CHAPTER IV

Rejection of Polynomials by Theorem A.

In this chapter we reject 31 polynomials with all real roots by the help of Theorem A stated in Chapter I, because the discriminant of their corresponding field is > 300125.

The prime ideal decomposition of a rational prime p, needed for Theorem A, is got by the method described below as given in Berwick [3], Chapter V-VIII.

§ 1 Method of Prime Ideal Decomposition of p.

Let f be an integer in an algebraic number field k, with field equation

\[ a(z) = z^n + a_1 z^{n-1} + \ldots + a_n = 0, \]

where \( a_1, a_2, \ldots, a_n \) are rational integers.

Then there is a unique representation of \( a(z) \mod p^{d+1}, d \geq 1 \), of the following form:

\[ a(z) \equiv \prod_{i=1}^{n} \left[ w_i(z) + p \eta_i(z) + \ldots + p^d \eta_{d_i}(z) \right] \mod p^{d+1}, \]

where \( w_i(z) = z^{g_i} + c_{11} z^{g_i-1} + \ldots + c_{g_1} \),

\[ f_1 g_1 + f_2 g_2 + \ldots + f_m g_m = n \]

and \( \eta_i(z), \ldots, \eta_{d_i}(z) \) are integral polynomials of degrees \( f_1 g_1 - 1, \ldots, f_m g_m - 1 \). [Berwick, Chapter V]

Moreover, \( \eta_j(z) \) are unique mod p. A factor of \( a(z) \mod p^{d+1} \) due to typical factor \( w(z) \mod p \) can be written as

\[ a_{d+1} = w^f(z) \cdot p \left[ \zeta^{(z)}_{f_{-1,1}} w^{f-1}(z) + \ldots + \zeta_{o_{-1}}(z) \right] \]

\[ + p^2 \zeta^{(z)}_{f_{-1,2}} w^{f-1}(z) + \ldots + \zeta_{o_{-2}}(z) \]

\[ + \ldots + p^d \zeta^{(z)}_{f_{-1,d}} w^{f-1}(z) + \ldots + \zeta_{o_{-d}}(z) \] (mod \( p^{d+1} \)),

\[ \ldots (1) \]
where each $\zeta_{i,j}$ is an integral polynomial of degree $g-1$ or lower and its coefficients lie in the interval $0$ to $p-1$ inclusive.

**Remark.** Some of the polynomials $\zeta_{i,j}(z)$ may vanish, but when $a(z)$ is irreducible, the set $\zeta_{0,1}(z), \zeta_{0,2}(z), \ldots, \zeta_{0,g}(z)$ cannot all vanish. We suppose that

$$\zeta_{0,1}(z) = \zeta_{0,2}(z) = \ldots = \zeta_{0,g}(z) = 0, \zeta_{0,g+1}(z) \neq 0$$

Let $\Theta$ be a root of $a(z) = 0$. It has been shown in Chapter VI of Berwick that every prime ideal factor of $p$ divides exactly one $w_1(\Theta)$.

We write $w(\Theta) = w_1(\Theta), w(z) = w_1(z), g_1 = g, f_1 = f$.

Let $|P_1, P_2, \ldots, P_n|$ be the prime ideal factors of $(p, w(\Theta))$.

Let $p$ be exactly divisible by $|P_1, P_2, \ldots, P_n|$ and $w(\Theta)$ be exactly divisible by $|P_1, P_2, \ldots, P_n|$, where are of degrees $d_1, d_2, \ldots, d_r$ respectively.

Then $g$ is an aliquot part of the degree of every prime ideal factor of $(p, w(\Theta))$, and

$$\frac{d_1}{g} u_1 + \frac{d_2}{g} u_2 + \ldots + \frac{d_r}{g} u_r = f$$

$$\frac{d_1}{g} v_1 + \frac{d_2}{g} v_2 + \ldots + \frac{d_r}{g} v_r = h$$

where $g$ is the degree of $w(z)$ and $f$ is the power of $w(z)$ to which it occurs as a factor in $a(z) \mod p$. (Berwick, Chapter VI).

**Second Dissection** [Berwick, Chapter VII].

Let the positive quadrant of a cartesian plane be cut into unit squares by lines parallel to the axes. The lattice points are called the nodes. The node $(x,y)$
is to be marked, when the expansion (1) contains a term 
$p^y \zeta_{xy}(z) w^x(z)$, with $\zeta_{xy}(z) \neq 0$. It is to be left 
unmarked if the corresponding term is missing from the 
expansion. On the axis of abscissas, the only marked 
node is $(f,0)$, and of the marked nodes on the axis of 
ordinates $(0,h)$ is nearest the origin. A straight edge, 
initially coinciding with the w-axis, is to be pivoted 
at $Q_0$, $(f,0)$ and turned in clockwise direction until it 
first crosses another marked node $Q'$. $Q_0$ is to be joined by 
a straight line to $Q_1$, the marked node on the line $Q_0Q'$ 
which is most distant from $Q_0$. The straight edge is then 
pivoted at $Q_1$ and turned in the same direction as before 
until it again crosses a marked node $Q''$. $Q_1$ is joined by 
a straight line to $Q_2$, the marked node on the line $Q_1Q''$ 
which is furthest from $Q_1$. Repeating the process until 
the node $Q_L,(0,h)$ is reached, a broken line $Q_0Q_1Q_2\cdots Q_L$ 
of $h$ are fewer segments, connects the two nodes $(f,0)$, 
$(0,h)$, and the closed polygon $Q_0Q_1\cdots Q_L$ 0 has no 
marked node inside it. The broken line $Q_0Q_1\cdots Q_L$ 
will be called the boundary, and the separate straight 
segments $E_1 = Q_0Q_1$, $E_2 = Q_1Q_2\cdots$, $E_L = Q_{L-1}Q_L$ 
the edges of the boundary.

In this dissection, it has been shown that an 
edge whose end nodes are $(x',u',y')$, $(x',y'+v')$ must 
attach a set of prime ideals whose contributions to the 
sums on the left of (2) and (3) are $u'$ and $v'$ respectively.

