) with \( N \subseteq K \) and \( \frac{K}{N} = \mathcal{L} \).

**Proof.** Proof follows by Theorem 4.12 and Theorem 4.8(3).

**Theorem 4.14.** Let \( N, K_1, K_2 \) be subtractive ternary subsemimodules of a ternary \( R \)-semimodule \( M \) with \( N \subseteq K_1, K_2 \). Then

1) \( \frac{K_1 \cap K_2}{N} = \frac{K_1}{N} \cap \frac{K_2}{N} \).

2) \( K_1 + K_2 \) is a subtractive ternary subsemimodule \( \Rightarrow \frac{K_1 + K_2}{N} = \frac{K_1}{N} \oplus \frac{K_2}{N} \).

**Proof.** (1) By Remark 3.2.1, \( K_1 \cap K_2 \) is a subtractive ternary subsemimodule of \( M \) with \( N \subseteq K_1 \cap K_2 \). Hence by Theorem 4.8, \( \frac{K_1 \cap K_2}{N} \subseteq \frac{K_1}{N} \) and \( \frac{K_1 \cap K_2}{N} \subseteq \frac{K_2}{N} \). So \( \frac{K_1 \cap K_2}{N} \subseteq \frac{K_1}{N} \cap \frac{K_2}{N} \). Now \( [x]_{N}^{M} \in \frac{K_1}{N} \cap \frac{K_2}{N} \Rightarrow [x]_{N}^{M} \in \frac{K_1}{N} \) and \( [x]_{N}^{M} \in \frac{K_2}{N} \Rightarrow x \in K_1 \) and
$x \in K_2 \Rightarrow x \in K_1 \cap K_2 \Rightarrow [x]_N^M \in \frac{K_1 \cap K_2}{N}$. So $\frac{K_1 \cap K_2}{N} \subseteq \frac{K_1 \cap K_2}{N}$. Thus $\frac{K_1 \cap K_2}{N} = \frac{K_1 \cap K_2}{N}$.

(2) Let $K_1 + K_2$ be a subtractive ternary subsemimodule of $M$ and $N \subseteq K_1 + K_2$. Now $[x]_N^M \in \frac{K_1 + K_2}{N}$. Then $x \in K_1 + K_2 \Rightarrow x = x_1 + x_2$ where $x_1 \in K_1$ and $x_2 \in K_2$. So $[x_1]_N^M \in \frac{K_1}{N}$ and $[x_2]_N^M \in \frac{K_2}{N}$. Hence $[x]_N^M = [x_1 + x_2]_N^M = [x_1]_N^M + [x_2]_N^M \in \frac{K_1}{N} + \frac{K_2}{N}$. So $\frac{K_1 + K_2}{N} \subseteq \frac{K_1}{N} + \frac{K_2}{N}$. For the other inclusion, $[y]_N^M \in \frac{K_1}{N} + \frac{K_2}{N}$. Then $[y]_N^M = [y_1]_N^M + [y_2]_N^M$ where $[y_1]_N^M \in \frac{K_1}{N}$ and $[y_2]_N^M \in \frac{K_2}{N}$. Hence $y_1 \in K_1$ and $y_2 \in K_2 \Rightarrow y_1 + y_2 \in K_1 + K_2 \Rightarrow [y]_N^M = [y_1]_N^M + [y_2]_N^M = [y_1 + y_2]_N^M \in \frac{K_1 + K_2}{N}$. Thus, $\frac{K_1}{N} + \frac{K_2}{N} \subseteq \frac{K_1 + K_2}{N}$.

**Theorem 4.15.** Let $N, K_1, K_2$ be subtractive ternary subsemimodules of a ternary $R$-semimodule $M$. Then

1) If $N + K_1$ is a subtractive ternary subsemimodule of $M$, then $\frac{N + K_1}{N}$ is a subtractive ternary subsemimodule of $\frac{M}{N}$.

2) If $N \subseteq K_1, K_2$, then $\frac{K_1}{N} = \frac{K_2}{N} \Rightarrow K_1 = K_2$.

**Proof.** (1) Clearly, $N \subseteq N + K_1$. By Theorem 4.8, $\frac{N + K_1}{N}$ is a subtractive ternary subsemimodule of $\frac{M}{N}$.

(2) Easy.

