1. Bending of an Infinite
Semi-Infinite Plate.

1.1. Introduction.

We consider the case of a semi-infinite
plate of isotropic material bent under transverse
loading. The load of intensity \( P \) is assumed to
act at the point \((0,y)\) of the plate and the
boundary is supposed to be either simply supported
or clamped or free. In article 1.3 we first obtain relations between the complex potentials
\( f(z) \) and \( \Phi(z) \) designed to give suitable deflection,
slope, moments or shear on the boundary and in
article 4 apply these formulae to find out the
desired results.

1.2. Relations Between Potentials \( f(z) \) and \( \Phi(z) \).

We write

\[
\omega = i f(z) + \overline{f}(z) + \phi(z) + \overline{\phi}(z)
\]

The solution of the deflection equation

\[
\omega^2 = 0
\]

This gives

\[
\frac{2\omega}{\partial z} - i \left[ \frac{\partial f}{\partial z} + \overline{f} \right] + \text{complex conjugate} \quad (1.2.1)
\]

\[
- \overline{D} \left[ \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} \right] = -i(1-y) D \left[ \frac{\partial f}{\partial z} + \overline{f} \right] + \text{complex conjugate} \quad (1.2.2)
\]

\[
- \overline{D} \left[ \frac{2}{\partial y} \left( \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} \right) \right] = -i(1-y) D \left[ \frac{\partial \phi}{\partial z} + \overline{\phi} \right] + \text{complex conjugate} \quad (1.2.3)
\]

Expressions (1.2.1) and (1.2.2) are expressions
for the moments \( M \) and shear along the boundary
\( y = \text{const} \).
1.3. If we employ the relation

\[ \psi (x) = - \varphi (x) \]  

the expression for \( \psi \) vanishes on the boundary when \( y = 0 \) and \( N \), the moment of inertia being arbitrary. The solution of the \( \psi \) equation \( y = 0 \) becomes

\[ (w)_{y=0} = -4R[\psi(x)]_{y=0} \]  

\[ -D \left[ \frac{2\omega}{y^2} \left( + \frac{2\omega}{y^2} \frac{2\omega}{3y^3} \right) \right]_{y=0} = -8D R[\psi(x)]_{y=0} \]  

\[ -D \left[ \frac{2\omega}{y^2} + \frac{2\omega}{3y^3} \right]_{y=0} = -4(1+2)D R[\psi(x)]_{y=0} \]  

1.3a. By employing

\[ \psi (x) = -x \int f(x) + x \int f(x) \]  

the expression for \( \psi \) vanishes on the boundary.

Here

\[ \kappa = \frac{y}{1+y} \]

The expression for moments on the boundary becomes

\[ (w)_{y=0} = 2 \kappa R \left[ \int f(x) \right]_{y=0} \]  

\[ (shears)_{y=0} = -4D(1+2)R[\psi(x)]_{y=0} \]
loads. For full thinckness,

\[ q(x) = -x f(x) + (x - \alpha) \int f(x) \, dx \]  \hspace{1cm} (1.2.16)

\[ \sigma = \rho \left( \frac{1}{2} \int f(x) \, dx \right) \]

\[ (w) y = (4 - 2x) \left[ R \int f(x) \, dx \right] y_0 \hspace{1cm} (1.2.17) \]

(Moments) \[ y_0 = -u \int (x^2) R[f'(x)] y_0 \hspace{1cm} (1.2.18) \]

1.3. For simply supported plate, we can write

\[ w/A = \left[ x^2 + (\alpha - 3) \right] \log \left[ x^2 + (\alpha - 3) \right] - \left[ x^2 + (\alpha + 3) \right] \log \left[ x^2 + (\alpha + 3) \right] \]  \hspace{1cm} (1.3.1)

Here

\[ A = -\pi/16 \mathrm{K} \]

For clamped plate we write

\[ w_1 = \left[ x^2 + (\alpha - 3) \right] \log \left[ x^2 + (\alpha - 3) \right] - \left[ x^2 + (\alpha + 3) \right] \log \left[ x^2 + (\alpha + 3) \right] \]  \hspace{1cm} (1.3.2)

This gives \( \gamma = C \) \( \psi = C' \) and

\[ \left( \frac{\partial w_1}{\partial y} \right) _{y_0} = -4a \log(x^2 + a^2) - 4a \]  \hspace{1cm} (1.3.3)

We are now to find a solution of \( \frac{\partial^2 w}{\partial y^2} = 0 \) that will give

\[ (w_1) y_0 = 0 \hspace{1cm} \left( \frac{\partial w_1}{\partial y} \right) _{y_0} = 4a \log(x^2 + a^2) + 4a \]

We assume the relation (1.3.2) and state that

\[ R \left[ 3a \log(x^2 + a^2) + 4a \right] y_0 = 4a \log(x^2 + a^2) + 4a \]  \hspace{1cm} (1.3.4)

Further \( 8a \log(x^2 + a^2) \) \( \alpha \) is a solution for \( \gamma > 0 \)

and therefore we write

\[ f(x) = 3a \log(x^2 + a^2) + 4a \]

\[ f(x) = 2a i \log(x^2 + a^2) + ai \]

and

\[ g(x) = -2a i \log(x^2 + a^2) - ai \]

\[ w_2 = 2ai x \log(x^2 + a^2) - 2ai x \log(x^2 + a^2) + ai x \]

\[ -ai x + \text{complex conjugate} \]  \hspace{1cm} (1.3.5)
\[ W = W_1 + W_2 = [x^3 + (y-a)^3] \log \left[ \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} \right] + i a y \quad (1.26) \]

1.8a. If the plate is free at \( y = 0 \), we write as before

\[ W_1 = [x^3 + (y-a)^3] \log \left[ \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} \right] \]

and assume the relation

\[ q(x) = -x f(x) + \phi \int f(x) \, dx \]

for some function \( \phi \). Let the moment (at \( y = 0 \)) be 0 and assume (at \( y = 0 \)) the shear given by

\[ W_1 \text{ at } y = 0. \]

Now,

\[ \frac{\partial W}{\partial y} \left[ \frac{2^3}{3} + (2-y) \frac{2^2}{2^2} \right] = \frac{-8a(1+y)x^2}{(x^2 + a^2)^3} - \frac{8a^3(3-y)}{(x^2 + a^2)^3} \quad (1.27) \]

We note that

\[ \left( \frac{1}{x+a} \right) \frac{1}{x-a} = \frac{2}{x^2 + a^2} \]

so that

\[ R \left[ \frac{1}{x+a} \right] y = \frac{1}{x^2 + a^2} \]

To find out the function \( q(x) \), let the \( \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} \) be its real part on the real axis and so we have

\[ \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{a^2 - b^2} \left[ \frac{x^2}{x^2 - a^2} - \frac{1}{a^2 - b^2} \right] \]

