CHAPTER X
ON MAHLER'S CONVEX SETS

[Published in the Research Bulletin (M.S.) of the Panjab University, Vol.14, Parts I-II, pp.87-91, June, 1963]

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§1. Let \( F \) be a field complete under a non-trivial non-Archimedean valuation \( |x|, \ x \in F \). Let \( P_n \) be the n-dimensional space over \( F \). Let \( X = (x_1, \ldots, x_n), x_i \in F \) denote the general element of \( P_n \).

Mahler (1941) defined a convex set in \( P_n \) as follows:

Let \( f(X) \) be a real valued function defined on \( P_n \) such that

(a) \( f(X) \geq 0 \) for all \( X \in P_n \).
(b) \( f(aX) = |a| f(X) \), for all \( a \in F, X \in P_n \).
(c) \( f(X + Y) \leq \max(f(X), f(Y)) \).

Then the subset of \( P_n \) satisfying \( f(X) \leq \tau \), where \( \tau \) is a given positive real number, is called a convex set in \( P_n \).

If \( f(X) = 0 \) if and only if \( X = 0 \), the convex set is called a convex body.

This definition is analogous to the definition of a convex body, with centre at the origin, through its gauge function in the ordinary n-dimensional Euclidean space. Thus if \( R_n \) denotes the Euclidean space and \( f(X) \) is a real valued function in \( R_n \) such that

(1) \( f(X) \geq 0 \) for all \( X \in R_n \)
(2) \( f(aX) = |a| f(X) \) for all real \( a \) and \( X \in R_n \)
(3) \( f(X + Y) \leq f(X) + f(Y) \)
(4) \( f(X) = 0 \) if and only if \( X = 0 \)
then \( f(X) < t, \) \( t \) real and positive, is a convex body in \( \mathbb{R}^n \).

In \( \mathbb{R}^n \) we normally define a convex set \( K \) by the geometrical property that if \( X, Y \) lie in \( K \), then the line segment joining \( X \) and \( Y \) lies in \( K \). We define a convex body as a non-empty bounded open convex set.

It is well known that the analytical and the geometrical definitions of a convex body in \( \mathbb{R}^n \) are equivalent (see, e.g. Cassels (1969), Chapter IV).

It appears worthwhile to give a geometrical characterization of the convex bodies of Mahler and the object of this note is to do so.

Our main theorem is as follows:-

**Theorem 1(a):** Let \( K \) be a subset of \( \mathbb{P}_n \). Then \( K \) is a convex set in the sense of Mahler if and only if it satisfies the following conditions:

1. \( X, Y \in K, a, b \in F, |a| < 1, |b| < 1 \Rightarrow aX + bY \in K \).
2. Given \( X \in \mathbb{P}_n \), there exists \( \varepsilon > 0 \) such that for all \( \lambda \in F \) with \( |\lambda| < \varepsilon \), \( \lambda X \in K \).
3. If \( X \in \mathbb{P}_n \) has the property that there exists a sequence \( \lambda_n \in F \) with \( |\lambda_n| \to 1 \) and \( X/\lambda_n \in K \), then \( X \in K \).

**Theorem 1(b):** \( K \) is a convex body if and only if in addition to (1), (ii) and (iii), we have

1. Given \( X \in \mathbb{P}_n \), \( X \neq 0 \), there exists \( \lambda \in F \) such that \( X/\lambda \) does not belong to \( K \).

In \( \S \S \) 2 and 3, we shall prove theorem 1. In \( \S \) 4, we shall show that if the valuation is discrete, then condition (iii) of theorem 1 is redundant. In \( \S \) 5, we shall show that convex sets are both closed and open in the topology of \( \mathbb{P}_n \).
induced by the valuation of $F$.

§ 2. In this section, we shall prove the necessity part of theorems 1(a) and 1(b).