**4.3 Morphisms of factor ternary semimodules**

In this section, we prove fundamental theorem of ternary semimodule homomorphism by using steady (or maximal or one-one) ternary semimodule homomorphism. Also we prove second isomorphism theorem for ternary semimodules.

**Lemma 4.16.** Let $N$ be a proper ternary subsemimodule of a ternary $R$-semimodule $M$ and $f : M \to \frac{M}{N}$ be a ternary $R$-semimodule homomorphism defined by $f(m) = [m]_N^M$. Then
1) If $K$ is a subtractive ternary subsemimodule of $M$ with $N \subseteq K$, then $f(K)$ is a subtractive ternary subsemimodule of $\frac{M}{N}$.

2) If $K$ is a subtractive ternary subsemimodule of $\frac{M}{N}$, then $f^{-1}(K)$ is a subtractive ternary subsemimodule of $M$.

Proof. (1) Let $K$ be a subtractive ternary subsemimodule of $M$ with $N \subseteq K$. Suppose that $[x]_N^M + [k_1]_N^M = [k_2]_N^M$ for some $k_1, k_2 \in K$ and $x \in M$. Therefore there exist $n_1, n_2 \in N$ such that $x + k_1 + n_1 = k_2 + n_2$. Since $k_1 + n_1, k_2 + n_2 \in K$ and $K$ is a subtractive ternary subsemimodule, $x \in K$. So $[x]_N^M = f(x) \in f(K)$. Hence $f(K)$ is a subtractive ternary subsemimodule of $\frac{M}{N}$.

(2) Suppose that $K$ is a subtractive ternary subsemimodule of $\frac{M}{N}$. Let $x + y_1 = y_2$ for some $y_1, y_2 \in f^{-1}(K)$ and $x \in M$. Therefore $f(x) + f(y_1) = [x]_N^M + [y_1]_N^M = [x + y_1]_N^M = [y_2]_N^M = f(y_2) \in K$. Hence $f(x) \in K$, since $K$ is a subtractive ternary subsemimodule and $f(y_1) \in K$. Therefore $x \in f^{-1}(K)$. □

Theorem 4.17. Let $f : M \to N$ be a ternary $R$-semimodule homomorphism. Then $f$ induces a ternary semimodule semi-isomorphism $\overline{f} : \frac{M}{\ker f} \to \text{Im} f$.

Proof. Define a map $\overline{f} : \frac{M}{\ker f} \to \text{Im} f$ by $\overline{f}([m]_M^\ker f) = f(m)$. Clearly, $\overline{f}$ is an onto ternary $R$-semimodule homomorphism with $\ker \overline{f} = \ker f = \{[0]_M^\ker f\}$. Hence $\overline{f}$ is a semi-isomorphism. □

The following example shows that the ternary semimodule homomorphism does not induce a ternary semimodule isomorphism.

Example 4.18. Consider an onto ternary $R$-semimodule homomorphism $f : M \to M'$ defined as in Example 2.11. Then $\ker f = \{0, -1, -2, -3, -4, -5\}$ and $\overline{f} : \frac{M}{\ker f} \to \text{Im} f > \text{Im} \overline{f}$.
$M'$ defined by $\overline{f} ([m]_{\ker f}^M) = f(m)$ is not one-one as $\overline{f} ([−6]_{\ker f}^M) = f(−6) = f(−7) = \overline{f} ([−7]_{\ker f}^M)$ but $[−6]_{\ker f}^M = \{−6\} \neq \{−7\} = [−7]_{\ker f}^M$. Hence $\overline{f}$ is not a ternary semimodule isomorphism.

The ternary semimodule homomorphism $f$ in Theorem 4.17 induces ternary isomorphism if $f$ is taken as steady (or maximal or one-one) ternary semimodule homomorphism. The proof is as follows.

**Theorem 4.19. (Fundamental theorem of ternary semimodule homomorphism)** Let $M, N$ be ternary $R$-semimodules and $f : M \rightarrow N$ be a steady (or maximal or one-one) ternary $R$-semimodule homomorphism. Then $f$ induces a ternary semimodule isomorphism $\overline{f} : \frac{M}{\ker f} \rightarrow Imf$ for which $\overline{f} ([m]_{\ker f}^M) = f(m)$ for all $m \in M$. In particular, if $f$ is an onto, then $\frac{M}{\ker f} \cong N$.

**Proof.** By Theorem 4.17, $\overline{f}$ is an onto ternary $R$-semimodule homomorphism. Now to show $\overline{f}$ is one-one.

