Notations and Basic Definitions

Let \( \Omega, \mathcal{A}, P \) be a probability space. Here \( \Omega, \mathcal{A} \) denotes measurable space, i.e., a set \( \Omega \) consisting of elementary events \( w \), with a distinguished system \( \mathcal{A} \) of its subsets (events) forming a \( \sigma \)-algebra and \( P \) denotes a probability measure defined on sets in \( \mathcal{A} \). Let \( T = [0, \infty) \). The family \( X = (X_t), t \in T \) of random variables \( X_t = X_t(w) \) is called a real random process with continuous time, \( t \in T \). In the case where the time parameter \( t \) is confined to the set \( N = \{0,1,\ldots\} \), the family \( X = (X_n), n \in N \), is called a random sequence or a random process with discrete time.

1. With \( w \in \Omega \) fixed, the time function \( X_t (t \in T \) or \( t \in N) \) is called a trajectory or realization or sample function corresponding to an elementary event \( w \).

2. The random process \( X = (X_t), t \in T \) is called measurable, if for all Borel sets \( B \in \mathcal{B} \) \( T \),
\[
\{(w, t) : X_t(w) \in B\} \in \mathcal{A} \times \mathcal{B} (T)
\]
where \( \mathcal{B} (T) \) is a \( \sigma \)-algebra of Borel sets on \( T = [0, \infty) \).

3. Let \( \mathcal{G}_t = (\mathcal{G}_s), t \in T \), be a non-decreasing family of \( \sigma \)-algebras, \( \mathcal{G}_s \subset \mathcal{G}_t \subset \mathcal{A}, s \leq t \). We say that a (measurable) random process \( X = (X_t), t \in T \) is adapted to a family \( \mathcal{G} \), if for any \( t \in T \), the random variables \( X_t \) are \( \mathcal{G}_t \)-measurable. For brevity such a random process will be denoted by \( X = (X_t, \mathcal{G}_t), t \in T \) and called \( \mathcal{G}_t \)-adapted.

4. The process \( X = (X_t), t \in T \) is continuous in probability or stochastically continuous at \( t_0 \in T \), if for any \( \varepsilon > 0 \),
X is stochastically continuous on $T$ if the above property holds for every $t_0 \in T$. A process $X = (X_t)$ is called continuous, right continuous or left continuous if for $P$-almost all $\omega$, the trajectory of $\omega$ has this property.

An increasing family $(\mathcal{B}_t)$ is right continuous if for each $t$, $\mathcal{B}_{t+} = \mathcal{B}_t$ for each $t$.

5. The random variable $\tau = \tau(\omega)$ taking on values in $\tau = [0, \infty]$ is called a Markov time (relative to the system $(\mathcal{B}_t)$, $t \in T$) if for all $t \in T$

$$\{ \omega : \tau(\omega) \leq t \} \in \mathcal{B}_t.$$ The Markov times are independent of the future. If $P(\tau(\omega) < \infty) = 1$, then it is a stopping time.

6. The real valued process $(X_t)$ will be called a martingale relative to $\mathcal{B}_t$ if

1) for each $t$, $X_t$ is integrable and $\mathcal{B}_t$ measurable
2) for $t > s$, $E(X_t / \mathcal{B}_s) = X_s$, a.s.

Similarly, the real valued process $(X_t)$ is called a supermartingale (submartingale) w.r.t $(\mathcal{B}_t)$ if (1) above is satisfied and (2) holds with '  ' replaced by "  " (respectively "  ").

7. The martingale $(X_t, \mathcal{B}_t)$, $t > 0$ is called regular if there exists an integrable random variable $\gamma$, $E|\gamma| < \infty$, such that

$$X_t = E(\gamma / \mathcal{B}_t), \text{ a.s. (P), } t > 0.$$ The regularity of the martingale $(X_t, \mathcal{B}_t)$, $t > 0$, is equivalent to the uniform integrability of the family of random variables $(X_t, t > 0)$.

8. A real stochastic process $(A_t)$, $t \in \mathbb{R}_+$ is called an increasing
process w.r.t an increasing right continuous family \((\mathcal{B}_t)\) if

a) \(A_t\) is \(\mathcal{B}_t\)-adapted

b) \(A_0 = 0\) and \(A_t(\omega)\) is an increasing right continuous function of \(t\), for almost all \(\omega\).

c) \(E(A_t) < \infty\), for each \(t \in \mathbb{R}_+\).

The increasing process \((A_t)\) is said to be integrable if \(\sup_t E(A_t) < \infty\).

9. Let \((A_t)\) be an increasing process and \((X_t)\) a measurable non-negative process. For each fixed \(\omega \in \mathcal{F}\), the function \(t \rightarrow X_t(\omega)\) is measurable. We can consider the Lebesgue-Stieltjes integral on \(\mathbb{R}_+\) for each \(\omega\):\[
\int_0^t X_t(\omega) \, dA_t(\omega).
\]

An increasing process is 'natural' if

\[
E\left(\int_0^t Y_s \, dA_s(\omega)\right) = E\left(\int_0^t Y_s \, dA_s\right)
\]

for every \(t \in \mathbb{R}_+\) and each non-negative bounded right continuous martingale \((Y_t)\).

