Upto now we have been concerned with the investigations which are essentially based on what is called the linear stability theory. This theory determines those conditions under which a disturbance grows exponentially with time. Whether this growth is without a limit is a question which remains unanswered there. Actually experiments have shown that after some time past the critical state the disturbances change their exponential form by advecting heat and momentum. This does not leave their amplitude to be small. So to know any thing after the marginal state is attained, we have to seek refuge under the so called non-linear stability theory. Before this theory as given by Malkus & Varnois (1958) is extended to the present case it seems worthwhile to study a comparatively simple problem viz. the effect of radiative transfer on the amplitude of convection. Such a problem in the absence of radiative transfer has been examined by Nakagawa (1960) who also estimated the effect of an impressed magnetic field. All these investigations are based on an experimental observation that the pattern of motion which appears at marginal stability manifest itself long past this state. Although no experimental investigation, which embodies radiative transfer effects within its frame-work, exists in literature yet this assumption can be used with justification in the present case too. This is because of the fact that radiative
transfer has a tendency to smooth out any disturbance arising out of temperature gradient which means that for a constant property medium the change from one state to another is accompanied by strong adverse temperature gradients (at least stronger than what would be in the absence of radiative transfer, an estimate of which is given in the calculations presented in chapters 1 to 3). This means that the pattern of motions will not effectively change before this temperature gradient is built up and so, in a way, justifies the use of the assumption mentioned above.

The basic equations of the problem are the usual non-linear equations (I) with \( \nabla^2 = 0 \). If we non-dimensionalize the variables as

\[
\begin{align*}
\bar{u}_i &= \frac{d\bar{u}}{d\bar{x}}; \quad (\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\frac{u}{\bar{u}}; \frac{v}{\bar{u}}, \frac{w}{\bar{u}}), \\
\bar{T}_i &= \frac{T}{\bar{T}_0}; \quad \bar{P}_i = \frac{d\bar{P}}{\bar{D}^2}; \quad \bar{V}^2 = \bar{F}^2/4\pi g \bar{R} \int (\bar{T}_i - \bar{T}_0) \, d\bar{x},
\end{align*}
\]

where \( \bar{T}_1 \) and \( \bar{T}_2 \) (\( \bar{T}_1 > \bar{T}_2 \)) are the temperatures of the lower and upper bounding surfaces and express the pressure and temperature fields as the sum of two fields - the horizontal mean field and the fluctuations associated with convection - as done for example by Nakagawa (1960), we obtain the following equations in the Boussinesq's approximation, for the steady case.

