CHAPTER V

BAYES RULES FOR COMPARISONS OF SCALE PARAMETERS OF NORMAL AND EXPONENTIAL POPULATIONS

1. Introduction

The problem of comparing scale parameters of distributions is of common interest in many applications. In this chapter, we give the Bayes rules for certain decision problems concerning scale parameters of several normal and of several exponential populations obtained by Haik (1967).

In the first instance, we consider the case when no initial information is available about the parameters, and therefore we assume that, a priori, the logarithms of scale parameters are uniformly distributed. See Jeffreys (1961), Lindley et al. (1960). We also consider the case where initial information in the form of known prior distribution is available.

We obtain the Bayes rules for deciding whether the variance of one normal population is greater than variance of another or not. We consider that the losses of wrong decisions depend on the ratio of the two variances compared. We also obtain
the Bayes solution of the three-decision problem concerning
two variances - the three decisions are: a specified variance
is significantly (1) greater than, (ii) not different from, and
(iii) less than the other. We also obtain the Bayes solution
of the multiple comparisons problem concerning several vari-
ances. We discuss similar problems concerning scale parameters
of several exponential populations. These are useful for com-
paring several Poisson processes and in life testing problems.

The two-decision problem

The two-decision problem is stated thus: Given
samples of size \( n_1 \) and \( n_2 \) from two normal populations -
\( \mathcal{N}(\mu_1, \sigma_1^2) \) and \( \mathcal{N}(\mu_2, \sigma_2^2) \) respectively, we are required to
choose one of the two decisions:

\[
d_0 : \text{decision that } V_1 = \left( \frac{\sigma_2^2}{\sigma_1^2} \right) \leq 1
\]

\[
d_1 : \text{decision that } V_1 = \left( \frac{\sigma_2^2}{\sigma_1^2} \right) > 1
\]

when the loss function is

\[
L_0(V_1) = L(d_0, V_1) = \begin{cases} 
0 & V_1 \leq 1 \\
k_0 & V_1 > 1
\end{cases}
\]

and

\[
L_1(V_1) = L(d_1, V_1) = \begin{cases} 
0 & V_1 \geq 1 \\
k_1 & V_1 < 1
\end{cases}
\]

(where \( k_0 \) and \( k_1 \) are positive constants) and \( k = k_1/k_0 \) is given.
When the means $\mu_1$ are known, following Jeffreys (1961) we assume that logarithms of $\sigma_i$ are uniformly distributed a priori, i.e.,

$$p(\log \sigma_i) \propto c \text{ or } p(\sigma_i) \propto \frac{1}{\sigma_i} \quad i=1,2 \quad \ldots \quad (2.4)$$

and that $\sigma_i$ when $i = 1, 2$ are independent. When any of the means are not known, we assume that, a priori, these means are uniformly distributed and are independent of the variances.

We obtain the minimum average risk rule of the two-decision problem with respect to the prior distribution (2.4).

Define for $i = 1, 2$,

$$S_i^2 = \begin{cases} \sum(x_i - \mu_i)^2 & \text{if } \mu_i \text{ is known} \\ \sum(x_i - \bar{x}_i)^2 & \text{if } \mu_i \text{ is not known}. \end{cases} \quad \ldots \quad (3.5)$$

The variable $S_i^2$ is a $\sigma_i^2 \chi^2$ variable where $f_i = n_i$ or $n_i - 1$.

We have for the posterior joint density of $\sigma_1$ and $\sigma_2$

$$p(\sigma_1, \sigma_2 | S_1^2, S_2^2) \propto \prod_{i=1}^{2} (\sigma_i)^{-(f_i+1)} \exp \left\{ -\frac{1}{2} \frac{S_i^2}{\sigma_i^2} \right\}. \quad (3.6)$$

Making the transformation $V_1 = \sigma_2^2 / \sigma_1^2$ and integrating over $\sigma_1$ we obtain posterior density of $V_1$

$$p(V_1 | S_1^2, S_2^2) \propto V_1^{1/2} \left( 1 + \frac{S_1^2}{S_2^2} V_1 \right)^{-(f_1 + f_2)/2}. \quad (3.7)$$
Thus \( y = V_1 \left( \frac{s_1^2 / f_1}{s_2^2 / f_2} \right) \) has a posteriori \( F \)-distribution with \( f_1 \) and \( f_2 \) degrees of freedom. Let us define

\[
F_2 = \frac{s_2^2 / f_2}{s_1^2 / f_1}.
\]