Case (i) $f$ is steady : Let $\overline{f} ([x]_{\ker f}^M) = \overline{f} ([y]_{\ker f}^M)$ where $x, y \in M$. Therefore $f(x) = f(y)$. Since $f$ is steady, $x + k_1 = y + k_2$ for some $k_1, k_2 \in \ker f$. So $[x]_{\ker f}^M = [y]_{\ker f}^M$. Hence $\overline{f}$ is one-one.

Case (ii) $f$ is maximal : By Lemma 2.17, $f$ is steady and hence by case (i), $\overline{f}$ is one-one.

Case (iii) $f$ is one-one: Then clearly $\overline{f}$ is one-one.

Thus in any case, $\overline{f}$ is one-one and hence $\overline{f} : \frac{M}{\ker f} \rightarrow Imf$ is a ternary semimodule isomorphism. \hfill $\Box$

**Proposition 4.20.** Let $M$ be a ternary $R$-semimodule. Then a subset $N$ of $M$ is a subtractive ternary subsemimodule of $M$ if and only if there exists a ternary $R$-semimodule homomorphism $f : M \rightarrow M'$ satisfying $N = ker f$ where $M'$ is a ternary
$R$-semimodule.

**Proof.** Suppose that $N$ is a subtractive ternary subsemimodule of $M$. Define $f : M \to \frac{M}{N}$ by $f(x) = \frac{[x]_N}{[x]_N}$. Then clearly $f$ is an onto ternary $R$-semimodule homomorphism and $\ker f = \{ x \in M : f(x) = 0 \frac{M}{N} \} = \{ x \in M : [x]_N^M = [0]_N^M \} = \{ x \in M : [x]_N^M = N \} = N$, since by Theorem 4.6. Now $\ker f = N$. Converse is easy. □

**Theorem 4.21.** If $f : M \to N$ is a ternary $R$-semimodule homomorphism and $f(M)$ is a subtractive ternary subsemimodule of $N$, then

1) $f^{-1}\{f(M)\}$ is a subtractive ternary subsemimodule of $M$ containing $\ker f$.

2) $f$ induces a ternary $R$-semimodule homomorphism $g : \frac{M}{f^{-1}(f(M))} \to \frac{N}{f(M)}$ having $\ker g = \{ [0]_{f^{-1}(f(M))}^M \}$.

**Proof.** (1) Proof is easy.

(2) Define $g : \frac{M}{f^{-1}(f(M))} \to \frac{N}{f(M)}$ by $g : \left( \frac{[x]_f}{f^{-1}(f(M))} \right) = \frac{[f(x)]}{f(M)}$. Clearly, $g$ is a well defined ternary $R$-semimodule homomorphism. Let $\frac{[x]_f}{f^{-1}(f(M))} \in \ker g$. Then $g \left( \frac{[x]_f}{f^{-1}(f(M))} \right) = 0 \in \frac{N}{f(M)} \Rightarrow [f(x)]_{f(M)}^N = [0]_{f(M)}^N$. Hence $f(x) \equiv_{f(M)} 0$. So there exist $n', n'' \in f(M)$ such that $f(x) + n' = 0 + n'' = n''$. Now $f(x) + n' = n'' \Rightarrow f(x) \in f(M) \Rightarrow x \in f^{-1}\{f(M)\}$. By Theorem 4.21 (1) and Theorem 4.6 (2) and (3), $\frac{[x]_f}{f^{-1}(f(M))} = \frac{[0]_f}{f^{-1}(f(M))}$. Hence $\ker g = \{ [0]_{f^{-1}(f(M))}^M \}$.

**Theorem 4.22.** Let $N, K$ be ternary subsemimodules of a ternary $R$-semimodule $M$ with $N \subseteq K$ and $K$ be subtractive. Then a function $f : \frac{M}{N} \to \frac{M}{K}$ defined by $f \left( \frac{[x]_N}{N} \right) = \frac{[x]_K}{K}$ is an onto steady ternary $R$-semimodule homomorphism.