\[
\begin{align*}
\langle (\bar{u} \cdot \nabla) \theta \rangle &= \frac{1}{\bar{R}} \frac{d^2 \bar{T}_0}{d\bar{x}^2} - \frac{3 \kappa \bar{L} \bar{x}^2}{\bar{R}^2} \frac{d \bar{P}}{d\bar{x}^2}; \\
(\bar{u} \cdot \nabla) \bar{u} - \langle (\bar{u} \cdot \nabla) \bar{u} \rangle &= -\bar{\nabla} (\frac{\bar{P}}{\bar{R}}) + \bar{\nabla}^2 \bar{u} + \frac{\bar{R}}{\bar{R}} \theta \bar{\nabla}^2 \bar{u}, \\
(\bar{u} \cdot \nabla) \theta - \langle (\bar{u} \cdot \nabla) \theta \rangle &= \frac{1}{\bar{R}} \bar{\nabla}^2 \theta - \omega \frac{d \bar{T}_0}{d\bar{x}^2} + \frac{3 \kappa \bar{L} \bar{x}^2}{\bar{R}^2} \phi.
\end{align*}
\]
where \( \mathbf{w} \) is the vertical component of the velocity and
\[
\Phi = \Phi_0 + \Phi; \quad \rho = \rho + 8\rho = \rho_0 - \alpha \rho_0 \theta; \quad \gamma = \gamma \alpha; \quad \mu = \mu_0 + \mu; \quad T = T_0 + \theta
\]
and the suffix '1' has been dropped while writing the above equations. The non-dimensional parameters appearing therein are:
\[
\mathcal{R} = \gamma(T_r - T_0) \frac{d^2}{\nu k} \quad \text{and} \quad \mathcal{P} = \frac{\nu}{k}
\]
the Rayleigh number and the Prandtl number respectively and other symbols have their usual meanings. In these equations the subscript '0' and \( < > \) represent the horizontal average. \( \Phi_0 \left( \Phi_0 = -\frac{dF_0}{dy} \right) \)
is to be obtained from the radiative transfer equation which after utilizing the Milne-Eddington approximation reduces to, after putting in the non-dimensional form,
\[
\frac{d^2 F_0}{dy^2} = \frac{dT_0}{dy} + 3\alpha^2 \frac{d^2 F_0}{dy^2} \quad (2)
\]
The solutions of equations (1) and (2) are required to satisfy the following boundary conditions,
\[
T_0 = \pm \frac{1}{2} \quad \frac{dT_0}{dy} = \pm 2\alpha \frac{dF_0}{dy} \quad \text{for} \ y = \pm \frac{1}{2}
\]
and \( \frac{\partial w}{\partial y} = 0 \) on a rigid surface
or \( \frac{\partial^2 w}{\partial y^2} = 0 \) on a free surface
\[
(3)
\]
The simultaneous solution of (1-1) and (2) will give the effect of motion as well as radiative transfer on the mean temperature gradient. In the absence of the former these equations have been solved by Goody (1956) and the solution is quoted in the 1st chapter.
Integral relations.

Multiplying equations (1-ii) by $\overline{u}$ and (1-iii) by $\Theta$, averaging over the horizontal plane and integrating the resulting equations with respect to $\overline{z}$, we get

$$\frac{R}{\overline{\nu}} \int_{-\overline{h}}^{\overline{h}} \langle w \Theta \rangle \, dz + \int_{-\overline{h}}^{\overline{h}} \langle (\overline{u} \cdot \nabla) \overline{u} \rangle \, dz = 0 \quad (4)$$

$$\frac{P}{\overline{\nu}} \int_{-\overline{h}}^{\overline{h}} \langle w \theta \rangle \frac{d}{dz} \frac{d}{dz} \frac{dz}{dz} = \int_{-\overline{h}}^{\overline{h}} \langle \Theta \frac{d}{dz} \theta \rangle dz + 3 \overline{h} \frac{d}{dz} \int_{-\overline{h}}^{\overline{h}} \langle \Theta \phi \rangle \, dz \quad (5)$$

The physical significance of equation (4) has already been discussed by Nakagawa (1960), on the other hand equation (5) represents a balance between the available thermal energy from the mean field and heat transferred to the bounding surfaces by conduction and radiation. As already mentioned, we assume that the solutions $w$ and $\Theta$ are the same as given for the case of marginal stability. Thus

$$w = A \, W(y) \cdot f(x,y) \quad ; \quad \Theta = A \, \theta(y) \cdot f(x,y)$$

In the linear theory 'A' remains an undetermined constant of proportionality and so we can without loss of generality put

$$\langle f^2 \rangle = 1$$

and since $f$ satisfies the Helmholtz equation (1.12), it may be seen that

$$\langle \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2 \rangle = -A^2 \langle f^2 \rangle = -A^2$$

Equations (4) and (5) may now be written as

$$\frac{R}{\overline{\nu}} \int_{-\overline{h}}^{\overline{h}} \theta w \, dz + \int_{-\overline{h}}^{\overline{h}} \left[ \frac{\partial}{\partial x} \left( \frac{D z}{a^2} \right) w + \frac{1}{a^2} \frac{D w}{D x} \left( \frac{D z}{a^2} \right) D w \right] \, dz = 0 \quad (6)$$
The two simultaneous equations (6) and (7) give us the amplitude of convection, for the solution of which we need \( \frac{dT_c}{dz} \) as well as the dependence of \( \phi \) (the fluctuation in radiative heating) on temperature. We shall again examine this problem for the two asymptotic cases of radiative transfer equation. Thus we have in the non-dimensional form

\[
\phi = \begin{bmatrix}
-\Theta + \text{const.} \\
\frac{1}{3k^2d^2} \nabla \Theta
\end{bmatrix}
\]

The expression for \( \frac{dT_c}{dz} \) is to be obtained from the solution of (1-1) and (2) subject to the relevant boundary conditions.