Let $K$ be a convex set of Mahler. Without loss of generality we can suppose that

$$K = \{ X \mid f(X) < 1, X \in \mathbb{P}_n \}$$

where $f(X)$ satisfies conditions (a), (b) and (c).

Then $K$ satisfies condition (i) of theorem 1. To prove this, let $X, Y \in K$, $a, b \in F$, $|a| < 1$, $|b| < 1$, then

$$f(X) < 1, f(Y) < 1 \text{ so that } f(aX + bY) \leq \max(|a|f(X), |b|f(Y))$$

$< 1$ and hence $(aX + bY) \in K$.

To prove condition (ii) of theorem 1, take any $X \in \mathbb{P}_n$. Choosing $\epsilon > 0$ such that $\epsilon f(X) < 1$, we can satisfy condition (ii).

We next prove condition (iii) of theorem 1 for $K$. Let $X \in \mathbb{P}_n$ and $\lambda_n$ be a sequence in $F$ such that $|\lambda_n| \to 1$ and $X/\lambda_n \in K$.

Since $|\lambda_n| \to 1$, therefore given $\epsilon > 0$, there exists $\lambda$ such that $|\lambda_n| < 1 + \epsilon$, so that

$$f(X) = f(0,_{\lambda_n} X_{\lambda_n}) = |\lambda_n| f(X_{\lambda_n}) < 1 + \epsilon$$

Thus $f(X) < 1 + \epsilon$ for all $\epsilon > 0$, so that $f(X) \leq 1$ and hence $X \in K$.

(b) In case $K$ is a convex body in the sense of Mahler and $X \neq 0$ is given in $\mathbb{P}_n$, then $f(X) \neq 0$. Since $|x|$ is non-trivial, there exists $\lambda \in F$ such that $f(X) = f(\lambda) > 1$ so that $X/\lambda \notin K$ and thus condition (iv) of theorem 1(b) is also satisfied.

§ 3. (a) Proof of Sufficiency

Let $K$ be a set in $\mathbb{P}_n$ satisfying conditions (i), (ii) and (iii) of theorem 1. By condition (ii) $K$ is not empty, for $0 \in K$.

Let $X \in \mathbb{P}_n$. Define

$$\lambda_X = \{ |\lambda| \mid \lambda \in F, \lambda \neq 0 \text{ and } X/\lambda \in K \}.$$
By condition (ii) there exists $\varepsilon > 0$ such that $|\lambda| < \varepsilon$ implies $\lambda X \in K$. Since $|x|$ is non-trivial, there exists $\mu$ in $F$ such that $|\mu| > \frac{1}{\varepsilon}$. Then $\mu \neq 0$ and $\frac{x}{\mu} \in K$ and so $A_x$ is not empty. Define

$$p(X) = \inf A_x.$$ 

We now prove

**Lemma 1:** $p(X)$ is a distance function in the sense of Mahler, i.e. $p(X)$ satisfies condition (a), (b), (c).

**Proof of Lemma 1:** By definition $p(X) \geq 0$ so that $p(X)$ satisfies condition (a). Consider $p(aX)$ when $a = 0$. Then

$$p(aX) = p(0) = \inf \{|\lambda| : \lambda \neq 0, \frac{0}{\lambda} = 0 \in K\} = 0.$$ 

Therefore

$$0 = p(aX) = |a| p(X).$$

Now suppose $a \neq 0$. Then

$$p(aX) = \inf \{|\lambda| : \lambda \in F, \lambda \neq 0 \text{ and } \frac{aX}{\lambda} \in K\}$$

$$= \inf \{|\mu| : \mu \in F, \mu \neq 0 \text{ and } \frac{x}{\mu} \in K\}$$

$$= |a| p(X).$$

Thus $p(aX) = |a| p(X)$ for all $a \in F$, $X \in P_n$ and hence condition (b) is also satisfied.