Putting \( a = b \) we get

\[ R \left[ \frac{1}{x^2 + (a+i)^2} \right] y = \frac{1}{x^2 + a^2} \]

the real part of \( \frac{1}{(x+i)^2} \) as \( y = 0 \) is \( \frac{x^2-a^2}{(x^2 + a^2)^2} \)

so that expression in (1.29) is the real part of

\[ -\frac{i}{2} \left[ \frac{x}{(x+i)^2} + \frac{1}{x+i} \right] + \frac{8a(2-y)}{(x+i)^2} \]
and we have

\[ u(3+i) \cdot f''(x) = \frac{-16i}{x+i} + \frac{8a(-1-i)}{x+i} \]

\[ f(x) = -\frac{4}{3+i} \left[ \log(x+i) - \log(x+i) \right] + \frac{2a(-1-i)}{3+i} \]

\[ \int f(x) \, dx = -\frac{4}{3+i} \left[ \log(x+i) - \log(x+i) \right] + \frac{2a(-1-i)}{3+i} \]

\[ (x+i) \left[ \log(x+i) - \log(x+i) \right] \]

and now

\[ \omega_k = (x - k) \left[ f(x) - f(x) \right] + \kappa \int f(x) \, dx + \kappa \int f(x) \, dx \]

The solution is

\[ W = A(\omega_1 - \omega_k) \]

2. Stress in an Orthotropic Half-plane

2.1 Basic Equations

The stress-strain relations are

\[ \varepsilon_x = \frac{1}{E_x} \sigma_x - \frac{v_y}{E_y} \gamma_y \quad \varepsilon_y = \frac{1}{E_y} \sigma_y - \frac{v_y}{E_x} \\sigma_x \quad \gamma_{xy} = G \gamma_{xy} \]

where the symbols used are self-explanatory.

Defining

\[ \sigma_x = \frac{\partial^2 \gamma}{\partial x^2}, \quad \sigma_y = \frac{\partial^2 \gamma}{\partial y^2}, \quad \gamma_{xy} = -\frac{\partial^2 \gamma}{\partial x \partial y} \]

to have, from the relation

\[ \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} \sigma_x \quad \frac{\partial^2 \gamma}{\partial x \partial y} \gamma_{xy} \]

the equation

\[ \left( \frac{\partial^2 \gamma}{\partial x^2} + \kappa^2 \frac{\partial^2 \gamma}{\partial y^2} \right) \gamma_{xy} = 0 \]

where

\[ \kappa^2 = \frac{E_y}{E_x} \]

\( \kappa^2 \) is essentially a positive quantity and we shall suppose it to be real and positive throughout.\[^{[24]}\]
2.2a. We have
\[ \eta = f_1(z) + \bar{f}_1(\overline{z}) + f_2(z) + \bar{f}_2(\overline{z}) \ldots \ (2.2a) \]
as the general solution of equation (2.2a), where
\[ z_1 = x + iy, \quad z_2 = x + iy/k \ldots \ (2.2c) \]
Now
\[ \sigma_y = \frac{\partial^2 \eta}{\partial x^2} = f''_1(z) + \bar{f}''(\overline{z}) + f''_2(z) + \bar{f}''(\overline{z}) \ldots \ (2.2d) \]
and this will vanish on \( y = 0 \) provided
\[ f_2(z) = -f_1(z) \ldots \ (2.2e) \]
so that with the relation
\[ \eta = f_1(z) + \bar{f}_1(\overline{z}) - f_2(z) - \bar{f}_2(\overline{z}) \ldots \ (2.2f) \]
the value of \( \sigma_y \) on \( y = 0 \) vanishes and then
\[ \sigma_y = -i \left[ f''_1(z) - \bar{f}''(\overline{z}) \right] + \frac{1}{k} \left[ f''_2(z) - \bar{f}''(\overline{z}) \right] \ldots \ (2.2g) \]
On \( y = 0 \) its value is
\[ \frac{2 \left( \frac{1-k}{k} \right) \mathcal{R} \left[ \left[ f''_1(z) \right] \right]} \ldots \ (2.2h) \]

2.2b. From
\[ \frac{\partial^2 \eta}{\partial x^2} = i \left[ f''_1(z) - \bar{f}''(\overline{z}) + \frac{1}{k} f''_2(z) - \frac{1}{k} \bar{f}''(\overline{z}) \right] \ldots \ (2.2i) \]
we get the relation
\[ f_2(z) = -k f_1(z) \ldots \ (2.2j) \]
which will give \( \sigma_y = 0 \) on the \( x \)-axis. The value of \( \sigma_y \) under this assumption is given by
\[ \sigma_y = k \mathcal{R} \left[ \left[ f''_1(z) \right] \right] \ldots \ (2.2k) \]

The problem of determining functions
which will give arbitrary stresses on \( y = 0 \) is
reduced by (2.2a) and (2.2b) to the problem of
determining functions of \( x + iy \) having given real
parts on \( x \) \( y = 0 \).

Similarly with
\[ \eta = f_1(z) + \bar{f}_1(\overline{z}) + f_2(z) + \bar{f}_2(\overline{z}) \]
and the relations

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{1}{E_x} \frac{\partial^2 u}{\partial x^2} - \frac{\gamma_{xy}}{E_x} \frac{\partial^2 u}{\partial x \partial y} \\
\varepsilon_y &= \frac{\partial u}{\partial y} = \frac{1}{E_y} \frac{\partial^2 u}{\partial y^2} - \frac{\gamma_{xy}}{E_y} \frac{\partial^2 u}{\partial x \partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = -\frac{1}{G} \frac{\partial^2 u}{\partial x \partial y}
\end{align*}
\]

(2.1.11) (2.1.12) (2.1.13)

we get

\[
\begin{align*}
u &= - \frac{1 + \nu_2}{E_x} \left[ f_2'(x) + f_3'(x) \right] - \frac{1 + \nu_2}{E_y} \left[ f_2'(x) + f_3'(x) \right] \\
v &= - \frac{i(1 + \nu_2)}{E_y} \left[ f_2'(x) f_3'(x) \right] - \frac{i(1 + \nu_2)}{E_x} \left[ f_2'(x) + f_3'(x) \right]
\end{align*}
\]

(2.2.14)

(2.2.15)

2.2c. The relation

\[
E_y (1 + \nu_2) f_2(x) = - E_x (1 + \nu_2) f_3(x)
\]

(2.2.16)
gives \( u = 0 \) on \( y = 0 \) and the expression for \( v \) becomes

\[
(v)_{y=0} = 2 \rho_2 R\left[ f_2'(x) \right]
\]

(2.2.17)

where \( R \) stands for the real part of

\[
\rho_2 = \frac{1}{k} \frac{E_y}{E_x (1 + \nu_2)} - \frac{1 + \nu_2}{E_y}
\]

(2.2.18)

2.2d. The relation

\[
E_y (1 + \nu_2) f_3(x) = - k E_x (1 + \nu_2) f_2(x)
\]

(2.2.19)
gives \( v = 0 \) on \( y = 0 \) and under this assumption

\[
(v)_{y=0} = 2 \rho_1 R\left[ f_2'(x) \right] \quad \text{where}
\]

\[
\rho_1 = k \frac{E_x (1 + \nu_2)^2}{E_y (1 + \nu_2)} - \frac{1 + \nu_2}{E_x}
\]

(2.2.20) (2.2.21)

2.3. As applications of the above theory we deduce the expressions for (i) stresses and (ii) displacements in a semi-infinite plate with boundary \( y = 0 \) (i) free from tractions (ii) free
from displacements, respectively under the action of an arbitrary force acting at an arbitrary point.

There is no loss of generality if we assume the point of application of the force to be \( x = 0, y = 0 \).

**Case:** An Isolated Force Parallel to the Boundary Acting in a Semi-Infinite Plate where Boundary \( y = 0 \) is Free from Tractions.