For a typical edge $E_x$, let $u' = j_{u'}$, $v' = j_{v'}$ 
such that $(u',v') = 1$. 
The edge $E_x$, whose end nodes are $(f, \sigma + j_x v_x)$,
$(f + j_x u_x, \sigma)$, attaches one or more prime ideals whose
contribution to the sums (2) and (3) are $j_x u_x$ and $j_x v_x$
respectively. When $j_x = 1$ there is only a single prime
ideal, of degree $g$, attached to the edge. But when $j_x > 1$,
there may be one or several, in which case the second
dissection fails. If, in particular $j_1 = j_2 = \ldots = j_L = 1$
the prime ideal decomposition of $p$ is completely deter-
mined by this dissection. When $j_x > 1$, we apply the
third dissection.

**Third dissection** (Chapter VIII, Berwick)

Putting $j_x = j_x$, $u = u$, $v_x = v$, the terms represent-
ed by nodes on $E_x$ are

$$p^c w^f(z) \{ \zeta_i(z) w^{j_1 u_1}(z) + p^v \zeta_i(z) w^{(j-1)u_1}(z)$$

$$+ \ldots + p^v \zeta_i(z) \} \equiv p^c w^f(z) Z(z),$$

where $\text{g.c.d.} (u, v) = 1$, the extreme coefficients

$\zeta_i(z), \zeta_j(z)$, and generally some others being different
from zero.

**Definition.** We define a new kind of irreducibility
mod $w^{muv+1}$ as follows:

$$w(z) = w^{m u}(z) + p^v \xi_i(z) w^{(m-1)u}(z) + \ldots + p^v \xi_m(z)$$

is irreducible mod $w^{muv+1}$ if it is not congruent to the
product of any two polynomials of the type

$$w(z) = w^{s u}(z) + p^v \xi_i'(z) w^{(s+1)u}(z) + \ldots + p^v \xi_i'(z),$$
\[
\Xi''(z) = w^u(z) + p^v \xi''(z) w^{(t-1)u}(z) + \cdots + p^{tv} \xi''(z),
\]

\[
\text{mod}\{ p^{1/u}, (w(z))^{1/v} \} \equiv (\text{mod} \ w^{\nu+1}),
\]

where \( s+t=m \), and the modulus includes all products of
the type \( p^\beta w^\alpha(z) \) with \( \alpha v + \beta u \geq \nu+1 \), \( \alpha > 0 \), \( \beta > 0 \).

Berwick has shown in Chapter VIII that \( Z(z) \)
is uniquely expressible as a product of irreducible divisors
\( \pmod{w^{\nu+1}} \), i.e., \( Z(z) = \zeta(z) \Xi''(z) \Xi''(z) \cdots \),
with \( \Xi(z) = w^{\nu+1}(z) + p^v \xi''(z) w^{(m-1)u}(z) + \cdots + p^{mv} \xi''(z) \),
and \( mm+M^1+\cdots = j \).

The presence of the divisor \( \Xi''(z) \) is shown
to involve the existence of an ideal \( \mathcal{A} \) of degree \( M \)
which divides \( p \) just \( u \) times and \( w(\mathcal{A}) \) just \( v \) times.
Further, \( mg \) is an aliquot part of the degree of every
prime ideal factor of \( \mathcal{A} \). The prime ideals due to the
factor \( \Xi''(z) \) are entirely distinct from those due to
\( \Xi''(z) \). In particular, when all the indices \( M, M^1, \ldots \) etc.
are 1, the prime ideal factorization of \( p \) is completely
determined. In this case we say that the third dissection
works.

We will be able to find the prime ideal factor-
ization needed, by the second or third dissection.

In this chapter, the second dissection suffices for
all the polynomials except the polynomial number 26,
where the third dissection is needed.
1. \(a(x) = x^6 + 2x^5 - 5x^4 - 10x^3 + 2x^2 + x + 1\), \(D_\mathfrak{p} = 8^2 \cdot 163323 = 8^2 \cdot 290352\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could be 163323 or 290352 = 2^8 \cdot 15147.

We obtain the prime ideal decomposition of \((2)\) by Berwick's method as follows:

\[
a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2}
\]

\[
\equiv (x^3 + x^2 + 1)^2 + 2[(x+1)(x^3 + x^2 + 1) + (x^2 + 1)] \pmod{2^2}
\]

[
This can be easily verified without reference to the method.

The nodes to be marked are \((2,0)\), \((1,1)\) and \((0,1)\).

Since there is no marked node on the edge joining \((2,0)\) and \((0,1)\), the second dissection of Berwick applies here.

We get \((2) = \mathfrak{p}^2\), where \(\mathfrak{p}\) is a prime ideal of degree 3.

By Theorem A, \(2^3(\delta - 1)\big| D\), where \(3 < \delta < 4\).

Therefore the power of 2 occurring in \(D\) is \(\geq 2^6\). So \(D\) cannot be 163323 or 290352. Hence this polynomial has to be rejected.

2. \(a(x) = x^6 + 2x^5 - 5x^4 - 8x^3 + 4x^2 + 4x - 1\), \(D_\mathfrak{p} = 8^2 \cdot 200479\).

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 200479.

We obtain the prime ideal decomposition of \((2)\) as follows:

\[
a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2}
\]

\[
\equiv (x^3 + x^2 + 1)^2 + 2[(x^3 + x^2 + 1) + (x^2 + 1)] \pmod{2^2}
\]

The nodes to be marked are \((2,0)\), \((1,1)\) and \((0,1)\). Since there is no marked node on the edge joining \((2,0)\) and \((0,1)\), the second dissection applies here. We get \((2) = \mathfrak{p}^2\), is
a prime ideal of degree 3.

By Theorem A, $2^3(\bar{e}-1) | D$, where $3 \leq \bar{e} \leq 4$.

Therefore the power of 2 occurring in the discriminant $D$ is $> 2^6$. So 200479 cannot be the discriminant of the field. Hence this polynomial has to be rejected.