The minimum average risk rule is given by choosing \( d_0 \) if

\[
\int_{0}^{\infty} \int_{0}^{\infty} f(V_1 p(V_1 s_1^2, s_2^2) d V_1 < \int_{0}^{\infty} \int_{0}^{\infty} f(V_1 p(V_1 s_1^2, s_2^2) d V_1
\]

or

\[
k_0 \int_{0}^{\infty} f(V_1 p(V_1 s_1^2, s_2^2) d V_1 p(V_1 s_1^2, s_2^2) d (V_1 f_2) < k_1 \int_{0}^{\infty} f(V_1 p(V_1 s_1^2, s_2^2) d (V_1 f_2)
\]

where \( f(x) \) is the density function of an \( F \)-variable with \( f_1 \) and \( f_2 \) degrees of freedom.

Thus the Bayes solution is

1. Accept \( d_0 \) if \( F_2 \geq F(\theta) \)
2. Accept \( d_1 \) if \( F_2 < F(\theta) \)

where \( F(\theta) \) is obtained from the \( F \)-distribution by

\[
Pr \{ F(f_1, f_2) \geq F(\theta) \} = \frac{k}{1+k}
\]

Example 1. Two independent samples give \( s_1^2 = 13431 \) and \( s_2^2 = 18061 \) with 14 and 16 degrees of freedom respectively. Find the Bayes solution of choosing between \( d_0 \) and \( d_1 \) of the decision problem defined above when \( k = 2 \) Here
\[
\frac{s_1^2 / f_1}{s_2^2 / f_2} = 0.850
\]

and

\[
\Pr \{ F(14,16) > 0.850 \} = 0.617 < 0.667 = \frac{k}{1+k}
\]

Therefore \( F_2 \) is greater than \( F(14,16) \), and the Bayes rule gives: Accept the decision that \( \sigma_1^2 \) is greater than \( \sigma_2^2 \).

3. The three-decision problem

This problem is concerned with choosing one of the following three decisions:

\( \bar{d}_1 \) : decision that \( v_1 = (\sigma_2^2 / \sigma_1^2) < \frac{1}{1+\delta} \)

\( \bar{d}_0 \) : decision that \( \frac{1}{1+\delta} \leq v_1 \leq 1+\delta \) \hspace{1cm} \cdots \hspace{1cm} (3.1) \)

\( \bar{d}_1 \) : decision that \( v_1 > 1+\delta \)

where \( \delta \) is some unspecified positive boundary value.

In the language of the experimenter, \( \bar{d}_1, \bar{d}_0 \) and \( \bar{d}_1 \) are the decisions that \( \sigma_1^2 \) is significantly greater than, not different from and less than \( \sigma_2^2 \).

Let \( \bar{d}_0^1 \) and \( \bar{d}_1^1 \) denote the decisions

\( \bar{d}_0^1 \) : decision that \( v_1 \leq 1+\delta \) \hspace{1cm} \cdots \hspace{1cm} (3.2) \)

\( \bar{d}_1^1 \) : decision that \( v_1 > 1+\delta \).
Similarly let \( d_{o}^{-1} \) and \( d_{1}^{-1} \) denote the decisions:

\[
\begin{align*}
\quad & d_{o}^{-1} : \text{decision that } V_{1}^{-1} = (\sigma_{1}^{2} / \sigma_{2}^{2}) \leq 1 + \varepsilon \\
\quad & d_{1}^{-1} : \text{decision that } V_{1}^{-1} = (\sigma_{1}^{2} / \sigma_{2}^{2}) > 1 + \varepsilon 
\end{align*}
\]  

(3.3)

Each of the three decisions \( d_{-1}, d_{o}, \) and \( d_{1} \) is equivalent to two joint decisions, one from (3.2) and one from (3.3) as:

\[
\begin{align*}
\quad & d_{-1} \text{ to } d_{o} \text{ with } d_{1}^{-1} \\
\quad & d_{o} \text{ to } d_{1} \text{ with } d_{o}^{-1} \\
\quad & d_{1} \text{ to } d_{1} \text{ with } d_{o}^{-1} \text{.} 
\end{align*}
\]

(3.6)

We obtain the Bayes solution of choosing between the three decisions in (3.1) by compounding two two-decision rules according to (3.4).