**Proof.** Clearly, $f$ is a well defined onto ternary $R$-semimodule homomorphism. To show $f$ is steady. Let $\frac{[x]_N}{N}, \frac{[y]_N}{N} \in \frac{M}{N}$ be such that $\frac{[x]_N}{N} \equiv f \left( \frac{[y]_N}{N} \right)$. Therefore $f \left( \frac{[x]_N}{N} \right) =
Proof. Clearly, \( \text{ker} f \) is one-one. Therefore \( [x]_K^M + [y]_K^M = [x+y]_K^M \). So there exist \( k_1, k_2 \in K \) such that \( x + k_1 = y + k_2 \). Hence \( k_1 \in M \). Then \( [k_1]_K^M = [0]_K^M = [k_2]_K^M \), since \( K \) is subtractive and by Theorem 4.6 (2) and (3). Also there exist \( [k_1]_N^M, [k_2]_N^M \) such that \( f ([k_1]_N^M) = [k_1]_K^M \) and \( f ([k_2]_N^M) = [k_2]_K^M \), as \( f \) is onto. But then \( f ([k_1]_N^M) = [k_1]_K^M = [0]_K^M = [k_2]_K^M = f ([k_2]_N^M) \) \( \Rightarrow [k_1]_N^M, [k_2]_N^M \in \text{ker} f \). So by (1), \( [x]_K^M \equiv_{\text{ker} f} [y]_K^M \). Thus \( f \) is an onto steady ternary \( R \)-semimodule homomorphism. □

Corollary 4.23. (Second Isomorphism Theorem) Let \( N, K \) be ternary subsemimodules of a ternary \( R \)-semimodule \( M \) with \( N \subseteq K \) and \( K \) be subtractive. Then

\[
\frac{M}{N} \cong \frac{M}{K}.
\]

Proof. Define \( f : \frac{M}{N} \to \frac{M}{K} \) by \( f ([x]_N^M) = [x]_K^M \). By Theorem 4.22, \( f \) is an onto steady ternary \( R \)-semimodule homomorphism. Now \( \text{ker} f = \{ [x]_N^M : f ([x]_N^M) = [0]_K^M \} = \{ [x]_N^M : [x]_K^M = [0]_K^M \} = \{ [x]_N^M : x \in K \} = \frac{K}{N} \). By Theorem 4.17, \( \overline{f} : \frac{M}{K} \to \frac{M}{N} \) defined by \( \overline{f} ([x]_K^M) = f ([x]_N^M) \) is a ternary semi-isomorphism. Let \( [x]_K^M, [y]_K^M \in \frac{M}{K} \) be such that \( \overline{f} ([x]_K^M) = \overline{f} ([y]_K^M) \). Therefore \( f ([x]_N^M) = f ([y]_N^M) \Rightarrow [x]_N^M \equiv f [y]_N^M \Rightarrow [x]_N^M \equiv_{\text{ker} f} [y]_N^M \), since \( f \) is steady. Hence there exist \( k_1, k_2 \in \text{ker} f \) such that \( [x]_N^M + k_1 = [y]_N^M + k_2 \). So \( [x]_N^M + k_1 = [y]_N^M + k_2 \). Hence \( [x]_N^M \equiv_{\text{ker} f} [y]_N^M \), since by (1). Hence \( \overline{f} \) is one-one. □

Proposition 4.24. If \( N, K \) are subtractive ternary subsemimodules of a ternary \( R \)-semimodule \( M \), then \( f : \frac{K}{N \cap K} \to \frac{N+K}{N} \) defined by \( f ([x]_{N \cap K}^K) = [x]_{N+K}^N \) is a ternary semimodule semi-isomorphism.

Proof. Clearly, \( f \) is a well defined onto ternary \( R \)-semimodule homomorphism. Now \( [x]_{N \cap K}^K \in \text{ker} f \Rightarrow f ([x]_{N \cap K}^K) = [0]_{N+K}^N \Rightarrow [x]_{N+K}^N = [0]_{N+K}^N \Rightarrow \) there exist \( n, n' \in N \)
such that $x + n = 0 + n' \Rightarrow x + n = n' \Rightarrow x \in N$, since $N$ is subtractive. Therefore $x \in N \cap K$. So $[x]^K_{N \cap K} = [0]^K_{N \cap K}$. Thus $\ker f = \{[0]^K_{N \cap K}\}$. Hence $f$ is a semi-isomorphism.

\[\square\]

**Definition 4.25.** A ternary $R$-semimodule homomorphism $f : M \rightarrow N$ is said to be monomorphism if whenever $g : M' \rightarrow M$ and $g' : M' \rightarrow M$ are ternary $R$-semimodule homomorphisms such that $g \neq g'$ for some ternary $R$-semimodule $M'$, then $fg \neq fg'$.