3. $a(x) = x^6 + 2x^5 - 5x^4 - 6x^3 + 6x^2 + 4x - 1$, 
$D_p = 4^2 \cdot 257808$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could only be $257808 = 2^5 \cdot 3157$.

We obtain the prime ideal decomposition of (2) as follows:

$a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2}$

$\equiv (x^3 + x^2 + 1)^2 + 2[(x+1)(x^3 + x^2 + 1) + (x^2 + x)] \pmod{2^2}$

The nodes to be marked are $(2,0)$, $(1,1)$ and $(0,1)$. Since there is no marked node on the edge joining $(2,0)$ and $(0,1)$, the second dissection applies and we get

$(2) = p^2$, where $p$ is a prime ideal of degree 3.

By Theorem A, $2^3(\bar{e}-1) | D$, where $3 \leq \bar{e} \leq 4$.

Therefore the power of 2 occurring in $D > 2^6$. So 257808 cannot be the discriminant of the field. Hence this polynomial has to be rejected.

4. $a(x) = x^6 - 7x^4 + 14x^2 - 7$, 
$D_p = 4^2 \cdot 538912$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could be only $538912 = 2^4 \cdot 134707$

We obtain the prime ideal decomposition of (2) as follows:

$a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2}$

$\equiv (x^3 + x^2 + 1)^2 + 2[(x^2 + x)(x^3 + x^2 + 1) + (x^2 + x)] \pmod{2^2}$

The nodes to be marked are $(2,0)$, $(1,1)$ and $(0,1)$.
Since there is no marked node on the edge joining $(2,0)$ and $(0,1)$, the second dissection applies and we get

$$(2) = \mathfrak{p}^2,$$

where $\mathfrak{p}$ is a prime ideal of degree 3.

By Theorem A, $2^3(6-1) \mid D$, where $3 \leq \ell \leq 4$.

Therefore the power of 2 occurring in $D$ is $2^6$. So $268912$ cannot be the discriminant of the field. Hence this polynomial has to be rejected.

5. $a(x) = x^6 - 6x^4 + 7x^2 - 1$, $D_\mathfrak{p} = 4^2 \cdot 254196$

If the discriminant of the corresponding field is $< 300125$, it could only be $254196 = 2^3 \cdot 66049$.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x^3 + x + 1)^2 \pmod{2}$$


$$\equiv (x^3 + x + 1)^2 + 2[(x^3 + x + 1)^3] \pmod{2^2}$$

The nodes to be marked are $(2,0)$, $(1,1)$ and $(0,1)$. Since there is no marked node on the edge joining $(2,0)$ and $(0,1)$, the second dissection applies and we get

$$(2) = \mathfrak{p}^3,$$

where $\mathfrak{p}$ is a prime ideal of degree 3.

By Theorem A, $2^3(6-1) \mid D$, where $3 \leq \ell \leq 4$.

Therefore the power of 2 occurring in $D$ is $2^6$. So $254196$ cannot be the discriminant of the field. Hence this polynomial has to be rejected.

6. $a(x) = x^6 - 6x^4 + 8x^2 - 1$, $D_\mathfrak{p} = 4^2 \cdot 209764$

If the discriminant $D$ of the corresponding field is $< 300125$, it could only be $209764 = 2^2 \cdot 52441$.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x+1)^2 (x^2 + x + 1)^2 \pmod{2}$$

$$\equiv [(x+1)^2 + 2[(x+1)]]((x^2 + x + 1)^2 + 2[x(x^2 + x + 1) + 1]) \pmod{2^2}.$$
For both the factors, the nodes to be marked are \((2,0), (1,1)\) and \((0,1)\). In each case there is no marked node on the edge joining \((2,0)\) and \((0,1)\). So the second dissection applies here and we get

\[(2) = \mathfrak{p}_1^2 \mathfrak{p}_2^2, \text{ where } \mathfrak{p}_1 \text{ and } \mathfrak{p}_2 \text{ are prime ideals of degrees 1 and 2 respectively.}
\]

By Theorem A, \(\prod_{i=1}^{2} f_i(e_i-1) \mid D\), where \(f_1=1, f_2=2, \epsilon_1=3, \epsilon_2^i = 1, 2, \ldots, 6\).

Therefore the power of 2 occurring in \(D\) is \(\geq 2^6\). So 209764 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

7. \(a(x) = x^6 + 2x^5 - 5x^4 - 8x^3 + 5x^2 + 2x + 1, \quad \Delta_f = 8^2 \cdot 254489\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 254489.

We obtain the prime ideal decomposition of (2) as follows:

\[a(x) \equiv (x+1)^6 \quad (\mod 2)\]

\[\equiv (x+1)^6 + 2[(x+1)^2 + (x+1) + 1] \quad (\mod 2^2)\]

The nodes to be marked are \((6,0), (2,1), (1,1)\) and \((0,1)\).

Since there is no marked node on the edge joining \((6,0)\) and \((0,1)\), the second dissection applies here and we get

\[(2) = \mathfrak{p}_1^6, \text{ where } \mathfrak{p}_1 \text{ is a prime ideal of degree 1.}
\]

By Theorem A, \(2^{(\tilde{e} - 1)} \mid D\), where \(7 \leq \tilde{e} \leq 12\).

Therefore the power of 2 occurring in \(D\) is \(\geq 2^6\). So 254489 cannot be the discriminant of the field. Hence the polynomial has to be rejected.
8. \( a(x) = x^6 - 7x^4 + 4x^3 + 6x^2 - 2x - 1 \), \( D_f = 4^2 \cdot 231668 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be \( 231668 \). We obtain the prime ideal decomposition of (2) as follows:

\[
a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2} \\
= (x^3 + x^2 + 1)^2 + 2[(x^2 + x)(x^3 + x^2 + 1) + (x^2 + 1)] \pmod{2^2}
\]

The nodes to be marked are \((2, 0), (1, 1)\) and \((0, 1)\). Since there is no marked node on the edge joining \((2, 0)\) and \((0, 1)\), the second dissection applies and we get

\[
(2) = \mathfrak{p}^2, \text{ where } \mathfrak{p} \text{ is a prime ideal of degree 3.}
\]

By Theorem A, \( 2^3 (\bar{e} - 1) \mid D \), where \( 3 \leq \bar{e} \leq 4 \). Therefore the power of 2 occurring in \( D \) is \( > 2^6 \). So 231668 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

9. \( a(x) = x^6 - 7x^4 + 12x^2 - 1 \), \( D_f = 4^2 \cdot 234196 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be \( 234196 = 2^2 \cdot 66049 \).