We next prove (c). Let $X \in P_n$, $Y \in P_n$. Without loss of generality we may assume that $p(X) < p(Y)$. Since $p(-Y) = p(Y)$, to prove $p(X + Y) < p(Y)$, it is enough to prove that $p(X + Y) \leq p(Y)$. For this we distinguish three cases:

1. $p(X) < p(Y)$.
2. $p(X) = p(Y)$ and $p(Y)$ is not attained.
3. $p(X) = p(Y)$ and $p(Y)$ is attained.

**Proof of (c) in case (i):** $p(X) < p(Y)$. Now $p(X) = \inf A_x$ and $p(Y) = \inf A_y$. Let $|\lambda| \in A_y$. Then $|\lambda| > p(Y) > p(X) = \inf A_x$. 


So there exists $|\mu|$ in $A_x$ such that $|\mu| < |\lambda|$, i.e., there exists $\mu \neq 0$ in $F$ such that $|\mu| < |\lambda|$ and $\frac{\lambda}{\mu} \in K$.

Now $\frac{X}{\lambda} = \frac{\mu}{\lambda} \cdot \frac{X}{\mu} + 1 \cdot 0$. Since $\frac{|\mu|}{|\lambda|} < 1$ and $0 \in K$, by condition (1) it follows that $\frac{X}{\lambda} \in K$. Also by our choice of $\lambda$, $\frac{Y}{\lambda} \in K$ so that $\frac{X}{\lambda} + 1 \cdot \frac{Y}{\lambda} = \frac{X+Y}{\lambda} \in K$ and consequently $p(X+Y) < |\lambda|$. Thus $p(X+Y) < |\lambda|$ for all $|\lambda| \in A_y$. Hence $p(X+Y) \leq \inf A_y = p(Y)$, which proves (c) in this case.

Proof of (c) in case (ii): In this case $p(X) = p(Y)$ and there exists no $|\lambda|$ in $A_y$ such that $p(Y) = \lambda$.

Let $|\mu| \in A_y$. Then $|\mu| > p(Y) = p(X)$. Therefore there exists $|\nu| \in A_y$ such that $|\nu| > |\mu|$ and $\frac{\lambda}{\nu} \in K$. Then as before we can show that $\frac{X}{\nu} \in K$. Now $\frac{X}{\mu}$ and $\frac{Y}{\mu}$ are in $K$ and consequently $\frac{X+Y}{\mu}$ is also in $K$. As in case (i) we can again show that $p(X+Y) < p(Y)$.

Proof of (c) in case (iii): We have $p(X) = p(Y)$ and $p(Y)$ is attained. If $p(X)$ is not attained, the proof is similar to that in case (ii). So assume that $p(X)$ and $p(Y)$ are both attained, i.e., there exists $\mu, \nu \in F$ such that $|\mu| = p(X) = p(Y) = |\nu|$ and $\frac{X}{\mu} \in K, \frac{Y}{\nu} \in K$.

Here $\frac{X}{\nu} = \frac{\mu}{\nu} \cdot \frac{X}{\mu} + 1 \cdot 0$ and $|\mu| = 1$ so that $\frac{X}{\nu} \in K$.

Since $\frac{X}{\mu}, \frac{Y}{\nu} \in K$, it follows that $\frac{X+Y}{\nu} \in K$ and so $p(X+Y) \leq |\nu| = p(Y)$ and thus lemma 1 is proved completely.

In order to prove that $K$ is a convex set and hence to complete the proof of the sufficiency part of theorem 1(a), it will be enough to prove

**Lemma 2**: $K$ is the set $\{x \mid p(x) < 1\}$.

**Proof of Lemma 2**: Denote the set $\{x \mid p(x) < 1\}$ by $C_{p,1}$.
We wish to prove that $K = C_{p,1}$. To prove this let $X \in K$. So $X \in K$ and hence $p(X) \leq |1| = 1$ and consequently $X \in C_{p,1}$. Thus $K \subseteq C_{p,1}$.