**Definition 4.26.** A ternary $R$-semimodule homomorphism $f : M \rightarrow N$ is said to be an epimorphism if whenever $g : N \rightarrow N'$ and $g' : N \rightarrow N'$ are ternary $R$-semimodule homomorphisms such that $g \neq g'$ for some ternary $R$-semimodule $N'$, then $gf \neq g'f$.

**Theorem 4.27.** Let $f : M \rightarrow N$ be a ternary $R$-semimodule homomorphism. Then

1) $f$ is injective if and only if $f$ is a monomorphism.

2) $f$ is surjective if and only if $f$ is an epimorphism.

**Proof.** (1) Suppose that $f$ is injective. Let $g : M' \rightarrow M$ and $g' : M' \rightarrow M$ be ternary $R$-semimodule homomorphisms such that $fg = fg'$. Now for $m' \in M'$, $f(g(m')) = (fg)(m') = (fg')(m') = f(g'(m'))$. Since $f$ is injective, $g(m') = g'(m')$ and hence $g = g'$. Now $f$ is a monomorphism. Conversely, suppose that $f$ is a monomorphism. Suppose that $f$ is not injective. Therefore there exist $m \neq m' \in M$ such that $f(m) = f(m')$. Define $g : R \rightarrow M$ by $g(a) = a1m$ and $g' : R \rightarrow M$ by $g'(a) = a1m'$. Then $g$ and $g'$ are ternary $R$-semimodule homomorphisms such that $g \neq g'$. Now for any $a \in R$, $(fg)(a) = f(g(a)) = f(a1m) = a1f(m) = a1f(m') = f(a1m') = f(g'(a)) = (fg')(a)$ and hence $fg = fg'$, a contradiction. Hence $f$ is injective.

(2) Suppose that $f$ is surjective. Let $g : N \rightarrow N'$ and $g' : N \rightarrow N'$ be ternary
$R$-semimodule homomorphisms such that $g \neq g'$. Therefore $g(n) \neq g'(n)$ for some $n \in N$. Since $f$ is surjective, there exists $m \in M$ such that $f(m) = n$. Now $g(n) \neq g(n') \Rightarrow g(f(m)) \neq g'(f(m)) \Rightarrow (gf)(m) \neq (g'f)(m) \Rightarrow gf \neq g'f$. Hence $f$ is an epimorphism. Conversely, suppose that $f : M \to N$ is an epimorphism. Take $N' = f(M)$. By Lemma 4.16, $N'$ is a subtractive ternary subsemimodule of $N$. Define $g : N \to \frac{N}{N'}$ by $g(n) = [n]_{N'}$ and $g' : N \to \frac{N}{N'}$ by $g'(n) = [u]_{N'}$. Then $g$ and $g'$ are ternary $R$-semimodule homomorphisms such that $gf = g'f$. So $g = g'$.

Suppose that $f$ is not surjective. Then there exists $n \in N \setminus N'$ such that $f(m) \neq n$ for any $m \in M$. Now $g = g' \Rightarrow g(n) = g'(n) \Rightarrow [n]_{N'} = [u]_{N'}$. So there exist $f(m), f(m') \in N'$ such that $n + f(m') = f(m) \in N'$. Since $N'$ is subtractive, $n \in N'$, a contradiction. Hence $f$ is surjective. \qed

Remark 4.3.1. A ternary $R$-semimodule homomorphism $f : M \to N$ is an isomorphism if and only if it is both monomorphism and epimorphism.

### 4.4 Relation between quotient ternary semimodules and factor ternary semimodules

In this section, we prove factor ternary semimodule and quotient ternary semimodule are equal as a sets as well as both are isomorphic to each other. Also by using the relation between factor ternary semimodule and quotient ternary semimodule we gives alternative proof of Theorem 2.4 and Theorem 2.5.

**Lemma 4.28.** Let $N$ be a partitioning ternary subsemimodule with respect to a subset $Q$ of a ternary $R$-semimodule $M$ and $m \in M$. If $m + N \subseteq q + N$ for some $q \in Q$, then $[m]_N^M = q + N$. 