10. A supermartingale \((X_n, \mathcal{G}_n), n \in \mathbb{N}\) has exactly one decomposition

\[
X_n = M_n - A_n, \quad \text{a.s.}
\]

where \((M_n, \mathcal{G}_n)\) is a martingale and \(A_n\) is a sequence of r.v.s such that \(A_0 = 0, \text{a.s.}\), \(A_n \leq A_{n+1}, \text{a.s.}\) and \(A_{n+1}\) is \(\mathcal{G}_n\)-measurable \((n \geq 0)\).

\(M_n\) and \(A_n\) are given by

\[
M_n = M_{n-1} + \left[ X_n - E(X_n / \mathcal{G}_{n-1}) \right], \quad n \geq 1
\]

\[
A_n = A_{n-1} + \left[ X_{n-1} - E(X_{n-1} / \mathcal{G}_{n-1}) \right]
\]

This representation is called the Doob decomposition for supermartingales, for sequences.
11. Let \((X_t, \mathcal{B}_t), t \in \mathbb{R}_+\) be a right continuous super martingale and let \(F\) be the collection of all the finite stopping times relative to this family \((\mathcal{B}_t)\). \(X_t\) is said to belong to the class \((\mathcal{D})\) if the collection \((X_{\tau}, \tau \in F)\) is uniformly integrable.

\((X_t)\) is said to belong to the class \((\mathcal{DL})\) or locally to the class \((\mathcal{D})\) if \((X_t)\) belongs to the class \(D\) for \(t \in [0, c]\) on every interval \([0, c]\), \((0 < c < \infty)\).

12. (Doob-Meyer decomposition for right-continuous super martingales).

A right continuous super martingale \(X_t\) has a D-M decomposition

\[X_t = M_t - A_t, \quad t \in \mathbb{R}_+\]

where \(M_t\) denotes a right continuous martingale and \(A_t\) is an increasing process if and only if \((X_t)\) is of class \((\mathcal{DL})\). In this case there is exactly one pair \((M_t, A_t)\) which yields such a decomposition, where \(A_t\) is also natural.

13. Let \(X\) denote the family of all real valued functions \(Y_t(\omega)\) defined on \(\mathbb{R}_+ \times \Omega\) which are measurable w.r.t \(\mathcal{B}(\mathbb{R}_+) \times \mathcal{A}\) and have the following properties:

1) \(Y = (Y_t)\) is adapted to \(\mathcal{B}_t\).

2) For each \(\omega\), the function \(t \rightarrow Y_t(\omega)\) is left continuous.

Let \(\mathcal{P}\) be the \(\sigma\)-field of subsets of \(\mathbb{R}_+ \times \Omega\), w.r.t which all the functions belonging to \(X\) are measurable.

Then a stochastic process \(X = (X_t(\omega))\) is said to be predictable relative to \(\mathcal{B}_t\) if the function \((t, \omega) \rightarrow X_t(\omega)\) is \(\mathcal{P}\) - measurable. In the case of sequences, \(A_n\) is predictable if \(A_n\) is \(\mathcal{B}_{n-1}\) - measurable.

It is shown in general that the class of predictable increasing processes...
coincides with the class of natural increasing processes.

14. A martingale $M = (M_t, \mathcal{F}_t), t \in \mathbb{R}^+ \,$ is said to be **square integrable** (or called an $L^2$-martingale) if
$$\sup_t E(M_t^2) < \infty.$$

The **quadratic variation process** of $M$ denoted by $\langle M \rangle$ is defined by
$$\langle M \rangle_t - \langle M \rangle_s = A_t,$$
where we set $\langle M \rangle_0 = M_0^2$ and $A_t$ is the unique natural increasing process as associated with the D-M decomposition of
$$Y_t = E(M_t^2 / \mathcal{G}_t) - M_t^2.$$

In particular, if $0 \leq s \leq t$
$$E[ (M_t - M_s)^2 / \mathcal{G}_s] = E(\langle M_t \rangle - \langle M_s \rangle / \mathcal{G}_s).$$

15. Let $(M_t, \mathcal{F}_t), 0 \leq t < T$ be a right-continuous square integrable martingale. Let $0 \leq t_0 < \ldots < t_n = T$. If $f_j$ are $\mathcal{F}_{t_j}$-measurable, we can define
$$\int_0^T f_s dM_s = \lim_{n \to \infty} \sum_{j=0}^n f_{t_j} (M_{t_{j+1}} - M_{t_j})$$
for functions $f$ with $E \int_0^T f_s^2 d\langle M_s \rangle < \infty$. This definition of stochastic integral for $L^2$-martingale, extends to all integrand processes which are predictable and satisfy the condition
$$E \int_0^T f_s^2 d\langle M_s \rangle < \infty.$$