Suppose now that $X \in C_{p,1}$, i.e., $p(X) \leq 1$.

In case $p(X) < 1$, i.e., $\inf A_X < 1$, there exists $\lambda \in F$ such that $0 < |\lambda| < 1$ and $X \in K$. Since $X = \lambda \frac{X}{\lambda} + 1 \in K$ and $|\lambda| < 1$, we see that $X \in K$. If $p(X) = 1$, i.e., if $\inf A_X = 1$, we can for every positive integer $n$, find $\lambda_n \in F$ such that $1 < |\lambda_n| < 1 + \frac{1}{n}$ and $\frac{X}{\lambda_n} \in K$. Therefore by condition (iii) $X \in K$. Thus $C_{p,1} \subseteq K$ and the proof of lemma 2 is complete.

(b) If in addition to (i), (ii) and (iii), (iv) is also satisfied, then given $X \neq 0$, there exists $\lambda \in F$ such that $\frac{X}{\lambda} \notin K$. We assert that $|\mu| < |\lambda| \Rightarrow \frac{X}{\mu} \notin K$. For otherwise $\frac{X}{\lambda} \cdot \frac{X}{\mu} + 0.1 = \frac{X}{\lambda}$ will belong to $K$, which is not true. Consequently $p(X) > |\lambda| > 0$ and so $p(X)$ is a special distance function in the sense of Mahler and hence $\{X \mid p(X) < 1\}$ is a convex body.

§4. We next prove

**Theorem 2:** In case $|x|$ is discrete, condition (iii) of theorem 1 is redundant.

**Proof:** We shall prove that in this case condition (i) implies condition (iii). Since the valuation is discrete $|\lambda_n| \to 1$ implies $|\lambda_n| = 1$ for at least one $n$, say for $n = m$. (In fact $|\lambda_n| = 1$ for all large $n$.) Therefore $\lambda_m \cdot \frac{X}{\lambda_m} + 1.0 = X \in K$ for $\frac{X}{\lambda_m} \in K$ and $|\lambda_m| = 1$.

§5. We now prove

**Theorem 3:** If $K$ is convex set, then $K$ is both open and closed in the topology induced by $|x|$ on $P_n$.

**Proof:** Let $|X| = \max (|x_1|, \ldots, |x_n|)$. Define
\[ d(X, Y) = |X - Y| \]

Then it can be shown that \( d(X, Y) \) is a metric in \( P_n \). Since \( p(X) \)
is a distance function in the sense of Mahler, it is continuous (Mahler (1941), p. 493) and hence the set

\[ K = \{ X \mid p(X) < 1 \} \]

is closed.

Also

\[ K = \{ X \mid p(X) < 1 \} \cup \{ X \mid p(X) = 1 \} \]

Since \( p(X) \) is continuous, \( \{ X \mid p(X) < 1 \} \) is open. So to prove \( K \) is open, it is enough to prove that \( V = \{ X \mid p(X) = 1 \} \) is open.

Let \( X_0 \in V \). \( p(X) \) being continuous at zero, there exists \( \epsilon > 0 \) such that \( |x| < \epsilon \Rightarrow p(x) < \frac{1}{2} \). We claim that for this \( \epsilon \)

\[ S(X_0, \epsilon) = \{ x \mid |x - X_0| < \epsilon \} \subset V. \]

Let \( X \in S(X_0, \epsilon) \). Now \( p(X) = p(X_0 + (X-X_0)) \). Since \( |X-X_0| < \epsilon \)
so that \( p(X-X_0) < \frac{1}{2} \) and since \( p(X_0) = 1 \) and \( p(X) \) is non-
Archimedean, therefore \( p(X) = p(X_0) = 1 \) which shows that \( X \in V \)
and thus \( S(X_0, \epsilon) \subset V \) which in turn proves that \( V \) is open and
hence the proof of the theorem 3 is complete.

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