Proof. Let \( x \in [m]_N^M \). Then there exist \( n_1, n_2 \in N \) such that \( x + n_1 = m + n_2 \)...(1). Now \( x \in M \). By Lemma 2.2, there exists a unique \( q' \in Q \) such that \( x + N \subseteq q' + N \)...(2). Then from (1) and (2) we get, \( (q + N) \cap (q' + N) \neq \phi \). Since \( N \) is a partitioning ternary subsemimodule, \( q = q' \). Therefore from (2), \( x \in q + N \). Now \( [m]_N^M \subseteq q + N \).

For the other inclusion, let \( s \in q + N \). Then \( s = q + n \) for some \( n \in N \). Since \( m \in q + N, m = q + w \) for some \( w \in N \). Thus \( m + n = q + w + n = s + w \). So \( s \in [m]_N^M \). Now \( q + N \subseteq [m]_N^M \).

Corollary 4.29. Let \( N \) be a partitioning ternary subsemimodule with respect to a subset \( Q \) of a ternary \( R \)-semimodule \( M \). Then the factor ternary semimodule \( M/N \) and the quotient ternary semimodule \( M/N(Q) \) are equal as sets.

Proof. Let \( q \in Q \). Since \( q + N \subseteq q + N \), we have \( [q]_N^M = q + N \). Now it follows from the Lemma 4.28.

Corollary 4.30. Let \( N \) be a partitioning ternary subsemimodule with respect to subsets \( Q_1 \) and \( Q_2 \) of a ternary \( R \)-semimodule \( M \). Then quotient ternary \( R \)-semimodules \( M/N(Q_1) \) and \( M/N(Q_2) \) are equal as sets.

Proof. We have \( M/N(Q_1) = M/N = M/N(Q_2) \) as sets.

Theorem 4.31. Let \( N \) be a partitioning ternary subsemimodule with respect to a subset \( Q \) of a ternary \( R \)-semimodule \( M \). Then the factor ternary \( R \)-semimodule \( M/N \) and the quotient ternary \( R \)-semimodule \( M/N(Q) \) are isomorphic.

Proof. Let \( x, y \in M \) and \( r_1, r_2 \in R \). So there exist unique \( q, q', q_1 \in Q \) such that \( x + N \subseteq q + N, y + N \subseteq q' + N, x + y + N \subseteq q_1 + N \)...(1). Then by Lemma 4.28, \( [x]_N^M = q + N, [y]_N^M = q' + N \) and \( [x + y]_N^M = q_1 + N \). Define \( \phi : \frac{M}{N} \rightarrow M/N(Q) \) by
\[ \phi ([x]_N^M) = q + N \text{ for all } [x]_N^M \in \frac{M}{N}. \]

Then \[ \phi ([x]_N^M + [y]_N^M) = \phi ([x + y]_N^M) = q_1 + N \]
and \[ \phi ([x]_N^M) \oplus \phi ([y]_N^M) = (q + N) \oplus (q' + N) = q_2 + N \] where \( q_2 \in Q \) is unique such that \( q + q' + N \leq q_2 + N \). We have \( x + y + N \subseteq q + q' + N \subseteq q_2 + N \). So \( q_1 = q_2 \).

Hence \[ \phi ([x]_N^M + [y]_N^M) = \phi ([x]_N^M) \oplus \phi ([y]_N^M). \] Similarly, \[ \phi ([r_1 r_2 x]_N^M) = r_1 r_2 \phi ([x]_N^M). \]

Now suppose that \( \phi ([x]_N^M) = \phi ([y]_N^M). \) Then \( q + N = q' + N. \) Hence \( q = q'. \) So by (1), \( x = q + n, y = q + n' \) for some \( n, n' \in N \). Now \( x + n' = q + n + n' = y + n. \) Hence \( [x]_N^M = [y]_N^M. \) So \( \phi \) is one-one. Let \( q + N \in M/N(Q) \). Then there exists \( [q]_N^M \in \frac{M}{N} \) such that \( \phi ([q]_N^M) = q + N. \) So \( \phi \) is onto.

**Corollary 4.32.** Let \( N \) be a partitioning ternary subsemimodule with respect to subsets \( Q_1 \) and \( Q_2 \) of a ternary \( R \)-semimodule \( M \). Then ternary \( R \)-semimodules \( M/N(Q_1) \) and \( M/N(Q_2) \) are isomorphic.

**Proof.** We have \( M/N(Q_1) \cong \frac{M}{N} \cong M/N(Q_2). \) \( \square \)