We write

\[
y_1 = \frac{-P}{2\pi(1-\nu^2)} \left[ \frac{k^2 + \nu \alpha}{\alpha} \log\left(\frac{z_1 - \alpha}{\alpha}\right) \right. \\
\left. + \frac{l^2 + \nu \beta}{\beta} \log\left(\frac{z_1 - \beta}{\beta}\right) + \text{complex conjugate} \right].
\]

which is the stress function of an infinite plate acted upon by a force of magnitude \( P \) in the direction of \( x \)-axis at the point \( z = 1a \).

This gives on \( y = 0 \)

\[
(\sigma_y)_{y=0} = \frac{-P}{2\pi(1-\nu^2)} \left[ \frac{k^2 + \nu \alpha}{\alpha} - \frac{l^2 + \nu \beta}{\beta} \right] \quad (2.1.2)
\]

\[
(\tau_{xy})_{y=0} = \frac{-P}{2\pi(1-\nu^2)} \left[ \frac{k^2 + \nu \alpha}{\alpha} - \frac{l^2 + \nu \beta}{\beta} \right] \quad (2.1.3)
\]

To cancel out these stresses, we write \( \tau_y = \tau_1 - \tau_2 - \tau_3 \),

where \( \tau_3 \) is the function that will give \( \tau_y = 0 \) as given above with \( \tau_y = 0 \) on \( y = 0 \).

\( \tau_1 \) is the function that will give \( \tau_{xy} = 0 \) as given above with \( \sigma_y = 0 \) on \( y = 0 \).

If we write

\[
\tau_1 = f_1(x) + \tilde{f}_1(z) - \kappa f_3(x) - \kappa \tilde{f}_3(z) \quad (2.3.4a)
\]

\[
\tau_3 = f_3(x) + \tilde{f}_3(z) - f_1(x) - \tilde{f}_1(z) \quad (2.3.4b)
\]

these considerations give from (2.2.7) and (2.2.10)
\[ 2(1-k)R \int_0^\infty \frac{-p}{2k(1-k^2)} \left[ \frac{(k^2 + y^2) x}{x^2 + a^2} - \frac{k(k + y)}{x^2 + a^2} \right] \, dx \tag{2.3.1} \]

Giving
\[ f_1(x) = \frac{-p}{4k(1-k^2)(1-k)} \left[ \frac{(k^2 + y^2)(x + ia)}{x^2 + a^2} \log(x + ia) \right] \]
\[ - \frac{k(k + y)}{x^2 + a^2} \log(x + ia) \tag{2.6} \]

and
\[ 2(1-k)R \left[ (f_1''(y)) = \frac{-p}{2k(1-k^2)} \left[ \frac{(k^2 + y^2) a}{x^2 + a^2} - \frac{iy}{x^2 + a^2} \right] \tag{2.6.1} \]

Giving
\[ f_2(x) = \frac{-p}{4k(1-k^2)(1-k)} \left[ \frac{(k^2 + y^2)(x + ia)}{x^2 + a^2} \log(x + ia) \right] \]
\[ - \frac{k(k + y)}{x^2 + a^2} \log(x + ia) \tag{2.3.2} \]

If we would have used image method, it is apparent from (2.3.2) that \( \sigma_y \) can be made to vanish on \( y = 0 \) by applying an equal and opposite force at \( (0, -a) \) so that we may write
\[ \sigma_y = \frac{-p}{2k(1-k^2)} \left[ \frac{k^2 + y^2}{2} \left\{ (x + ia) \log(x + ia) - (x + ia) \log(x - ia) \right\} \right] \]
\[ - \frac{k(k + y)}{x^2 + a^2} \left\{ (x + ia) - (x - ia) \right\} \left( \log(x + ia) + \log(x - ia) \right) \tag{2.3.3} \]

Giving on \( y = 0 \)
\[ \sigma_y = \frac{-p}{2k(1-k^2)} \left[ \frac{k^2 + y^2}{2} \left\{ (x + ia) \log(x + ia) - (x + ia) \log(x - ia) \right\} \right] \]
\[ - \frac{k(k + y)}{x^2 + a^2} \left\{ (x + ia) + (x - ia) \right\} \left( \log(x + ia) + \log(x - ia) \right) \tag{2.3.4} \]

And
\[ \sigma_y = 0 \tag{2.3.5} \]

So that if we write \( \sigma_y = \sigma_y - \sigma_y \), we get
\[ \sigma_y = f_1(x) + f_2(x) - f_1(x) \]
\[ f_1(x) = \frac{-p}{2k(1-k^2)(1-k)} \left[ \frac{(k^2 + y^2)(x + ia)}{x^2 + a^2} \log(x + ia) \right] \]
\[ - \frac{k(k + y)}{x^2 + a^2} \log(x + ia) \tag{2.3.6} \]

2.3b. Plate acted upon at \( x = 0 \) by a force \( P \) in the \( y \)-direction, the boundary line \( y = 0 \) being free from tractions.
In this case we write
\[ \eta_1 = \frac{\rho}{4\pi(x-a)^3} \left[ \left( \frac{(x+a)}{(x+a)^2} \right) \log \left( \frac{x+a}{x-a} \right) - \left( \frac{(x+a)}{(x-a)^2} \right) \log \left( \frac{x+a}{x-a} \right) \right] \]

This gives \( \bar{\eta}_1 = 0 \) on \( y = 0 \) and we write
\[ \eta_2 = f_1(x) + \tilde{f}_1(x) - \kappa \tilde{f}_1(x-a) - \kappa \tilde{f}_1(x-a) \overline{f}(x) \]

giving
\[ (\sigma_y)_{y=0} = 2\kappa \tilde{f}_1(x) \]

From \( \eta_1 \), the value of \( \sigma_2 \) at \( y = 0 \) is
\[ \frac{\rho}{\pi(\kappa-k^2)} \left[ \left( \frac{(x+a)}{(x+a)^2} \right) \log \left( \frac{x+a}{x-a} \right) - \left( \frac{(x+a)}{(x-a)^2} \right) \log \left( \frac{x+a}{x-a} \right) \right] \]

so that we get
\[ 2(\kappa-k) \tilde{f}_1(x) = \frac{\rho}{\pi(\kappa-k^2)} \left[ \left( \frac{(x+a)}{(x+a)^2} \right) \log \left( \frac{x+a}{x-a} \right) - \left( \frac{(x+a)}{(x-a)^2} \right) \log \left( \frac{x+a}{x-a} \right) \right] \]

or
\[ \tilde{f}_1(x) = \frac{\rho}{2\pi(\kappa-k)^2} \left[ \left( \frac{(x+a)}{(x+a)^2} \right) \log \left( \frac{x+a}{x-a} \right) - \left( \frac{(x+a)}{(x-a)^2} \right) \log \left( \frac{x+a}{x-a} \right) \right] \]

\( \eta_1 - \eta_2 \) gives the required solution.

2.3c. Plate Acted Upon At \( z = 0 \) By a Force \( P \) in the \( x \)-direction. The Boundary Line \( y = 0 \) Being Free From Displacements.