We obtain the prime ideal decomposition of (2) as follows:

\[
a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2} \\
= (x^3 + x^2 + 1)^2 + 2[(x^2 + x)(x^3 + x^2 + 1) + (x + 1)] \pmod{2^2}
\]

The nodes to be marked are \((2, 0), (1, 1)\) and \((0, 1)\). Since there is no marked node on the edge joining \((2, 0)\) and \((0, 1)\), the second dissection applies and we get

\[
(2) = \mathfrak{p}^2, \text{ where } \mathfrak{p} \text{ is a prime ideal of degree 3.}
\]
By Theorem A, $2^3(\bar{e}-1) \mid D$, where $3 \leq \bar{e} \leq 4$.

Therefore the power of 2 occurring in $D$ is $> 2^6$. So 254196 cannot be the discriminant of the field. Hence this polynomial has to be rejected.

10. $a(x) = x^6 - 2x + 2x^2 - 2x - 1$, $D_f = 8^2 \cdot 155029$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could only be 155029.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x+1)^6 \pmod{2}$$

$$\equiv (x+1)^6 + 2[(x+1)^5 + (x+1) + 1] \pmod{2^2}$$

The nodes to be marked are (6,0), (5,1), (1,1) and (0,1).

Since there is no marked node on the edge joining (6,0) and (0,1), the second dissection applies and we get

$$(2) = \frac{p^6}{\mathfrak{p}}, \text{where } \mathfrak{p} \text{ is a prime ideal of degree 1.}$$

By Theorem A, $2^{(\bar{e}-1)} \mid D$, where $7 < \bar{e} < 12$.

Therefore the power of 2 occurring in $D$ is $> 2^6$.

So 155029 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

11. $a(x) = x^6 - 7x^4 + 11x^2 - 2x - 1$, $D_f = 8^2 \cdot 155029$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could only be 155029.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x+1)^6 \pmod{2}$$

$$\equiv (x+1)^6 + 2[(x+1)^5 + (x+1) + 1] \pmod{2^2}$$

The nodes to be marked are (6,0), (5,1), (1,1) and (0,1).

Since there is no marked node on the edge joining (6,0) and
(0,1), the second dissection applies and we get
\[
(2) = \mathfrak{p}^6, \text{ where } \mathfrak{p} \text{ is a prime ideal of degree } 1.
\]
By Theorem A, \(2^6(\bar{e}-1) \mid D\), where \(7 \leq \bar{e} \leq 12\).
Therefore the power of 2 occurring in \(D\) is \(\geq 2^6\). So 155029 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

12. \(a(x) = x^6 - 7x^4 + 10x^2 + 2x - 1\), \(D_f = 10^2 \cdot 224912\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 224912 = \(4^2 \cdot 14057\).
We obtain the prime ideal decomposition of \(2\) as follows:
\[
a(x) \equiv (x^3 + x^2 + 1)^2 \quad \pmod{2}
\]
\[
\equiv (x^3 + x^2 + 1)^2 + 2[(x^2 + x)(x^3 + x^2 + 1) + (x^2 + 1)] \quad \pmod{2^2}
\]
The nodes to be marked are \((2,0), (1,1)\) and \((0,1)\). Since there is no marked node on the edge joining \((2,0)\) and \((0,1)\), the second dissection applies and we get
\[
(2) = \mathfrak{p}^6, \text{ where } \mathfrak{p} \text{ is a prime ideal of degree } 3.
\]
By Theorem A, \(2^3(\bar{e}-1) \mid D\), where \(3 \leq \bar{e} \leq 4\).
Therefore the power of 2 occurring in \(D\) is \(\geq 2^6\).
So 224912 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

13. \(a(x) = x^6 - 7x^4 + 10x^2 - 2x - 1\), \(D_f = 10^2 \cdot 224912\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 224912 = \(4^2 \cdot 14057\).
We obtain the prime ideal decomposition of \(2\) as follows:
\[
a(x) \equiv (x^3 + x^2 + 1)^2 \quad \pmod{2}
\]
\[
\equiv (x^3 + x^2 + 1)^2 + 2[(x^2 + x)(x^3 + x^2 + 1) + (x^2 + 1)] \quad \pmod{2^2}
\]
The nodes to be marked are \((2,0), (1,1)\) and \((0,1)\).
Since there is no marked node on the edge joining (2,0) and (0,1), the second dissection applies and we get
\[(2) = \mathfrak{p}^2,\] where \(\mathfrak{p}\) is a prime ideal of degree 3.
By Theorem A, \(2^3(\bar{e}-1) \mid D\), where \(3 < \bar{e} < 4\).
Therefore the power of 2 occurring in \(D\) is \(> 2^5\). So 224912 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

14. \(a(x) = x^6 - 7x^4 + 11x^2 - 4\), \(D_p = 2^2 \cdot 209764\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be \(209764 = 2^2 \cdot 52441\).
We obtain the prime ideal decomposition of \(2\) as follows:

\[a(x) \equiv x^2(x^2+x+1)^2 \pmod{2}\]
\[\equiv \{x^2+2.0+2.1\\} \{x^2+x+1\}^2 \pmod{2}\]
\[\equiv 2^2[(x^2+x+1)(x+1)+x] \pmod{2^3}\]

For the first factor, the nodes to be marked are (2,0) and (0,2). Since there is no marked node on the edge joining (2,0) and (0,2), the second dissection applies and we get \(\mathfrak{p}_1^2\) as the prime ideal factor of \(2\) corresponding to this edge, where \(\mathfrak{p}_1\) is a prime ideal of degree 1.
For the second factor, the nodes to be marked are (2,0), (1,1) and (0,1). Since there is no marked node on the edge joining (2,0) and (0,1), the second dissection applies and we get
\[(2) = \mathfrak{p}_1^2 \mathfrak{p}_2^2,\] where \(\mathfrak{p}_1\) and \(\mathfrak{p}_2\) are prime ideals of degrees 1 and 2 respectively.
By Theorem A, \(\prod_{i=1}^{2} f_1(\bar{e}_i-1) \mid D\), where \(f_1=1, f_2=2,\)
\[3 < \bar{e}_i < 4, \ i = 1,2.\]
Therefore the power of 2 occurring in \(D\) is \(> 2^5\).
So 209764 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.
15. \( a(x) = x^6 - 7x^4 + 10x^2 - 2 \), \( D_F = 16^2.161312 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be \( 161312 = 2^5.5041 \).