Further if $Y_t = \int_0^t f_s dM_s \,, (Y_t, \mathcal{F}_t)$ is an $L^2$-martingale and
$$\langle Y \rangle_t = \int_0^t f_s^2 d\langle M_s \rangle.$$

**Central Limit Theorems for Martingale Sequences**

Let $\{Z_k\}$ be a sequence of martingale differences such that $E(Z_k^2) < \infty, \, k > 1.$

Let
\[ S_n = Z_1 + \ldots + Z_n, \]
\[ \xi_n = \sum_{i=1}^{n} E (Z_k^* / \mathcal{F}_{k-1}), \]

and
\[ I_n = \sum_{i=1}^{n} E (Z_k^2). \]

Where \( \mathcal{F}_n \) is the \( \sigma \)-field induced by \( Z_1, \ldots, Z_n \).

a) C.L.T. of Hall (1977)

If (1) \( I_n^{-1} \xi_n \overset{P}{\rightarrow} \gamma \), where \( P(\gamma > 0) = 1 \),

and

(2) \( I_n^{-1} \sum_{i=1}^{n} E (Z_k^2 I[|Z_k| > \varepsilon I_n^{1/2}]) / \mathcal{F}_{k-1} \overset{P}{\rightarrow} 0 \), for all \( \varepsilon > 0 \).

Then,

\[ I_n^{-1/2} S_n \overset{L}{\rightarrow} Z^*, \]

and

\[ \xi_n S_n \overset{L}{\rightarrow} Z, \]

where \( Z^* = \gamma^{1/2} Z \) and \( Z \) is a standard normal variate independent of \( \gamma \).

b) C.L.T. of Basawa and Scott (1977)

If (1) \( I_n^{-1} \sum_{i=1}^{n} Z_k^2 \overset{P}{\rightarrow} \gamma \), where \( P(\gamma > 0) = 1 \),

and

(2) \( I_n^{-1} E( \max_{1 \leq k \leq n} Z_k^2 ) \rightarrow 0 \) as \( n \rightarrow \infty \),

then \( I_n^{-1/2} S_n \overset{L}{\rightarrow} Z^* \),

where \( Z^* \) is as defined in (a) above.

17. Strong law of large numbers for continuous time martingales.

Let \( V_t \) be a continuous time martingale with a quadratic
variation process \( \langle V_t \rangle = \int_0^t f_u^2 \, du \), a.s., where \( f = \{ f_u, \ u \geq 0 \} \) is \( \{ \mathcal{B}_u, \ u \geq 0 \} \) adapted. Then

\[ \langle V_T \rangle^{-1} V_T \to 0, \ a.s., \ \text{on } [\langle V_T \rangle \to \infty]. \]

18. C.L.T for Continuous Time Martingales.

**Theorem 1.** Let \( V_t = Y_t \), a.s., where \( Y \) is a stationary process with independent increments satisfying \( \mathbb{E}(Y_t^2) = t, \ t > 0 \) and \( \tau = \{ \tau_t, \ t > 0 \} \) satisfies

1) \( \tau \) is a non-decreasing function of \( t \), a.s.

2) for some function \( m(t) \to \infty \), as \( t \to \infty \),

\[
\frac{\tau_t}{m(t)} \xrightarrow{P} \gamma^2
\]

where \( P(\gamma^2 > 0) > 0 \). Let \( \tau_t(a) = \tau_{t}^{-\frac{1}{2}} Y(t) \), \( 0 \leq a \leq 1 \).

Then the stochastic process \( V_t \) converges in distribution to the Wiener process on \([0,1]\) and

\[ V_t(1) = \tau_t^{-1/2} V_t \xrightarrow{L} N(0,1). \]

Convergence here holds w.r.t any probability measure \( \mu \) absolutely continuous w.r.t

\[ P_B(. \) = P(. / B) \text{ where } B = \{ \gamma^2 > 0 \}. \]

**Theorem 2:** Let \( V_t \) be a continuous time martingale and have continuous sample paths, a.s. Then there exists a standard Wiener process \( \{ \omega_t, t \geq 0 \} \) such that

\[ V_t = \omega_{\langle V_t \rangle}, \ a.s., \ t \geq 0. \]
19. A strong law for square integrable martingales (Neveu (1975) page 151)

If \((X_n, n \in \mathbb{N})\) is a square integrable martingale such that \(X_0 = 0\) and if \((A_n, n \in \mathbb{N})\) denotes the increasing process associated with the submartingale \((X^2_n, n \in \mathbb{N})\) by the Doob decomposition, then

\[
X_n = 0(f(A_n)) \quad \text{a.s on } \{A_\infty = \infty\}
\]

for every increasing function \(f : \mathbb{R}_+ \to \mathbb{R}_+\) increasing sufficiently rapidly at infinity that \(\int_0^\infty (1 + f(t))^2 \, dt < \infty\). In particular, the functions \(f_\infty(t) = t^\alpha\) and \(f'_\infty(t) = t^{1/2} (\log^+ t)^\alpha\) are suitable if \(\alpha > 1/2\) (in both cases).