If we write
\[ \eta_1 = \eta_1 - \eta_2 \]

with
\[ \eta_1 = -\frac{\rho}{2\pi(\kappa-k^2)} \left[ \frac{\kappa_x}{x} \left\{ \log \left( x-a \right) - \log \left( x+a \right) \right\} \right] \]

we get \( u = 0 \) on the boundary and
\[ (w)_{y=0} = \frac{P(x^2+y^2)^{1/2}}{x} \left( \overline{\eta}_1 - \overline{\eta}_2 \right) - \frac{P(x+y)(x+y)}{\kappa(x-k)} \cdot \frac{y}{x} \]

Writing
\[ \eta_2 = f_2(x) + \overline{f}_2(x) \]

\[ \eta_2 = f_2(x) + \overline{f}_2(x) - \frac{E_0(x+y)^{1/2}}{x} \left[ f_2(x) + \overline{f}_2(x) \right] \]
we get

\[ P_1 f'_1(x) = \frac{P(x^2 + y^2)}{\pi (1 - k^2)} \cdot \frac{i x}{Ey} \left[ i \log(x + ia) \right] - \frac{P(x^2 + y^2)}{\pi (1 - k^2)} \cdot \frac{i y}{Ex} \left[ i \log(x + ia) \right] \]

giving

\[ P_1 f'_1(x) = - \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{i (x + ia) \log(x + i a)}{Ex} \]

\[ + \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{i (x + ia) \log(x + i a)}{Ey} \]

2.3d. An Isolated Force Perpendicular to the Boundary Which is Free From Displacement.

We write

\[ \gamma = \gamma_1 - \gamma_1 \]

Where \( \gamma_1 = \frac{P}{\pi (1 - k^2)} \cdot \left[ i (x^2 + y^2) \log(x^2 + y^2) - (x - i \alpha) \log(x + ia) - (x + i \alpha) \log(x + ia) - (x + i \alpha) \log(x + ia) \right] \)

This gives \( \gamma = 0 \) on the boundary and

\[ (\omega)g_1 = \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{\tan^{-1} \frac{a}{x}}{\pi (1 - k^2)} - \frac{k^2 P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{\tan^{-1} \frac{a}{y}}{\pi (1 - k^2)} \]

so that if we write

\[ \gamma_1 = f_1(x_1) + \overline{f}_1(x_1) - k \cdot \frac{Ex(x^2 + y^2)}{Ey(1 - k^2)} \cdot \left[ f_1(x_1) + \overline{f}_1(x_1) \right] \]

we get

\[ \omega P_1 f'_1(x) = - \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{i (x + ia) \log(x + ia)}{Ex} \]

\[ + \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{i (x + ia) \log(x + ia)}{Ey} \]

giving

\[ \omega P_1 f'_1(x) = - \frac{P(x^2 + y^2)(1 + iy)}{\pi (1 - k^2)} \cdot \frac{i (x + ia) \log(x + ia) + Pk^2(1 + iy)(1 + iy) x}{Ey(1 - k^2)} \]

This completes the solution.

2.1. The Problem of Distribution of Stress in the Neighbourhood of a Griffith Crack in an Orthotropic Plate Under the Assumption of Plane Stress.
If the crack is opened up by a pressure
$k(x)(x < c)$ along the boundary, the boundary conditions are

$$
\begin{align*}
\tau_{xy} &= 0 \quad \text{for } y = 0 \\
\sigma_y &= \tau_{xy} = 0 \quad \text{for } (x < c, y) \\
\sigma_x &= 0 \quad \text{for } (x > c, y).
\end{align*}
$$

Because $\tau_{xy} = 0$ on the boundary we take

$$
\begin{align*}
y &= f_1(x) + \tilde{f}_1(x) - k f_2(x) - k \tilde{f}_2(x) \\
\sigma_y &= \frac{\partial f_1}{\partial x} - k \frac{\partial \tilde{f}_2}{\partial x}
\end{align*}
$$

so that we get on $y = 0$

$$
\begin{align*}
\left(\sigma_y\right)_{y=0} &= \frac{\partial f_1}{\partial x} - k \frac{\partial \tilde{f}_2}{\partial x} \\
\left(\tau_{xy}\right)_{y=0} &= \frac{\partial^2 f_1}{\partial x^2} - k \frac{\partial^2 \tilde{f}_2}{\partial x^2}
\end{align*}
$$

and

$$
\begin{align*}
\left(\sigma_y\right)_{y=0} &= \frac{\partial f_1}{\partial x} - k \frac{\partial \tilde{f}_2}{\partial x} \\
\left(\tau_{xy}\right)_{y=0} &= \frac{\partial^2 f_1}{\partial x^2} - k \frac{\partial^2 \tilde{f}_2}{\partial x^2}
\end{align*}
$$

The problem now reduces to that of finding a function of $z$ satisfying on $y = 0$

$$
\begin{align*}
\tau_{xy} &= 0, \quad (x < c) \\
\sigma_y &= \frac{\partial f_1}{\partial x} - k \frac{\partial \tilde{f}_2}{\partial x}, \quad (x < c)
\end{align*}
$$

If we write

$$
\begin{align*}
f'(x) &= f + \gamma \\
\tilde{f}'(x) &= \frac{2i}{\pi} - \frac{2c}{\pi}
\end{align*}
$$

so that

$$
\begin{align*}
f''(x) &= \frac{2i}{\pi} + \frac{2c}{\pi}
\end{align*}
$$

our problem reduces to determining $\gamma$, a solution of $\nabla^2 \eta = 0$ with the boundary conditions

$$
\begin{align*}
\eta &= 0, \quad (x > c) \\
\frac{\partial \eta}{\partial y} &= \frac{\partial f_1}{\partial x} - k \frac{\partial \tilde{f}_2}{\partial x}, \quad (x > c)
\end{align*}
$$

To find $\gamma$, we apply Fourier transform

$$
\tilde{\eta}(k, y) = \int_{-\infty}^{\infty} \eta(x, y) e^{-ikx} dx
$$

and, assuming $\gamma$ to be even in $x$, gives

$$
\tilde{\eta}(k, y) = 2 \int_{0}^{\infty} \eta(x, y) \cos kx dx
$$
so that $\eta$ satisfies
\[ \frac{\partial^2 \eta}{\partial y^2} - \frac{\partial \eta}{\partial x} = 0 \]

Solving
\[ \eta = A e^{ry} + B e^{-ry} \]

neglecting $B e^{-ry}$, we get
\[ \eta = \eta_0 e^{ry} \]

where \[ \eta_0 = 2 \int_0^c \eta(x, 0) \cos \pi x \, dx \]

we now have
\[ \eta = \frac{1}{\pi} \int_0^c \eta(x, y) \cos \pi x \, dx \]
\[ = \frac{1}{\pi} \int_0^c \eta(x, 0) e^{ry} \cos \pi x \, dx \]  
(2.4.12)

and
\[ \frac{\partial \eta}{\partial y} = \frac{1}{\pi} \int_0^c \eta(x, y) \cos \pi x \, dx \]

the boundary conditions, therefore, give
\[ \int_0^c \eta_0 \cos \pi x \, dx = 0 \quad \text{for } |x| > c \]
\[ \int_0^c \eta_0 \cos \pi x \, dx = \frac{\pi}{\lambda^2} \int_{-c}^c p(x) \cos \pi x \, dx \quad \text{for } |x| < c \]

these equations are similar to equations in Article 14.12 of [10] and writing
\[ p(x) = p_0 \sum a_n \left( \frac{x}{c} \right) \]

we get the solution.
\[ \eta_0 = \frac{1}{\pi a_n(x, y)} p_0 e^{ry/2} \int_0^c \frac{T(\bar{q} + \bar{r})}{T(\bar{r} + \bar{q})} a_n \left( \frac{x}{c} (c + e) + e \right) \sum e^{\alpha^2 (c + e)} d\alpha \]

hence from (2.4.12) we get $\eta$ and then $\nu$.