We obtain the prime ideal decomposition of \((2)\) as follows:

\[
a(x) \equiv x^4(x+1)^2 \quad \pmod 2
\]

\[
\equiv \{x^4+2.1\}\{(x+1)^2+2(x+1)+2.1\} \quad \pmod 2^\circ
\]

For the first factor, the nodes to be marked are \((4,0)\) and \((0,1)\),
For the second factor, the nodes to be marked are \((2,0)\), \((1,1)\), \((0,1)\).
In each case, there is no marked node on the edge joining \((4,0)\)
and \((0,1)\); \((2,0)\) and \((0,1)\). So the second dissection applies
in each case and we get

\[
(2) = \| \mathbb{P}^2 \| \mathbb{P}^2, \text{ where } \| \mathbb{P}, \| \mathbb{P} \text{ are prime ideals of degree 1 each. By Theorem A, } \prod_{i=1}^{2} f_i(\mathbb{P}^1 \mathbb{P}^1), D, \text{ where } f_1=1, \ f_2=1,
5 \leq \mathbb{P} \leq 12, 3 \leq \mathbb{P} \leq 4.
\]

Therefore, the power of 2 occurring in \( D \) is \( > 2^5 \). So \( 161312 \)
cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

16. \( a(x) = x^6 - 7x^4 + 6x^2 - 1 \), \( D_F = 4^2.264195 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be \( 264195 = 2^2.66049 \).

We obtain the prime ideal decomposition of \((2)\) as follows:

\[
a(x) \equiv (x^3+x^2+1)^2 \quad \pmod 2
\]

\[
\equiv (x^3+x^2+1)^2+2[(x^2+x)(x^3+x^2+1)+(x^2+x+1)] \quad \pmod 2^\circ
\]

The nodes to be marked are \((2,0)\), \((1,1)\) and \((0,1)\). Since
there is no marked node on the edge joining \((2,0)\) and \((0,1)\),
the second dissection applies and we get

\[
(2) = \| \mathbb{P}^2 \, \| \mathbb{P} \text{ is a prime ideal of degree 1.}
\]
By Theorem A, $2^{3(\bar{e}-1)} \mid D$, where $3 \leq \bar{e} \leq 4$.
Therefore, the power of 2 occurring in $D$ is $\geq 2^3$.
So 264196 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

17. $a(x) = x^6 + 7x^4 + 9x^2 + 1$, $D_f = 16^2 \cdot 155236$

If the discriminant $D$ of the corresponding field is
$\leq 300125$, it could only be 155236 = 438209.
We obtain the prime ideal decomposition of (2) as follows:
\[ a(x) \equiv (x+1)^6 \pmod{2} \]
\[ \equiv (x+1)^6 + 2[(x+1)^5 + (x+1)^2 + 1] \pmod{2^2} \]
The nodes to be marked are (6,0), (5,1), (2,1) and (0,1).
Since there is no marked node on the edge joining (6,0) and
(0,1), the second dissection applies and we get
\[ (2) = \mathfrak{p}^6 \], where $\mathfrak{p}$ is a prime ideal of degree 1.

By Theorem A, $2^{(\bar{e}-1)} \mid D$, where $7 \leq \bar{e} \leq 12$.
Therefore, the power of 2 occurring in $D$ is $\geq 2^7$.
So 155236 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

18. $a(x) = x^6 - 7x^4 + 9x^2 + 11x + 1$, $D_f = 5^2 \cdot 282820$

If the discriminant $D$ of the corresponding field
is $\leq 300125$, it could only be 282820 = 55554.
We obtain the prime ideal decomposition of (5) as follows:
\[ a(x) \equiv (x^3 - x + 2)^2 \pmod{5} \]
\[ \equiv (x^3 - x + 2)^2 + 5[(x^3 - x + 2)(4x + 1) + (x^2 + 3x + 1)] \pmod{5^2} \]
The nodes to be marked are (2,0), (1,1) and (0,1).
Since there is no marked node on the edge joining (2,0) and
(0,1), the second dissection applies and we get
(5) = \( \mathbf{p}^2 \), where \( \mathbf{p} \) is a prime ideal of degree 3.

Therefore, the power of 5 occurring in \( D \) is \( 5^3 \).

So 282820 cannot be the discriminant of the field.

Hence the polynomial is rejected.

\[ 19. \ a(x) = x^6 - 7x^4 + 2x^3 + 10x^2 - 2x - 3 \quad , \quad D_\mathbf{p} = 2^2 \cdot 171853 \]

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be 171853.

We obtain the prime ideal decomposition of \( (2) \) as follows:

\[ a(x) = (x^3 + x^2 + 1)^2 \quad (\text{mod } 2) \]

\[ = (x^3 + x^2 + 1)^2 + 2[(x^3 + x^2 + 1)(x^3 + x^2 + 1) + 1] \quad (\text{mod } 2^2) \]

The nodes to be marked are \((2,0), (1,1)\) and \((0,1)\). Since there is no marked node on the edge joining \((2,0)\) and \((0,1)\), the second dissection applies and we get

\[ (2) = \mathbf{p}^2 \], where \( \mathbf{p} \) is a prime ideal of degree 3.

