CHAPTER VI

An upper bound for the minimum discriminant of sixth degree fields with two real and four imaginary conjugate fields.

In this chapter we prove the following

Theorem. The minimum discriminant of fields of degree 6 with 2 real and 4 imaginary conjugate fields is \( \Delta \leq 77461 \).

Proof. From table II, it can be checked that

\[ x^6 + 2x^5 - 3x^4 - 3x^2 + 1 \]

is an irreducible polynomial with 2 real and 4 imaginary roots, having \( D = 5^2 \cdot 77461 \).

Our theorem will follow if we prove that the discriminant \( D \) of the corresponding field of the above polynomial is 77461.

We obtain the exact power of 5 occurring in the discriminant \( D \) with the help of Theorem A.

The prime ideal decomposition of \( (5) \) is obtained by Berwick's method as follows:

\[
a(x) \equiv (x-2)^2(x^4 + x^3 + 2x^2 + 3x - 1) \pmod{5}
\]

\[
\equiv \{(x-2)^2 + 5.2(x-2) + 5^2[3(x-2) + 4]\}[x^4 + x^3 + 2x^2 + 3x - 1]
+ 5(4x^3 + 2x^2 + 2x + 3) + 5^2(x^2 - 2x^2 + 4x - 2) \pmod{5^3}
\]

For the first factor, there is a marked node namely \((1,1)\) on the edge joining \((2,0)\) and \((0,2)\). So the second dissection fails here and we apply the third dissection.

Comparing the end points \((2,0)\) and \((0,2)\) of this edge with \((\rho + j\sigma, \sigma)\), \((\rho, \sigma + j\tau)\), we get \(\sigma = \rho = 0, u = v = 1, j = 2\).

The terms represented by nodes on this edge are

\[(x-2)^2 + 5.2(x-2) + 5^2.4\]

We show that it is irreducible modd \((5, x+2)^3\).

For otherwise
(x-2)^2 + 6(x-2)x + 8^2 \equiv [(x-2)+6\alpha][(x-2)+6\beta] \pmod{(5, x-2)^3},

where \( \alpha, \beta \) are rational integers. Their difference

\[ 5(2-\alpha-\beta)(x-2)+8^2(4-\alpha \beta) = 2^2 \beta (x-2)^2, \]

with \( a_1 + b_1 > 3, a_1 > 0, b_1 > 0 \).

Clearly \( a_1 < 2, b_1 < 1 \). The only possible solution is \( a_1=2, b_1=1 \).

So \( 5(2-\alpha-\beta)(x-2)+8^2(4-\alpha \beta) = 2^2 \beta (x-2)^2 \).

Comparing coefficients we get \( \alpha \beta = 4 \) and \( 2-\alpha-\beta = 5 \).

\[ \alpha \beta = 4 \Rightarrow \alpha+\beta=0 \text{ or } \pm 1 \pmod{5}; \]

\[ 2-\alpha-\beta = 5 \Rightarrow \alpha+\beta=2 \pmod{5}, \]

which is a contradiction. Hence \((x-2)^2 + 6(x-2)x + 8^2 \) is irreducible \( \pmod{(5, x-2)^3} \). So Berwick's third dissection applies.

We get \( \mathfrak{p} \) as the prime ideal factor of \((5)\) corresponding to this edge, where \( \mathfrak{p} \) is of degree \( M_{(1,1)} = 2 \) (for \( M=1, m=2, q=1 \)).

For the second factor of \( a(x) \pmod{5^3} \), the nodes to be marked are \((1,0)\) and \((0,1)\). Since there is no marked node on the edge joining \((1,0)\) and \((0,1)\), the second dissection applies and we get \( \mathfrak{p}_2 \) as the prime ideal factor of \((5)\) corresponding to this edge, where \( \mathfrak{p}_2 \) is a prime ideal of degree 4.

So \((5) = \mathfrak{p} \cdot \mathfrak{p}_2 \), where \( \mathfrak{p}, \mathfrak{p}_2 \) are prime ideals of degrees 2 and 4 respectively.

So by Theorem A, 5 does not occur in the discriminant of the field. Since 77461 = 7 \times 1091 is square free and \( D_f = a^2 D \), so 77461 is the discriminant of the corresponding field.

Hence the minimum discriminant of fields of degree 6 with two real and four imaginary conjugate fields is 77461.