The value of $\nu$ at $y = 0$ for $|x| < c$ is given by
\[ u(x, 0) = \frac{1 + \lambda}{\sqrt{2} E} p_0 \int_0^c \frac{T(\bar{q} + \bar{r})}{T(\bar{r} + \bar{q})} a_n \left[ e^{\lambda \left( \frac{x}{c} (c + e) + 1 \right)} \right] \sum e^{\alpha^2 \left( \frac{x}{c} (c + e) \right)} d\alpha \]

For the case of a uniform pressure $p_0$, we take $a_n = 1$, $a_0 = 0$ (other $a_n$) and get
\[ u(x, 0) = \frac{1 + \lambda}{\sqrt{2} E} p_0 \left( c^2 - x^2 \right)^{1/2} \]
The result is very similar to the one obtained in the case of isotropic plates.

Also

\[
\eta_0 = \frac{\alpha}{2(I-K)} \int \psi_{1}(\xi_{g}) \frac{d\xi}{(\xi_{g})^{1/2}}
\]

this gives

\[
\eta_1 = \frac{\alpha}{2(I-K)} \int \psi_{1}(\xi_{g}) \frac{d\xi}{(\xi_{g})^{1/2}}
\]

giving

\[
\psi_{1}(x) = \frac{\alpha}{2(I-K)} \int \psi_{1}(\xi_{g}) \frac{d\xi}{(\xi_{g})^{1/2}}
\]

We note that even the real axis for \( x \geq 0 \) is a displacement variable.

The result is analytic for \( x > 0 \) and holds true for \( x > 0 \), so that one may write

\[
\psi_{1}(x) = \frac{\alpha}{2(I-K)} \left[ x \sqrt{\xi_{g}-x^2} + c \sinh \frac{x}{2} \right]
\]

\( \sinh \frac{x}{2} \) being defined as the branch of the function which is zero at the origin and \( \sqrt{\xi_{g}-x^2} \) that one which is real to the right of the origin. This completes the solution.

2.6 Imagination Problem

We consider the problem of the medium being deformed by application of small stresses along the

\[
\sigma_{11} = 0, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0, \quad \tau_{12} = 0, \quad \tau_{23} = 0, \quad \tau_{31} = 0
\]

If the boundary is defined by the surface against it of \( x = 0 \), \( y = 0 \), \( z = 0 \), a suitable choice of co-ordinates system, may assume, at least in first approximation, that the body \( \mathcal{C} \) is in contact with the elastic solid along the plane \( y = 0, \quad z = 0 \). In this case we may assume the surface showing stress is zero, and that

\[
\sigma_{11} = \sigma_{22} = \sigma_{33} = 0, \quad \tau_{12} = 0, \quad \tau_{23} = 0, \quad \tau_{31} = 0
\]
the band $\omega < 1$, the normal component $v$ of the
surface displacement is prescribed but outside
$\omega < 1$, we have, instead of a knowledge of $v$,
the condition that the normal component of stress
is zero. Along $y = 0$, we therefore have the condi­
tions

$$\begin{align*}
T_{x,y} &= 0 \quad \text{at} \quad y = 0 \\
\sigma_y &= \sigma_x(x) \ , \ \text{at} \quad 1 \omega < 1 \ , \ y = 0 \\
\sigma_y &= 0 \quad \text{at} \quad 1 \omega > 1 \ , \ y = 0
\end{align*}$$

where $\sigma_x(x)$ is a prescribed function of $x$. (See p.
431 of [10]). Writing as before

$$\begin{align*}
y &= f_1(x) + \frac{f_2(x)}{2} \omega_x(x) - \omega_0(x) \\
&= \frac{f_1(x)}{2} \omega_x(x) + \omega_0(x)
\end{align*}$$

we get our boundary conditions as

$$\begin{align*}
f'_1(x) &= -\frac{E_0}{\omega_x(x)} \quad \text{for} \quad 1 \omega < 1 \\
f''_1(x) &= 0 \quad \text{for} \quad 1 \omega > 1
\end{align*}$$

If we write

$$\begin{align*}
f'_1(x) &= f_1(x,y) + i \gamma(x,y) \\
f''_1(x) &= \frac{2}{\omega_y(x)} \gamma(x,y)
\end{align*}$$

we get the conditions

$$\begin{align*}
\gamma(x,\omega) &= \frac{E_0}{\omega_x(x)} \quad 1 \omega < 1 \\
\frac{d^2 \gamma(x,\omega)}{dx^2} &= 0 \quad 1 \omega > 1
\end{align*}$$

writing

$$\begin{align*}
\gamma_1(\omega,\gamma) &= \gamma_0(\omega) \cdot C^2 \gamma + \int_{-\infty}^{\infty} \gamma(x,\omega) \cdot C \omega_x(\omega) \cdot d\omega
\end{align*}$$

($\gamma_0(x)$ is supposed to be even), we get

$$\begin{align*}
\gamma &= \gamma_0(x) \cdot C \omega_x(\omega) \\
\gamma &= \frac{1}{\omega_x(\omega)} \int_{-\infty}^{\infty} \gamma_0(x) \cdot C \omega_x(\omega) \cdot d\omega
\end{align*}$$

The boundary conditions now read

$$\begin{align*}
\int_{-\infty}^{\infty} \gamma_0(x) \cdot C \omega_x(\omega) \cdot d\omega &= \frac{x}{2(\cdot \omega^2)} \cdot \omega_0(x) \quad 1 \omega < 1
\end{align*}$$

and

$$\begin{align*}
\int_{-\infty}^{\infty} \omega \gamma_0(x) \cdot C \omega_x(\omega) \cdot d\omega &= 0 \quad 1 \omega > 1
\end{align*}$$
The solution of these equations is given by Sneddon (see p. 414 of [10]) as
\[ \bar{\eta} = \sum_{n=0}^{\infty} A_n \int_0^1 t^n \eta(1) dt \]
where
\[ A_n = -\frac{\sqrt{\pi} E_{1/2} \left( \frac{\eta}{x} \right)}{\left( x^2 - 1 \right)^{1/2}} \]
and
\[ u_0(x) = \sum_{n=0}^{\infty} A_n x^n \left( x^2 - 1 \right)^n \]
Further analysis proceeds as in [10].

art. 48.3.

2.5a. Asymmetric Distribution.