By Theorem A, \( 2^3(\mathbf{e} - 1) \mid D \), where \( 3 \leq \mathbf{e} \leq 4 \).

Therefore, the power of 2 occurring in \( D \) is \( \geq 2^6 \).

So 171853 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

\[ 20. \ a(x) = x^6 - 7x^4 + 2x^3 + 10x^2 - 2x - 2 \quad , \quad D_\mathbf{p} = 2^2 \cdot 189851 \]

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be 189851.

We obtain the prime ideal decomposition of \( (2) \) as follows:

\[ a(x) = x^4(x+1)^2 \quad (\text{mod } 2) \]

\[ = [x^4 + 2(x+1)](x+1)^2 + 2(x+1) + 1] \quad (\text{mod } 2^2) \]

For the first factor, the nodes to be marked are \((4,0), (1,1)\) and \((0,1)\). For the second factor, the nodes to be marked are \((2,0), (1,1)\) and \((0,1)\). Since there is no marked node on the edge joining \((4,0)\) and \((0,1)\); and on the edge joining
(2,0) and (0,1), the second dissection applies and we get

\[
(2) = \left| p_1^4 \right| p_2^2 , \text{ where } |p_1| \text{ and } |p_2| \text{ are prime ideals of degree 1 each. By Theorem A, } 2 \prod_{i=1}^{2} f_i(\bar{e}_i-1) \mid D ,
\]

\[
f_1 f_2 = 1 , 5 \leq \bar{e}_1 \leq 12 , 3 \leq \bar{e}_2 \leq 4 .
\]

Therefore, the power of 2 occurring in D is \( \geq 2^6 \).

So 1893851 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

21. \( a(x) = x^6 + x^5 - 7x^4 - 8x^3 + 13x^2 + 5x + 7 \), \( D_p = 8^2 \cdot 224669 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be 224669.

We obtain the prime ideal decomposition of (2) as follows:

\[
a(x) \equiv x^4 (x^2 + x + 1) \pmod{2}
\]

\[
= [x^4 + 2(x^3 + x^2 + 1)](x^2 + x + 1) + 2x \pmod{2^2}
\]

For the first factor, the nodes to be marked are (4,0), (3,1), (2,1), (0,1). For the second factor, the nodes to be marked are (1,0) and (0,1). Since there are no marked nodes on the edges joining (4,0) and (0,1); (1,0) and (0,1), the second dissection applies and we get

\[
(2) = \left| p_1^4 \right| p_2^2 , \text{ where } |p_1| \text{ is a prime ideal of degree 1 and } |p_2| \text{ is a prime ideal of degree 2.}
\]

By Theorem A, \( 2(\bar{e}-1) \mid D \), where \( 5 \leq \bar{e} \leq 12 \).

Therefore, the power of 2 occurring in D is \( \geq 2^4 \).

So 224669 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

22. \( a(x) = x^6 + x^5 - 7x^4 - 5x^3 + 13x^2 + 5x - 7 \), \( D_p = 5^2 \cdot 243377 \)

If the discriminant \( D \) of the corresponding field is \( \leq 300125 \), it could only be 243377.

We obtain the prime ideal decomposition of (5) as follows:
\[ a(x) = (x+2)^3(x^2+x+1) \quad (\text{mod } 5) \]
\[ = [(x+2)^3+5(4(x+2)^2+4(x+2)+4)](x^2+x+1)+5(x+3) \quad (\text{mod } 5^2) \]
For the first factor, the nodes to be marked are \((0,0),(2,1),(1,1)\) and \((0,1)\). For the second factor, the nodes to be marked are \((1,0)\) and \((0,1)\). Since there are no marked nodes on the edges joining \((3,0)\) and \((0,1)\); \((1,0)\) and \((0,1)\), the second dissection applies and we get

\[ (5) = \|p_1^3 \|p_2^2 \|p_3^1 \text{, where } \|p_1 \text{ and } \|p_2 \text{ are prime ideals of degrees 1 and 3 respectively.} \]
By Theorem A, \(5^{(5-1)} \mid D\), where \(\bar{e} = 3\). Therefore, \(5^2\) is the exact power of 5 occurring in \(D\).
So 243377 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

23. \[ a(x) = x^6+x^5+6x^4+5x^3+13x^2+5x-4 \quad \text{, } D_f = 8^2 \cdot 277331 \]
If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 277331.
We obtain the prime ideal decomposition of \((2)\) as follows:
\[ a(x) = x(x+1) (x^2+x+1)^2 \quad (\text{mod } 2) \]
\[ = [(x+2,0+2^2,1)](x+1,0+2^1,1)(x^2+x+1)^2+2,1 \]
\[ +2^2[x(x^2+x+1)+1] \quad (\text{mod } 2^3) \]
Corresponding to each factor, the nodes to be marked are \((1,0)\) and \((0,2)\); \((1,0)\) and \((0,1)\); \((2,0),(0,1)\) respectively.
In each case, there are no marked nodes on the edges joining \((1,0)\) and \((0,2)\); \((1,0)\) and \((0,1)\); \((2,0)\) and \((0,1)\), so the second dissection applies and we get

\[ (2) = \|p_1 \|p_2^2 \|p_3^1 \text{, where } \|p_1 \text{, } \|p_2 \text{, } \|p_3 \text{ are prime ideals of degrees 1, 1 and 2 respectively.} \]
By Theorem A, \(2^{2(5-1)} \mid D\), where \(3 \leq \bar{e} \leq 4\).
Therefore, the power of 2 occurring in $D$ is $\geq 2^4$.
So 277331 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

24. $a(x) = x^5 + x^5 - 7x^4 - 5x^3 + 12x^2 + 3x - 3$, $D_f = 2^2 \cdot 257551$

If the discriminant $D$ of the corresponding field is $< 300125$, it could only be 257551.