The three-decision problem is stated thus — Given samples of size \( n_{1} \) and \( n_{2} \) from two normal populations

\[ N(\mu_{1}, \sigma_{1}^{2}) \] and \[ N(\mu_{2}, \sigma_{2}^{2}) \] respectively, it is required to choose one of the three decisions \( d_{-1}, d_{o}, \) and \( d_{1} \) given in (3.1), when the loss function is given by

\[
\begin{align*}
L(d_{-1}) & = \begin{cases} 
0 & V_{1} \leq 1 \\
k_{o} + k_{1} & V_{1} > 1 
\end{cases} \\
L(d_{o}) & = \begin{cases} 
k_{o} & V_{1} < 1 \\
0 & V_{1} = 1 \\
k_{o} & V_{1} > 1 
\end{cases} \\
L(d_{1}) & = \begin{cases} 
k_{o} + k_{1} & V_{1} > 1 \\
0 & V_{1} \leq 1 
\end{cases} 
\end{align*}
\]

(3.6)
with \( k = k_1/k_0 \geq 1 \) specified. We assume that the prior distribution of \( \alpha_1, \alpha_2 \) is as given in section 2.

The Bayes solution is obtained as follows:

We first express the losses of each of the three decisions \( d_{-1}, d_0 \) and \( d_1 \) as sum of the losses of its components defined in (3.4):

Consider that the losses of the two decisions \( d_0^1 \) and \( d_1^1 \) of (3.2) are

\[
L(d_0^1) = \begin{cases} 
0 & v_1 \leq 1 \\
k_0 & v_1 > 1 
\end{cases} \quad (3.6)
\]

\[
L(d_1^1) = \begin{cases} 
0 & v_1 \geq 1 \\
k_1 & v_1 < 1 
\end{cases}
\]

and that the losses of the two decisions \( d_0^{-1} \) and \( d_1^{-1} \) are

\[
L(d_0^{-1}) = \begin{cases} 
0 & v_1^{-1} \leq 1 \\
k_0 & v_1^{-1} > 1 
\end{cases} \quad (3.7)
\]

\[
L(d_1^{-1}) = \begin{cases} 
0 & v_1^{-1} \geq 1 \\
k_1 & v_1^{-1} < 1 
\end{cases}
\]

It is easy to verify that the losses of each of the three decisions \( d_{-1}, d_0 \) and \( d_1 \) are the sum of the losses of its two components defined in (3.4).

Let \( s_1^1, s_2^1 \) be as defined in (2.6). Let \( \varphi_1 = \varphi_1(s_1^1, s_2^1) = 0 \) or 1 denote the not making or making the decisions \( d_1, i = -1, 0 \)
and 1. Let $\varphi^2$ be defined similarly for the decisions $\xi^2$.

Because of (34), we have

$$\varphi = (\varphi_0^1 \varphi_1 \varphi_2) = (\varphi_0^1 \varphi_0^2, \varphi_0^1 \varphi_0^{-1}, \varphi_1 \varphi_0^{-1}) \quad \ldots \quad (3.2)$$

When the compatibility condition holds, i.e., when

$$\varphi_1 \varphi_1^{-1} = 0 \quad \ldots \quad (3.3)$$

the minimum average risk rule $\varphi$ is obtained from the minimum average risk rules $\varphi^1 = (\varphi_0^1, \varphi_1^1)$ and $\varphi^{-1} = (\varphi_0^{-1}, \varphi_1^{-1})$.