In this case the boundary conditions are
\[ u = u_0(x) , \quad -1 < x < 1 , \quad y = 0 \]
where
\[ u_0(-x) = -u_0(x) \]
and
\[ \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0 \]
writing
\[ \eta = \tilde{f}(x,y) + \tilde{L}(x,y) \cdot \int \tilde{f}(x,y) - k \tilde{f}(x,y) \]
the boundary conditions read
\[ \eta = \frac{E_0}{2(1-k^2)} u \quad \text{for} \quad 1 < x < 1 , \quad y = 0 \]
writing
\[ \bar{\eta}(\omega,\gamma) = 2 \int_0^\infty \eta(x,y) \sin \omega x dx \]
we get
\[ \eta(x,y) = \frac{1}{\pi} \int_0^\infty \eta(\omega,\gamma) \sin \omega x dw \]
and
\[ \frac{\partial \eta}{\partial \gamma} = \frac{-1}{\pi} \int_0^\infty \omega \eta(\omega,0) \sin \omega x dw \]
so that the boundary conditions become
\[ \int_0^\infty \eta(\omega,0) \sin \omega x dw = \frac{\lambda E_0}{2(1-k^2)} u_0(x) \quad \text{for} \quad 1 < x < 1 \]
\[ \int_0^\infty \omega \eta(\omega,0) \sin \omega x dw = 0 \quad \text{for} \quad x = 1 \]
\[ J_1 = \int_0^\infty \tilde{\eta}_0(\omega, \omega) \omega^{2/3} J_{3/2}^2(\omega) \, d\omega = \frac{E_0^2}{2\pi(1-\nu^2)} \frac{\nu(1-\nu)}{\nu(1-\nu^2)} \]

and

\[ \int_0^\infty \tilde{\eta}_0(\omega, \omega) \omega^{2/3} J_{3/2}^2(\omega) \, d\omega = 0 \]

For \( \nu_0(\omega) = \frac{\rho}{\rho_0} \lambda \)

The solution of the differential equation is

\[ \tilde{\eta}_0(\omega, \omega) = \sum D_n \int_0^\infty \tilde{\eta}_0(\omega, t) \, dt \]

where

\[ D_n = \frac{\nu_0(\omega)}{2\pi(1-\nu^2)} \frac{(\nu/2)!}{(\nu^2/2)!} \]

With the particular case of \( \nu_0(\omega) = 1 \), \( \nu_0 \) const.

Thus

\[ \tilde{\eta}_0(\omega, \omega) = 2B_0 J_0(\omega) \]

Therefore, we have

\[ \eta(x, y) = 4 \left( \frac{\nu E_0}{\pi^2(1-\nu^2)} \right) \int_0^\infty J_0(\omega) \sin \omega x \omega \, d\omega \]

\[ f(x) = 4 \left( \frac{\nu E_0}{\pi^2(1-\nu^2)} \right) \int_0^\infty J_0(\omega) \cos \omega x \omega \, d\omega \]

\[ = \frac{\nu E_0}{\pi^2(1-\nu^2)} \frac{1}{x} \]

So

\[ q = \frac{\nu E_0}{\pi^2(1-\nu^2)} \left[ \frac{1}{x^2} + \frac{1}{x^2} - \frac{1}{x^2} - \frac{1}{x^2} \right] \]

and the solution is complete.

3. Bending of Orthotropic Plates

3.1 Basic Equations

In the case of orthotropic plates under
plan strain assumption, stress-strain relations

\[ \sigma_y = \frac{E_y}{1-\nu_{xy}} \left( \epsilon_y + \nu_{xy} \epsilon_x \right) \quad \text{(3.1.1a)} \]

\[ \tau_{xy} = \frac{E_{xy}}{1-\nu_{xy}} \left( \epsilon_x + \nu_{xy} \epsilon_y \right) \quad \text{(3.1.1b)} \]

\[ \tau_{yx} = \frac{1}{\mu} \nu_{xy} = \frac{1}{\mu} \left( \frac{E_{xy}}{E_y} + \frac{E_y}{E_{xy}} \right) \quad \text{(3.1.1e)} \]
\[
\lambda = \frac{E_x + E_y + 2\nu_x E_x}{E_x E_y} \quad (2.1.2)
\]

For small approximation of stress, we have

\[
e_x = \frac{x}{\lambda} \quad e_y = \frac{y}{\lambda} \quad (2.1.3)
\]

where L is normal stress. Let \( \lambda \) be the thickness of the plate, \( \nu_x, \nu_y \) are the ratio of deviation of deflection in one of the principal directions and \( e_x, e_y \)

are displaced in the direction.

\[
\frac{1}{\nu_x} = -\frac{2w}{2x^2}, \quad \frac{1}{\nu_y} = -\frac{2w}{2y^2} \quad (2.1.4)
\]

Then the equation

\[
e_x = -\frac{E_x x}{1-\nu_x \nu_y} \left( \frac{2w}{2x^2} + \nu_y \frac{2w}{2y^2} \right) \quad (2.1.5a)
\]

\[
e_y = -\frac{E_x x}{1-\nu_x \nu_y} \left( \frac{2w}{2y^2} + \nu_x \frac{2w}{2x^2} \right) \quad (2.1.5b)
\]

For \( x \) and \( y \):

\[
\int_{-h/2}^{h/2} e_x \, dx = M_x, \quad \int_{-h/2}^{h/2} e_y \, dx = M_y \quad (2.1.6a)
\]

\[
M_x = -D_x \left( \frac{2w}{2x^2} + \nu_y \frac{2w}{2y^2} \right) \quad (2.1.7a)
\]

\[
M_y = -D_y \left( \frac{2w}{2y^2} + \nu_x \frac{2w}{2x^2} \right) \quad (2.1.7b)
\]

\[
D_x = \frac{E_x h^3}{12(1-\nu_x \nu_y)}, \quad D_y = \frac{E_y h^3}{12(1-\nu_x \nu_y)} \quad (2.1.8)
\]

Introducing the notation of the above:

\[
H_{xt} = -\int_{-h/2}^{h/2} x \, T_{xt} \, dx \quad (2.1.9a)
\]

\[
H_{yt} = D_x \nu_y \left( \frac{k^2}{\nu_x \nu_y} - 1 \right) \frac{2w}{2x^2} \quad (2.1.9b)
\]

\[
k^2 = \nu_x \nu_y \left( \frac{2 + \nu_x + \nu_y}{\nu_x + \nu_y + 2 \nu_x \nu_y} \right) \quad (2.1.9c)
\]
\[
\frac{\partial^2}{\partial x^2} N_x + \frac{\partial^2}{\partial y^2} N_y = \frac{\partial^2}{\partial x^2} M_{xy} \quad (3.1.10)
\]
\[
\gamma_y \frac{\partial^2}{\partial x^2} W + 2 \gamma_y^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial W}{\partial x} \right)^2 - \frac{\partial}{\partial x} \left( \frac{\partial W}{\partial x} \right) \frac{\partial}{\partial y} \left( \frac{\partial W}{\partial y} \right) = \frac{\partial^2}{\partial x^2} \left( \frac{\partial W}{\partial y} \right) \quad (3.1.11)
\]

We write this equation as the form
\[
\left( \frac{\partial^2}{\partial x^2} + k_x^2 \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + k_y^2 \frac{\partial^2}{\partial y^2} \right) W = \frac{\partial^2}{\partial x^2} \left( \frac{\partial W}{\partial y} \right) = 0 \quad (2.1.12)
\]
For:
\[
\begin{align*}
\gamma_y k_x^2 &= k_x^2 + \sqrt{k_x^2 - \gamma_y} \\
\gamma_y k_y^2 &= k_y^2 - \sqrt{k_y^2 - \gamma_y}
\end{align*}
\]
In the particular case of \( \gamma_y = 0 \) for simplicity we have:
\[
\left( \frac{\partial^2}{\partial x^2} + k_x^2 \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} + k_y^2 \frac{\partial^2}{\partial y^2} \right) W = 0 \quad (2.1.14)
\]

The general solution of this equation can be written in the form:
\[
W = f_1(x) + f_2(y) \quad (2.1.15)
\]
For:
\[
x = a y + b, \quad y = a + b
\]
We shall assume throughout that \( k_x, k_y \) are both positive quantities. The solution \( W \) is essentially of the form \( f(x) = k_x + k_y \) to complex quantities, in which case they are complex conjugates.