We obtain the prime ideal decomposition of $(2)$ as follows:

$a(x) \equiv x^3(3x^2 + x^2 + x + 1) (\text{mod } 3)$

$\equiv \{x^3 + 3(2x^2 + 2)\} + 3(x^2 + 2x + 1)$ (mod $3^2$)

For each factor, the nodes to be marked are $(3,0)$, $(2,1)$ and $(0,1)$, $(1,0)$ and $(0,1)$. In each case, there are no marked nodes on the edges joining $(3,0)$ and $(0,1)$; $(1,0)$ and $(0,1)$, so the second dissection applies and we get

$$(2) = \mathbb{P}_1 \cdot \mathbb{P}_2 \cdot \mathbb{P}_3$$

where $\mathbb{P}_1$, $\mathbb{P}_2$ are prime ideals of degrees 1 and 3 respectively.

By Theorem A, $3^{(e-1)} | D$, where $4 \leq e \leq 6$.

Therefore, the power of 3 occurring in $D$ is $\geq 3^3$.

So 257551 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

25. $a(x) = x^6 + x^5 - 7x^4 - 5x^3 + 12x^2 + 4x - 2$, $D_f = 18^2 \cdot 258619$

If the discriminant $D$ of the corresponding field is $< 300125$, it could only be 258619.

We obtain the prime ideal decomposition of $(2)$ as follows:

$a(x) \equiv x^2(x^4 + x^3 + x^2 + x + 1) (\text{mod } 2)$

$\equiv \{x^2 + 2[x+1]\} + 2(x^3 + 2x^2 + x + 1)$ (mod $2^2$)

For each factor, the nodes to be marked are $(2,0)$, $(1,1)$ and
(0,1); (1,0) and (0,1) respectively. Since there are no marked nodes on the edges joining (2,0) and (0,1); (1,0) and (0,1), so the second dissection applies and we get 
\[ (2) = \mathcal{P}_1^2 \mathcal{P}_2, \]
where \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are prime ideals of degrees 1 and 4 respectively.

By Theorem A, \( 2^{(5-1)} \mid D \), where \( 3 \leq e \leq 4 \).

Therefore, the power of 2 occurring in \( D \) is \( > 2^2 \).
So 259619 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

25. \[ a(x) = x^6 + x^5 - 7x^4 - 4x^3 + 11x^2 + 3x - 1, \quad D_\rho = 16^2 \cdot 219500 \]

If the discriminant \( D \) of the corresponding field is \( < 300125 \), it could only be \( 219500 = 2^2 \cdot 54375 \).

We obtain the prime ideal decomposition of \( (2) \) as follows:
\[ a(x) \equiv (x+1)^4 (x^2+x+1) \pmod{2} \]
\[ \equiv [(x+1)^4 + 2(x+1)^2 + 2^2(x+1)^3 + (x+1)^2 + 1] + 2^3(x+1)^3 + (x+1)^2 + 2^4(x+1)^3 + 2^4 \cdot 1 \pmod{2^5} \]

For the first factor, the nodes to be marked are (4,0), (2,1), (3,2), (2,2) and (0,2). Since there is a marked node, namely (2,1), on the edge joining (4,0) and (0,2), so the second dissection fails here and we apply the third dissection.

The points joining this edge are (0+2,2,0) and (0,0+2,1).
Comparing them with the end points \((\sigma+ju, \sigma), (\gamma, \sigma+jv)\) we get \( \sigma = \sigma = 0, j = 2, u = 2, v = 1 \). Here the degree \( g \) of \( x+1 \) is 1.

The terms represented by the nodes on this edge are \( (x+1)^4 + 2(x+1)^2 + 2^2 \cdot 1 \).

We show that this is irreducible \( \text{mod}(\sqrt{2}, x+1)^5 \), i.e. it cannot be written as a product of two factors of the type \[ [(x+1)^2 + \alpha] [(x+1)^2 + \beta] \pmod{\sqrt{2}, x+1} \], where \( \alpha \) and \( \beta \) are rational integers.
To prove this, let
\[(x+1)^4 + 2(x+1)^2 + 2 \equiv [(x+1)^2 + 2\alpha][(x+1)^2 + 2\beta) \mod (\sqrt{2}, x+1)^5].\]

i.e. their difference
\[2(1-\alpha\beta)(x+1)^2 + 2^2(1-\alpha\beta) = \sum \gamma \cdot 2^a \cdot (1+x)^b_i,
\]
where \(b_1 + 2a_1 > 5, a_1 > 0, b_1 > 0\).

Clearly \(b_1 \leq 2, a_1 \leq 2\). Since \(b_1=1, a_1 = 2\) is not possible, so the only possible values of \(a_1\) and \(b_1\) are \(a_1 = 2, b_1 = 2\).

So \(2(1-\alpha\beta)(x+1)^2 + 2^2(1-\alpha\beta) = \gamma \cdot 2^2(x+1)^2\)

Comparing coefficients we get \(\alpha\beta = 1\) and \(1-\alpha\beta = 2\gamma\).

Since \(\alpha\beta = 1\) so \(\alpha\beta = \text{even and } 1-\alpha\beta \text{ being odd cannot be } 2\gamma\).

Hence \((x+1)^4 + 2(x+1)^2 + 2\) is irreducible \(\mod (\sqrt{2}, x+1)^5\).

So Berwick's third dissection applies and we get \(\mathcal{P}_1^2\) as the prime ideal factor of \(\mathfrak{p}\) corresponding to this edge, where \(\mathcal{P}_1\) is of degree \(M=2, g=1\).

For the second factor of \(\alpha(x) \mod 2^5\), since there is no marked node on the edge joining \((1,0)\) and \((0,4)\), the second dissection applies and we get \(\mathcal{P}_2\) as the prime ideal factor of \(\mathfrak{p}\) corresponding to this edge, where \(\mathcal{P}_2\) is a prime ideal of degree 2.

Thus \((2) = \mathcal{P}_1^2 \mathcal{P}_2\), where \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are prime ideals of degree 2 each.

By Theorem A, \(2^2(\bar{\alpha}-1) \mid D\), where \(3 \leq \bar{\alpha} \leq 4\).

Therefore, the power of 2 occurring in \(D\) is \(\geq 2^4\).

So 219500 cannot be the discriminant of the field.

Hence the polynomial has to be rejected.