The two-decision rule $\varphi^1$ is as given in Section 2:

$$\varphi^1 = \begin{cases} 0 & \text{if } F_2 = \frac{S_1^2 / f_1}{S_2^2 / f_2} > F_2(o) \\ 1 & \text{if } F_2 < F_2(o) \end{cases} \quad \ldots \quad (3.10)$$

where $F_2(o)$ is obtained from the equation

$$\Pr \{F(f_1, f_2) > F_2(o)\} = \frac{k}{1 + k} \quad \ldots \quad (3.11)$$

Also it can be easily verified that the two-decision rule $\varphi^{-1}$ is

$$\varphi^{-1} = \begin{cases} 0 & \text{if } F_2 < F_2(o) \\ 1 & \text{if } F_2 > F_2(o) \end{cases} \quad \ldots \quad (3.12)$$

where $F_2(o)$ is obtained from the equation

$$\Pr \{F(f_1, f_2) < F_2(o)\} = \frac{k}{1 + k} \quad \ldots \quad (3.13)$$
Since \( k > 1 \), it is clear that \( F_2^{(\infty)} > F_2(0) \) and therefore the condition of compatibility \( \Phi_1 \Phi_1^{-1} = \Phi_1 \Phi_1^{-1} = 0 \) holds for all \( \Phi_1 \) and \( \Phi_2 \).
Thus the Bayes rule \( \Phi \) is
\[
\Phi \equiv (\Phi_1, \Phi_2, \Phi_0) = \begin{cases} 
(1, 0, 0) & \text{if } F_2^{(\infty)} > F_2(0) \\
(0, 1, 0) & \text{if } 0 < F_2(0) < F_2^{(\infty)} \\
(0, 0, 1) & \text{if } F_2 < F_2(0) 
\end{cases} \tag{3.14}
\]

Example 2. Given \( \Phi_1^2 = 13431 \) and \( \Phi_2^2 = 18061 \) based on 14 and 16 degrees of freedom as in example 1, find the Bayes solution of choosing between \( d_{-1} \), \( d_0 \) and \( d_1 \) defined above when \( k = 3 \) in the above loss function.

Of the two decisions \( d_1^1 \) and \( d_1^1 \), we see from the solution of example 1 that the rule chooses \( d_0^1 \). And since
\[
\Pr \{ F(14, 16) < 0.350 \} = 0.383 < 0.667 = \frac{k}{1 + k}
\]
\( F_2 \) is less than \( F_2^{(\infty)} \), and of the two decisions \( d_0 = 1 \) and \( d_1 = 1 \), the rule chooses \( d_0 = 1 \). Therefore the Bayes rule chooses the decision that \( \Phi_1^2 \) and \( \Phi_2^2 \) are not significantly different.

4. Decision problems where losses depend on the ratio of the variances

In this section we consider two types of loss functions.

(a) First type of loss function
The two-decision problem in this case is stated thus:

Given two independent variables $S_1$ and $S_2$, it is required to choose one of the two decisions $d_0$ and $d_1$ given in (2.1) when the loss function is

$$L(d_0) = \begin{cases} 0 & \frac{\sigma_2^2}{\sigma_1^2} \leq 1 \\ k_0 v_1 & \frac{\sigma_2^2}{\sigma_1^2} > 1 \end{cases} \quad (4.1)$$

and

$$L(d_1) = \begin{cases} 0 & v_1 \geq 1 \\ \frac{k_1}{v_1} & v_1 < 1 \end{cases} \quad (4.2)$$

where the constant $k = k_1/k_0$ is specified and the prior distributions of $\sigma_1$ and $\sigma_2$ are as given in Section 2.

The Bayes rule states: Choose $d_0$ if

$$\int k_0 v_1 p(V_1 \mid S_1, S_2) dv_1 < \int \frac{1}{v_1} p(V_1 \mid S_1, S_2) dv_1 \quad (4.3)$$

and choose $d_1$ otherwise.

Now we have

$$\int V_1 p(V_1 \mid S_1, S_2) dv_1 = \frac{1}{F_2} \int_{F_1/2}^{F_1/2} y \left( \frac{F_1}{F_2} \right)^{y/2} \left( 1 + \frac{F_1}{F_2} \right)^{1/2} dy$$

where $F_1$ and $F_2$ are the parameters of the prior distributions of $\sigma_1^2$ and $\sigma_2^2$ respectively.
\[
\frac{1}{f_2} \frac{f_2}{B \left( \frac{f_2}{2} - 1, \frac{f_1}{2} + 1 \right) \Gamma_x \left( \frac{f_2}{2} - 1, -\frac{f_1}{2} + 1 \right)}{B \left( \frac{f_1}{2}, \frac{f_2}{2} \right)} = h_1(f_1, f_2, f_2) \text{ say}.
\]

Here \(I_x(p, q)\) is the incomplete beta function tabulated by Pearson (1948) and

\[
x = \frac{f_2}{f_2 + f_1 f_2} \quad \text{..} \quad (4.4)
\]

and \(F_2\) is as defined in (2.9).