Our fundamental solution of a given (3.1.24) field gives the deflection at any point in an arbitrary manner. In the form of intensity of the field we regard to be:
\[
W = \frac{A(1 + k_x^2)}{2k_y(k_x^2 - k_y)} \left[ (k_x + iy)^2 \log(k_x + iy) + \text{complex conjugate} \right] + \frac{A(1 + k_y^2)}{2k_x(k_y^2 - k_x)} \left[ (k_y + iy)^2 \log(k_y + iy) + \text{complex conjugate} \right] \quad (2.1.17)
\]
where the value of \( A \) is to be found from the consideration that the shear along any curve surrounding the origin must be \( P \). It is found that the value of \( A \) is given by the equation.

\[
A = \frac{P}{2} (k_1^2 - k_2^2)
\]

Where

\[
y = \frac{4D_x(1+k_1^2)}{\gamma_2 k_1} \left[ \frac{\pi}{2k_1^2} (v_{k_1} - k_2 v_{k_2}) - \left( \frac{v_{k_1}}{k_1} + v_{k_2} k_1^2 - 2k_1^2 \right) \tan^{-1} \frac{1}{k_1} \right]
\]

\[
y - \frac{4D_x(1+k_2^2)}{\gamma_2 k_2} \left[ \frac{\pi}{2k_2^2} (v_{k_2} - k_1 v_{k_1}) - \left( \frac{v_{k_2}}{k_2} + v_{k_1} k_2^2 - 2k_2^2 \right) \tan^{-1} \frac{1}{k_2} \right]
\]

We note that \( A \) is real whether \( k_1 \) and \( k_2 \) be real or complex conjugates.

The result (3.1.17) is observed to be symmetrical in \( k_1 \) and \( k_2 \) as it should be and is valid for all values of \( k_1 \) and \( k_2 \) real or complex.

Now for a simply supported semi-infinite plate with a load \( P \) applied at the point \( x = 0 \), \( y = a \), we write, employing the method of images

\[
\frac{1}{A} (k_1^2 - k_2^2) \omega = \frac{1}{\alpha k_1^2} \left[ \left\{ k_1^2 - \omega_1 \left( \begin{array}{c} v_{k_1} - k_2 v_{k_2} \\ k_1^2 - 2k_1^2 \end{array} \right) \right\} \log \left( \frac{k_1^2 + \omega_1 - 2k_1^2 \omega_1}{\omega_1} \right) \right]
\]

\[
-4 k_1 x (y-a) \tan^{-1} \frac{y-a}{k_1} - \frac{1}{\alpha k_1^2} \left[ \left\{ k_1^2 - 2k_1^2 (y-a)^2 \right\} \log \left( \frac{k_1^2 + (y-a)^2}{(y-a)^2} \right) \right]
\]

\[
-4 k_2 x (y-a) \tan^{-1} \frac{y-a}{k_2} - \frac{1}{\alpha k_2^2} \left[ \left\{ k_2^2 - 2k_2^2 (y-a)^2 \right\} \log \left( \frac{k_2^2 + (y-a)^2}{(y-a)^2} \right) \right]
\]

\[
-4 k_1 x (y+a) \tan^{-1} \frac{y+a}{k_1} + \frac{1}{\alpha k_1^2} \left[ \left\{ k_1^2 - 2k_1^2 (y+a)^2 \right\} \log \left( \frac{k_1^2 + (y+a)^2}{(y+a)^2} \right) \right]
\]

\[
-4 k_2 x (y+a) \tan^{-1} \frac{y+a}{k_2} + \frac{1}{\alpha k_2^2} \left[ \left\{ k_2^2 - 2k_2^2 (y+a)^2 \right\} \log \left( \frac{k_2^2 + (y+a)^2}{(y+a)^2} \right) \right]
\]

This expression gives \( w, \omega \) at \( y = 0 \) so that the two boundary conditions for a simply supported plate are satisfied.
\begin{align*}
&\text{S.C. Electronic Laboratory, Kharagpur.}
\end{align*}

where \( w_1 \) is the expression on the right-hand side of (2.2.3) along \( y = 0 \) and \( w_2 \) is such that \( z = 0 + \frac{3w_2}{\beta y} \) along \( y = 0 \).

Since \( w_3 = \frac{3}{2} \log y \) and \( w_4 = e^{-\frac{3w_4}{2y}} \)

\begin{align*}
\frac{2w_1}{\beta y} & = \left( \frac{1}{w_1} - \frac{1}{w_2} \right) = R \left( f'(x) \right) \\
\text{also} \quad \frac{k^2 - k_2^2}{\beta} \left( \frac{2w_1}{\beta y} \right) & = \frac{k_2^2}{\beta} \quad \text{along the real axis.}
\end{align*}

\begin{align*}
&2 \left( \frac{1}{w_1} - \frac{1}{w_2} \right) R \left( f'(x) \right) = \frac{a^2}{k^2 - k_2^2} \left[ \begin{align*}
&z + z a \log k^2 + z a \log \left( z^2 + a^2/k^2 \right) \\
&+ 4 a x \tan^{-1} \frac{a}{a x} \end{align*} \right] \\
&- \frac{a}{k_2^2 - k_2^2} \left[ 2a + 2a \log k_2^2 + 2a \log \left( z^2 + a^2/k_2^2 \right) \\
&+ 4 k_2 x \tan^{-1} \frac{a}{a x} \right] \\
&\text{and} \quad 2 \left( \frac{1}{w_1} - \frac{1}{w_2} \right) i f \left( x \right) = \frac{a}{k^2 - k_2^2} \left[ \begin{align*}
&z + z a \log k^2 + i k_2 \left( z^2 + a^2/k_2^2 \right) \\
&- 2 i k_2 \left( z^2 + a^2/k_2^2 \right) \log \left( z + i a/k_2 \right) \end{align*} \right]
\end{align*}

This completes the solution.
In case we consider \(k_1, k_2\) as complex conjugate quantities \((x_1, y_1, x_2, y_2)\) we get
\[
\left(\frac{1}{k_1} - \frac{1}{k_2}\right) f_1(x, y) = 0
\]
\[
-xi k_s (x - i y) \left[ (x+i \frac{a}{k_1})^2 \log(x+i \frac{a}{k_1}) - \frac{1}{4} (x+i \frac{a}{k_1})^4 \right]
\]
\[
-2i k_s (x + i y) \left[ (x+i \frac{a}{k_1})^2 \log(x+i \frac{a}{k_1}) - \frac{1}{4} (x+i \frac{a}{k_1})^4 \right]
\]
\[
+ a \log k_s \left[ 2 a x (x + i y) + (x^2 - \gamma_x y) \right]
\]
\[
+ a \log k_s \left[ 2 a x (x - i y) + (x^2 - \gamma_x y) \right]
\]

where
\[
\omega_s = f_1(x + i \frac{a}{k_1}) + \bar{f}_1(x - i \frac{a}{k_1}) - f_1(x + i \frac{a}{k_2}) - \bar{f}_1(x - i \frac{a}{k_2})
\]
\[
k_s = \alpha + i \beta, \quad k_1 = \alpha - i \beta
\]
and \(x + iy = \frac{a (1 + k_s^2)}{ak_s(k_1^2 - k_2^2)}
\]

\(u = \omega_1 - \omega_2\) gives the required solution.

