CHAPTER III:- A STUDY OF THE EFFECT OF RADIATIVE TRANSFER ON THE GRAVITATIONAL CONVECTION OF A HOT FLUID ENCLOSED BETWEEN TWO PARALLEL RIGID BLACK SURFACES.

In the last two chapters we had discussed the problem of convective instability between two parallel surfaces for the case of free boundaries in the presence of radiative transfer. Now we shall investigate such a case for rigid bounding surfaces. Here it so happens that due to the variable nature of \( \left( \frac{\rho}{\bar{\rho}} \right) \) and the boundary conditions, the solution cannot be based on a variational principle. The magnetic field and rotation etc. are neglected. We shall here employ a method originally due to Chandrasekhar (1954a), the variational basis of which has been found by Roberts (1960) (see Chandrasekhar 1961 also) by defining another system of differential equations, which are adjoint to the given equations of the problem. It may be pointed out that in such situations the validity of the principle of exchange of stabilities still remains unsettled. We shall here, only, discuss the steady state problem. Thus in the absence of magnetic field and rotation i.e. for an ordinary hot radiating fluid, we may put

\[
(D^2 - a^2) w = F \\
(D^2 - a^2 - 3b^2 \dot{a}^2 \dot{a}) F = -Ra^2 \frac{\beta}{\bar{\rho}} w
\]

The boundary conditions of the problem are

\[
w = 0, \quad F = 0, \quad D w = 0 \quad \text{for} \quad \beta = \pm \frac{1}{2}
\]
where \( D = \frac{d}{dy} \) and other symbols have their usual meanings. The solutions of equations (1) subject to the boundary conditions (2) are either even or odd. We first consider the even solutions. Since, now \( F \) is even and is required to vanish at \( y = \pm \frac{1}{2} \), we may expand it in a cosine series of the following form

\[
F = \sum_{m=0}^{\infty} A_m \cos((2m+1)\pi y) \tag{3}
\]

where the summation over \( m \) may go from zero to infinity.

Again, we may put

\[
w = \sum_{m=0}^{\infty} A_m w_m \tag{4}
\]

The form of equation (4) for \( w \) owes its origin to the linearity of the differential equations. Now combining (1), (3) and (4) we may put

\[
(D^2 - a^2)^2 w_m = \cos((2m+1)\pi y) \tag{5}
\]

where \( w_m \) satisfies the boundary conditions

\[w_m = 0, \quad Dw_m = 0 \quad \text{for} \quad y = \pm \frac{1}{2}\]

Solution of equation (5), satisfying the boundary conditions, is given by

\[
w_m = P_m \cosh ay + A_m y \sinh ay + \gamma_{2m+1}^2 \cos((2m+1)\pi y) \tag{6}
\]

where

\[
P_m = (-1)^{m+1} \frac{(2m+1)\pi \gamma_{2m+1}}{a + \sinh a} \sinh ay/2
\]

\[
A_m = (-1)^{m} \frac{2(2m+1)\pi \gamma_{2m+1}}{a + \sinh a} \cosh ay/2
\]

\[
\gamma_{2m+1} = \frac{1}{(2m+1)^2 \pi^2 + a^2}
\]
Substituting from (3) and (4) in (i-ii), we get

\[ \sum_m A_m \left[ \left( (2m+1) \pi^2 + a^2 + 3 \lambda c^2 \right) \cos (2m+1) \pi \right] \cos (2m+1) \pi \gamma - R a^2 \frac{B}{B} \sum_m = 0 \]  

(7)

Multiplying equation (7) by \( \cos (2n+1) \pi \gamma \) and integrating between the limits \( -\frac{1}{2} \leq \gamma \leq \frac{1}{2} \), we obtain

\[ \sum_m A_m \left[ \left( (2m+1) \pi^2 + a^2 + 3 \lambda c^2 \right) \delta_{mn} - 2 R a^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_m \cos (2m+1) \pi \gamma \right] = 0 \]

(8)

utilizing equations (1.16) & (6), we have

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{B}{B} \right) \sum_m \cos (2n+1) \pi \gamma = L G_{mn} + M (n|m) \]

where \( G_{mn} = H_m \left( \gamma_{2n+1} \cos \frac{a+\lambda}{2} - \gamma_{2n+1} \cosh \frac{a-\lambda}{2} \right) \)

\[ \frac{A_{mn}}{\pi^2} = \sum_m \left( \gamma_{2n+1} \cosh \frac{a-\lambda}{2} \right) + \frac{A_{mn}}{\pi^2} \]

and

\[ A_{mn} = \lambda \left[ \frac{\cos (m+n+1) \pi \sinh \lambda}{4 \lambda^2 (m+n+1)^2 + \lambda^2} + \frac{\cos (m-n) \pi \sinh \lambda}{4 \lambda^2 (m-n)^2 + \lambda^2} \right] \]
Thus equation (8) take the form

$$\sum_m A_m \left\{ (2m+1)\pi^2 + a^2 + 3k^2 d^2 \right\} \delta_{mn} - 2 R a^2 \left[ L G_{nm} + M (n|m) \right] = 0$$

(9)