Also it can be shown that

\[
\int_0^1 \frac{p(v_1 s_1^2, s_2^2)}{v_1} dv_1 = h_1(f_2, f_1, F_2^{-1}) \quad \text{..} \quad (4.5)
\]

where \(h_1(f_1, f_2, F_2)\) is defined in (4.4)

The inequality (4.5) can be written as

\[
\alpha_1(F_2) = \frac{h_1(f_1, f_2, F_2)}{h_1(f_2, s_1, F_2^{-1})} < k \quad \text{..} \quad (4.6)
\]

Since \(\alpha_1(F_2)\) is a monotonic decreasing function of \(F_1\),

the Bayes solution is given by
Accept \( d_0 \) if \( P_2 > P_2^{(c)} \) \hspace{1cm} (4.3)

Accept \( d_1 \) if \( P_2 < P_2^{(c)} \) \hspace{1cm} (4.4)

where \( P_2^{(c)} \) is the solution of

\[ g_1(P_2) = k . \] \hspace{1cm} (4.5)

Now consider the three-decision problem of choosing one of the three decisions \( d_{-1}, d_0 \) and \( d_1 \) defined in (3.1) when the loss function is

\[
L(d_{-1}) = \begin{cases} 
0 & V_1 \leq 1 \\
(k_0 + k_1) V_1 & V_1 > 1 
\end{cases}
\]

\[
L(d_0) = \begin{cases} 
\frac{k_0}{V_1} & V_1 < 1 \\
0 & V_1 = 1 \\
k_0 V_1 & V_1 > 1 
\end{cases}
\]

\[
L(d_1) = \begin{cases} 
\frac{(k_0 + k_1)}{V_1} & V_1 < 1 \\
0 & V_1 \geq 1 
\end{cases}
\]

where the constant \( k = k_1 / k_0 > 1 \) is specified.

The Bayes solution of this three-decision problem is

Accept \( d_{-1} \) if \( P_2 > P_2^{(\infty)} \)

Accept \( d_0 \) if \( P_2^{(\infty)} < P_2 < P_2^{(0)} \) \hspace{1cm} (4.6)

Accept \( d_1 \) if \( P_2 < P_2^{(0)} \)
where \( f_2^{(0)} \) and \( f_2^{(\infty)} \) are respectively the solutions of 
\[ g_1(f_2) = k \quad \text{and} \quad g_2(f_2) = 1/k. \]
Alternatively, the three conditions on the right side of (4.11) are given by 
\[ 1/k < g_1(f_2) < k \quad \text{and} \quad g_2(f_2) > k \]
in the same order.

Example 3. Given \( S_1^2 \) and \( S_2^2 \) of example 1, use the Bayes rule for choosing between \( d_0 \) and \( d_1 \) when \( k = k_1/k_0 \) equals 2 in the loss function (4.10).

We have
\[
g_1(f_2) = \frac{h_1(f_1, f_2, f_2)}{h_1(f_2, f_1, f_2, f_2)}
\]
\[
= \frac{f_2^{(2)} - 1}{f_1 f_2^{(2)} - 1} \times \frac{I_x\left(\frac{f_2^{(2)} - 1}{2}, \frac{1}{2} + 1\right)}{I_{1-x}\left(\frac{f_1^{(2)} - 1}{2}, \frac{f_2^{(2)} + 1}{2}\right) = 1.556.}
\]

Since \( 1/k < g_1(f_2) < k \), we accept decision \( d_0 \).