3.3. Condition for No Moments Along \(y = 0\)

We have
\[
\omega = f_1(x) + \bar{f}_1(x) + \xi(x) + \bar{\xi}(x)
\]
\[
\begin{align*}
\left( \frac{2}{\gamma_x} + \frac{2}{\gamma_y} \right) \gamma_x (x) + \left( \frac{2}{\gamma_x} + \frac{2}{\gamma_y} \right) \gamma_y (x) \\
\text{with complex conjugate}
\end{align*}
\]
so that if we write
\[
\omega = f_1(x) + \bar{f}_1(x) - \frac{\gamma_x}{\gamma_y} \frac{\gamma_x}{\gamma_y} \left( f_1(x) + \bar{f}_1(x) \right)
\]

The condition of no moment along \(y = 0\) is satisfied and the expression for their becomes
\[
a P_2 R \left\{ \left| f''(x) \right| \right\}
\]
where

\[ P = \frac{1}{k_1} \left[ \frac{2k_1^3 - \nu_x}{k_1^3} \right] \frac{1}{k_2} + \frac{\nu_y - k_2^2}{k_2} \left[ \frac{2k_2^2}{k_2^2} \right] \]

3.3a Plate with Free Edges Along \( y = 0 \)

**Loaded at** \( x = C, y = a \).

We write

\[ W = W_1 - W_2, \]

where

\[ \frac{\kappa_1^2 - \kappa_2^2}{2k_1^2} W_1 = \frac{1 + k_1^2}{2k_1^2} \left[ \left\{ \kappa_1 x + i(y-a) \right\}^2 \log \left\{ \kappa_1 x + i(y-a) \right\} + \text{complex conjugate} \right] \]

\[ - \frac{1 + k_2^2}{2k_2^2} \left[ \left\{ \kappa_2 x + i(y-a) \right\}^2 \log \left\{ \kappa_2 x + i(y-a) \right\} + \text{complex conjugate} \right] \]

\[ - \frac{1 + k_1^2}{2k_1^2} \left[ \left\{ \kappa_1 x + i(y+a) \right\}^2 \log \left\{ \kappa_1 x + i(y+a) \right\} \right] \]

and \( W_2 \) is chosen so as to cancel out the shears due to \( W_1 \) along \( y = C \).

The moments along \( y = 0 \) due to \( W_2 \) being zero, we write

\[ W_2 = \Sigma f_1 (x) + \Sigma f_2 (x) - \frac{\gamma_1 - \gamma_2}{\gamma_1 - \gamma_2} \left[ \Sigma f_1 (x) - \Sigma f_2 (x) \right] \]

A simple calculation gives

\[ 2 \frac{P_0 \gamma_1}{k_1 - k_2} \left\{ \gamma_1 \left( \frac{1}{k_1^2} - \frac{2k_1^2}{\gamma_1} + \gamma_2 \right) \left\{ \frac{1}{2} \left( x + \frac{i a}{k_1} \right)^2 \log \left( x + \frac{i a}{k_1} \right) \right\} \right\} \]

\[ \left\{ \frac{1}{4} \left( x + \frac{i a}{k_2} \right)^2 \log \left( x + \frac{i a}{k_2} \right) \right\} \]

This completes the solution.
4. Bending of an equilateral triangular Plate Two of whose edges are simply supported and the third clamped.

We consider the case of an equilateral triangle $\triangle CAB$ whose sides $CA$, $CB$ are simply supported and the side $AB$ is clamped. (See fig. 4.1).

An isolated load of intensity $P$ is assumed to be acting at the centre $H$ of the triangle so that $OP = \frac{1}{\sqrt{3}} (OA = 1)$.

The equation giving the deflection $w$ at any point except $P$, of the triangle is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w = 0 \quad \ldots \quad (4.1)$$

with the boundary conditions

$$w = 0, \quad w = 0 \quad \text{on} \quad \theta = \pm \frac{\pi}{6} \quad \ldots \quad (4.2a)$$

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{on} \quad AB \quad \ldots \quad (4.2b)$$

**SOLUTION**

We write $u = w_0 + w_1$ \ldots \quad (4.3)

where $w_0$ is a solution satisfying \ldots satisfying (4.2) so that

$$\frac{w_0}{\pi} = (x - a_x)(\overline{a} - a_x) \log (x - a_x)(\overline{x} - a_x) - (x + a_x \overline{x})(x + a_x) \log (x + a_x)(\overline{x} + a_x)$$

$$- (x + a_x)(\overline{x} + a_x) \log (x + a_x)(\overline{x} + a_x) + (x - a_x \overline{x})(x - a_x) \log (x - a_x)(\overline{x} - a_x)$$

$$- (x - a_x \overline{x})(\overline{a} - a_x) - (x - a_x)(\overline{x} - a_x) \log (x - a_x)(\overline{x} - a_x) \ldots \quad (4.4)$$

where

$$A = - \frac{P}{4 \pi D}.$$
FIG. 6
We require \( u \) to be nullify \( v_a \) and \( \partial u/\partial x \) on the straight line \( x = \sqrt{3}/2 \), while keeping the moments and deflections on \( \theta = \pm \pi/6 \) arcs. We could proceed by
\[
\frac{w}{A} = \sum_{m=0}^{\infty} \left( a_n \sqrt{n+3} + b_n \sqrt{n+2} \right) \cos(\pi m + 3) \gamma \quad (4.6)
\]
and proceed to determine \( a_n \) and \( b_n \) so as to nullify \( u \) and \( \partial u/\partial x \) at \( x = \sqrt{3}/2 \), \( (\sqrt{n+3} \cos(\pi m + 3) \gamma \) and \( \sqrt{n+2} \cos(\pi m + 2) \gamma \) are the eigenfunctions of the problem). But this would be quite cumbersome.

Instead, we shall write
\[
\frac{w}{A} = (A \sqrt{2} \cos 2 \gamma + B \sqrt{2} \cos 3 \gamma) + (C \sqrt{2} \cos 2 \gamma + D \sqrt{2} \cos 3 \gamma) \quad (4.7)
\]
and demand
\[
\int_0^{\sqrt{3}/2} w \, dy = 0 \quad \text{at} \quad x = \sqrt{3}/2 \quad (4.8a)
\]
\[
\int_0^{\sqrt{3}/2} \frac{\partial w}{\partial x} \, dy = 0 \quad \text{at} \quad x = \sqrt{3}/2 \quad (4.8b)
\]
\[
w \left( \frac{\sqrt{3}}{2}, 0 \right) = \frac{\partial w}{\partial x} \left( \frac{\sqrt{3}}{2}, 0 \right) = 0 \quad (4.8c)
\]
These give the equations
\[
2.26A + 2.837B + 2.612C + 2.610D = 0.02626 \\
0.647A + 0.273B + 0.437C + 0.2053D = 0.2568 \\
0.216A + 0.367B + 0.176C + 0.0707D = 0.1049 \\
A - B + 1.2C - 1.0035D = 0.0787
\]
Solving these equations and substituting in (4.7) we get
\[
\frac{w}{A} = \left( -1.3255 \sqrt{2} \cos 2 \gamma + 0.042 \sqrt{2} \cos 3 \gamma \right) + \left( -0.0596 \sqrt{2} \cos 2 \gamma + 0.0032 \sqrt{2} \cos 3 \gamma \right) \quad (4.9)
\]
In fact \( r \) is throughout the region less than unity and therefore all the first terms in the two series may be kept.