**Solution for the mean temperature gradient.**

Since equation (1-1) includes the effect of motions, it is not possible to integrate these equations unless the variation of \( \omega \) and \( \Theta \) is known and for which we have to make a choice about the nature of the bounding surfaces. We shall here present the case of free bounding surfaces. Since solutions for \( \omega \) and \( \Theta \) are to be the same as in the case of marginal stability, we may put

\[
\omega = \sin \pi (3r + \frac{1}{2})
\]

The last equation does not represent the exact solution of the steady state linear problem when radiative transfer...
effects have also been included, but, in fact, is the
trial function satisfying the boundary conditions, which
has been used for the evaluation of marginal conditions.
This in the absence of exact solution will give us approx­
imate value of the amplitude of convection. Now the
expression for $\Theta$ comes out to be

$$\Theta = \frac{R (x^2 + a^2)^{1/2}}{R a^2} \sin \pi (y + \frac{1}{2})$$  \hspace{1cm} (10)

Utilizing equations (8), (9) and (10), the required solution
for the temperature gradient comes out to be, for optically
thin case,

$$\frac{d T_0}{d y} = - (\beta/\beta_c) \left[ 1 + \frac{P \lambda^2}{(1+\lambda)(4\pi^2 + \lambda^2)} \right]$$

$$+ \frac{P \lambda^2}{(1+\lambda)(4\pi^2 + \lambda^2)} - \left[ 1 - \frac{\lambda^2 \lambda}{(1+\lambda)(4\pi^2 + \lambda^2)} \right] P \cos \pi (y + \frac{1}{2})$$  \hspace{1cm} (11)

where $$(\beta/\beta_c)_c = L \cosh \lambda \gamma + M$$  \hspace{1cm} (11.16)

and

$$P = \frac{\rho A^2}{\gamma R} \left( \frac{x^2 + a^2}{a^2} \right)^{1/2}$$

Clearly in the limit of $P \to 0$, which amounts to going
back to linear theory, equation (11) reduces to Goody's case.

Amplitude of convection.

Substituting from equations (8) to (11) in (6) and
(7), we have for the optically thin case.

$$A^2 = \frac{4 D_1 a^2}{R (\pi^2 a^2)^{1/2}} \left[ 1 - \frac{\lambda^2}{4\pi^2 + \lambda^2} \frac{\chi + 2 D_1}{1 + \chi} \right] (R - R_c)$$  \hspace{1cm} (12)
where \[ D_i = \left( R \frac{\pi^2}{\lambda (\pi^2 + \lambda^2)} \right) L \sinh \lambda \sqrt{\lambda} \]
and \( R_c \) is the critical Rayleigh number for marginal stability case. In the limit of \( \lambda \to 0 \),
equation (12), giving the value of the amplitude for convection, reduces to that for no radiative transfer case.

The case of optically thick medium can similarly be dealt with. The amplitude in this case decreases by an amount \( \left( 1 + \lambda \right)^{\frac{1}{\sqrt{\lambda}}} \). The variation of amplitude remains unchanged as far as \( (R - R_c)^{\frac{1}{\sqrt{\lambda}}} \) law is concerned and is true for both the asymptotic cases. The reason for this is that this simple law is more due to the basic structure of the differential equations rather than the exact solution of the corresponding linear problem and as such the validity of the law in the optically thin case seems to be unquestionable. However the error involved because of the use of an approximate expressions for \( \omega \) and \( \Theta \) seems to change the result by a simple numerical factor rather than the basic \( (R - R_c)^{\frac{1}{\sqrt{\lambda}}} \) law which is also true in the absence of radiative transfer (see Chandrasekhar 1961, Appendix).