In usual practice, experience have shown that a first approximation to $R$ as given by setting $(0,0)$ element of the secular matrix (9) to zero, gives sufficiently accurate values. This, in fact, corresponds to $\cos \pi z$ as a trial function for $F$. Thus, we have

$$R a^2 = \frac{\pi^2 + a^2 + 3k^2 d^2}{2L G + 2M (n|m)_{0,0}}$$

(10)

where $(n|m)_{0,0} = \frac{1}{2} \gamma_1^2 \frac{8\pi^2 \gamma_1^4 \cosh^2 a/2}{\alpha + \sinh a}$

$$2G = P_0 \pi \left( \gamma_1^+ \cosh \frac{a+\lambda}{2} + \gamma_1^- \cosh \frac{a-\lambda}{2} \right) + Q_0 \pi \left[ (\gamma_1^+ \sinh \frac{a+\lambda}{2} + (\gamma_1^- \sinh \frac{a-\lambda}{2}) \right]$$

$$+ \gamma_1^2 \frac{8\pi^2 \sinh \lambda/2}{\lambda \left[ 4\pi^2 + \lambda^2 \right]}$$

$$\gamma_1^+ = \frac{1}{\pi^2 + (\alpha+\lambda)^2}; \quad \gamma_1^- = \frac{1}{\pi^2 + (\alpha-\lambda)^2}$$

$$\gamma_1^2 = \frac{1}{\pi^2 + a^2}$$

$$P_0 = -\pi \gamma_1^2 \sinh a/2 \frac{\cosh a}{a + \sinh a}; \quad Q_0 = \frac{2\pi \gamma_1^2 \cosh a}{a + \sinh a}$$

Substituting these expressions in equation (10) and after some algebraic adjustment, we have...
\[
R = \frac{(\pi^2 + a^2)^2(\pi^2 + a^2 + 3d^2d^2)}{M^2} \left[ 2 \frac{\lambda^2}{\pi^2 \lambda^2} \left\{ \frac{2 \cosh \lambda z}{a + \sinh \lambda z} \zeta_{a,\lambda} - \frac{\sinh \lambda z}{a + \sinh \lambda z} \zeta_{a,\lambda} - \frac{8 \cosh \lambda z \zeta_{a,\lambda}}{a + \sinh \lambda z} \right\} + \frac{8 \sinh \lambda \lambda}{\lambda(4\lambda^2 + \lambda^2)} \right] \right. \\
\left. + \left\{ 1 - \frac{16 \pi^2 a \cosh^2 \lambda z}{(\pi^2 + a^2)^2 (a + \sinh \lambda z)} \right\} \right]
\]

where \( \zeta_{a,\lambda} = \frac{\cosh a + \frac{\lambda}{2}}{\pi^2 + (a + \lambda)^2} + \frac{\cosh a - \frac{\lambda}{2}}{\pi^2 + (a - \lambda)^2} \)

\( \zeta_{a,\lambda}^{**} = \frac{\sinh a + \frac{\lambda}{2}}{\pi^2 + (a + \lambda)^2} + \frac{\sinh a - \frac{\lambda}{2}}{\pi^2 + (a - \lambda)^2} \)

\( \zeta_{a,\lambda}^{***} = \frac{(a + \lambda) \sinh a + \frac{\lambda}{2}}{[\pi^2 + (a + \lambda)^2]^2} + \frac{(a - \lambda) \sinh a - \frac{\lambda}{2}}{[\pi^2 + (a - \lambda)^2]^2} \)

The expression in the first curly bracket of equation (11) represents the contribution due to variable temperature gradient which also shows that the wave number of the disturbance at marginal stability is also affected by it.

As such its effect is different from what we found in the case of free bounding surfaces where wave number is not influenced by \((\beta/\overline{\beta})\). In the limit of \( \lambda \) and or \( \lambda \rightarrow 0 \) equation (11) reduces to the corresponding relation obtained by Pellew & Southwell (see Chandrasekhar 1961).

Upto now we have examined the even modes. The investigation for the odd modes can similarly be carried out by assuming the trial function for \( F \) to be

\[
F = \sum_{m} A_m \sin m\pi y
\]

(12)
For the first approximation, the expression for the Rayleigh number comes out to be

\[
R = \frac{(4\pi^2 + \alpha^2) \left(4\pi^2 + a^2 + 3\kappa^2\lambda^2\right)}{\left\{ \frac{g L \lambda^2}{M} \left[ \frac{\sinh \alpha_2}{\sinh a - a} \xi_{a_2}^2 + \frac{\cosh \alpha_2}{\sinh a - a} \xi_{a_1} \right] - \frac{4}{\sinh a - a} \xi_{a_2} \right\}} \left[ \frac{4 \sinh \lambda/2}{\lambda (4\pi^2 + \lambda^2)} \right] \left[ 1 - \frac{64 \pi^2 a \sinh \lambda/2}{(4\pi^2 + \lambda^2)^2 (\sinh a - a)} \right]^{**} (13)
\]

The above analysis only refers to the first type of approximation of the radiative transfer equation. For case (b), (optically thick case) the differential equations are self-adjoint because of the constant temperature gradient prevailing in that case. The Rayleigh number characterizing the marginal state comes out to be \((1 + \kappa)R_{oc}\) where \(R_{oc}\) is the value of critical Rayleigh number in the absence of radiative transfer and is the same as obtained by Pellew & Southwell (1940).

Calculations based on equation (11) are performed and \(R\) is evaluated for different values of \(\alpha^2\) and for \(\kappa = 10^3\), \(\lambda = 10\). It is found that instability first arises for \(R = 31.5 \times 10^3\) and \(\alpha = 5.2\) approximately. In these calculations the maximum value of \(\alpha^2\) taken is 10. However, a large number of cases, which could have enabled to draw some concrete conclusions, could not be evaluated because of the obvious reasons.
Although radiation seems to stabilize the fluid motion, the behaviour of variable temperature gradient seems to be quite curious.