27. \(a(x) = 6x^6 + 7x^4 - 3x^2 + 13x - 2\), \(D = 8^2 \cdot 238727\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 238727.
We obtain the prime ideal decomposition of \((2)\) as follows:
\[
a(x) \equiv x(x+1)(x^2+x+1)^2 \pmod{2}
\]
\[
\equiv [x+2,1+2^2,1]\{(x+1)+2^3,1\}\{(x^2+x+1)^2+2[(x^2+x+1)+(x+1)]\} \pmod{2^4}
\]

For each factor, the nodes to be marked are \((1,0)\) and \((0,1)\); \((1,0)\) and \((0,3)\); \((2,0),(1,1)\) and \((0,1)\) respectively. Since there are no marked nodes on the edges joining \((1,0)\) and \((0,1)\); \((1,0)\) and \((0,3)\); \((2,0)\) and \((0,1)\), so the second dissection applies and we get
\[
(2) = \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3^2, \text{ where } \mathcal{P}_1, \mathcal{P}_2 \text{ and } \mathcal{P}_3 \text{ are prime ideals of degrees } 1, 1 \text{ and } 2 \text{ respectively.}
\]

By Theorem A, \(2^2(e-1) \mid D\), where \(3 \leq e \leq 4\). Therefore, the power of 2 occurring in \(D\) is \(\geq 2^4\). So \(238727\) cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

28. \(a(x) = x^6+3x^5-3x^4+3x^3+5x^2+4x-2\), \(D = 2^2 \cdot 263175\)

If the discriminant \(D\) of the corresponding field is \(\leq 300125\), it could only be 263175.

We obtain the prime ideal decomposition of \((2)\) as follows:
\[
a(x) \equiv x^2(x^4+x^3+x^2+x+1) \pmod{2}
\]
\[
\equiv [x^2+2(x+1)][(x^4+x^3+x^2+x+1)+2x] \pmod{2^2}
\]

For each factor, the nodes to be marked are \((2,0)\), \((1,1)\) and \((0,1)\); \((2,0)\) and \((0,1)\) respectively. Since there are no marked nodes on the edges joining \((2,0)\) and \((0,1)\); \((1,0)\) and \((0,1)\) respectively, so the second dissection applies and we get
\[
(2) = \mathcal{P}_1^2 \mathcal{P}_2, \text{ where } \mathcal{P}_1 \text{ and } \mathcal{P}_2 \text{ are prime ideals of degrees } 1 \text{ and } 4 \text{ respectively.}
\]
By Theorem A, $2^{(6-1)} \mid D$, where $3 < \delta < 4$.

Therefore, the power of 2 occurring in $D$ is $\geq 2^2$.

So 263175 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

29. $a(x) = x^6 + 2x^5 - 6x^4 - 10x^3 + 4x^2 + 6x + 1$, $D_p = 2^2 \cdot 228896$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could only be $228896 = 2^5 \cdot 7153$.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x^3 + x^2 + 1)^2 \pmod{2}$$

$$\equiv (x^3 + x^2 + 1)^2 + 2[(x + 1)(x^3 + x^2 + 1) + x] \pmod{2^2}$$

The nodes to be marked are $(2,0)$, $(1,1)$ and $(0,1)$. Since there is no marked node on the edge joining $(2,0)$ and $(0,1)$, the second dissection applies and we get

$$(2) = \mathfrak{p}^2$$

where $\mathfrak{p}$ is a prime ideal of degree 3.

By Theorem A, $2^{3(\delta-1)} \mid D$, where $3 < \delta < 4$.

Therefore, the power of 2 occurring in $D$ is $\geq 2^6$.

So 228896 cannot be the discriminant of the field.

Hence this polynomial has to be rejected.

30. $a(x) = x^6 + x^5 - 6x^4 - 7x^3 + 4x^2 + 5x + 1$, $D_p = 2^2 \cdot 130500$

If the discriminant $D$ of the corresponding field is $\leq 300125$, it could only be $130500 = 2^2 \cdot 45125$.

We obtain the prime ideal decomposition of $(2)$ as follows:

$$a(x) \equiv (x^2 + x + 1)^3 \pmod{2}$$

$$\equiv (x^2 + x + 1)^3 + 2[x(x^2 + x + 1)^2 + (x^2 + x + 1) + (x + 1)] \pmod{2^2}$$

The nodes to be marked are $(3,0)$, $(2,1)$, $(1,1)$, and $(0,1)$. Since there is no marked node on the edge joining $(3,0)$ and $(0,1)$, the second dissection applies and we get

$$(2) = \mathfrak{p}^3$$

where $\mathfrak{p}$ is a prime ideal of degree 2.
By Theorem A, 2^4 is the exact power of 2 occurring in D.
So 180500 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

31. \( a(x) = x^6 + x^5 - 7x^4 + 6x^3 + 9x^2 + 7x + 1 \), \( D = 2^2 \cdot 285508 \)

If the discriminant D of the corresponding field is \( \leq 300125 \), it could only be 285508 = 2^2 \cdot 71377.
We obtain the prime ideal decomposition of (2) as follows:
\[
a(x) \equiv (x+1)^4(x^2-x+1) \pmod{2}
\equiv \{(x+1)^4+2 \cdot 1\}(x^2-x+1)+2 \cdot 1 \pmod{2^2}
\]
For each factor, the nodes to be marked are (4,0) and (0,1); (1,0) and (0,1) respectively. Since there are no marked nodes on the edges joining (4,0) and (0,1); (1,0) and (0,1) respectively, the second dissection applies and we get
\[
(2) = \mathfrak{p}_1^4 \mathfrak{p}_2^2, \text{ where } \mathfrak{p}_1 \text{ and } \mathfrak{p}_2 \text{ are prime ideals of degrees 1 and 2 respectively.}
\]
By Theorem A, \( 2^{(e-1)} \mid D \), where \( 5 \leq e \leq 12 \).
Therefore, the power of 2 occurring in D is \( > 2^4 \).
So 285508 cannot be the discriminant of the field.
Hence this polynomial has to be rejected.

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