(b) Second type of loss function

We here consider the two-decision problem stated above, when the loss function given by (4.1) and (4.2) is replaced by

\[
L(d_0) = \begin{cases} 
0 & V_1 \leq 1 \\
kv_1(V_1-1) & V_1 > 1
\end{cases} \quad (4.12)
\]

and

\[
L(d_1) = \begin{cases} 
0 & V_1 \geq 1 \\
k_1 V_1(V_1-1)V_1 < 1
\end{cases} \quad (4.13)
\]
The Bayes rule is to choose \( d_0 \) if
\[
g_2(f_2) = \frac{h_2(f_1, f_2, f_2)}{h_2(f_2, f_1, f_2 - 1)} < k \quad \ldots \quad (4.14)
\]
where
\[
h_2(f_1, f_2, f_2)
\]
\[
= \frac{1}{f_2} \frac{B\left(\frac{f_2}{2} - 1, \frac{f_1}{2} + 1\right)}{B\left(\frac{f_2}{2}, \frac{f_2}{2}\right)} I_x\left(\frac{f_2}{2} - 1, \frac{f_1}{2} + 1\right) - I_x\left(\frac{f_2}{2}, \frac{f_1}{2}\right) \quad \ldots \quad (4.15)
\]
Since \( g_2(f_2) \) is a monotonic decreasing function of \( f_2 \), the Bayes rule is given by

Accept \( d_0 \) if \( f_2 > f_2^{(o)} \) \ldots \quad (4.16)

Accept \( d_1 \) if \( f_2 < f_2^{(o)} \)

where \( f_2^{(o)} \) is the solution of
\[
g_2(f_2) = k \quad \ldots \quad (4.17)
\]

For the three-decision problem of choosing between \( d_{-1}, d_0, \) and \( d_1 \) when the loss function
\[
L(d_0) = \begin{cases} 
0 & V_1 \leq 1 \\
(k_0 + k_1)(V_1 - 1) & V_1 > 1 
\end{cases} \quad \ldots \quad (4.18)
\]
\[
L(d_1) = \begin{cases} 
k_0(V_1 - 1) & V_1 < 1 \\
0 & V_1 = 1 \\
k_0(V_1 - 1) & V_1 > 1 
\end{cases} \quad \ldots \quad (4.18)
\]
\[
L(d_2) = \begin{cases} 
(k_0 + k_1)(V_1 - 1) & V_1 < 1 \\
0 & V_1 \geq 1 
\end{cases}
\]
with a specified \( k = k_1/k_0 > 1 \), the Bayes rule is as given in (4.11) with \( f_2(o) \) and \( F_2(oo) \) being respectively the solutions of \( g_2(F_2) = k \) and \( g_2(F_2) = 1/k \).

Example 4. Given \( s_1^2 \) and \( s_2^2 \) of example 1, use the Bayes rule for choosing between the three decisions \( d_1, d_0 \) and \( d_1 \) when the loss function is (4.18) with \( k = 2 \).

We have

\[
g_2(F_2) = \frac{f_2}{(f_2 - 2)} x_2^{-1} x_2^{-1} + 1 - x_2^{-1} = \frac{1}{x_2^{-1}}
\]

and therefore we accept decision \( d_1 \).

5. Decision problems where losses depend on the ratio of standard deviations

(a) First type of loss function

For the two-decision problem of Section 2, we consider that the losses of \( d_0 \) and \( d_1 \) are given by

\[
L(d_o) = \begin{cases} 0 & \text{if } v_1 = \sigma_2^2/\sigma_1^2 \leq 1 \\ k_0 v_1^{1/2} & \text{if } v_1 > 1 \end{cases} \quad \cdots \quad (5.1)
\]

and

\[
L(d_1) = \begin{cases} 0 & \text{if } v_1 = \sigma_2^2/\sigma_1^2 = 1 \\ k_1 v_1^{-1/2} & \text{if } v_1 < 1 \end{cases} \quad \cdots \quad (5.2)
\]

and \( k = k_1/k_0 \) is specified.
Since

\[ \int_{V_1} v_1^{1/2} p(v_1, v_2^2, s_2^2) d v_1 \]

\[ = \frac{1}{\sqrt{2}} \frac{f_2^{1/2}}{f_1} \frac{B\left(\frac{f_2-1}{2}, \frac{f_1+1}{2}\right)}{B\left(\frac{f_1}{2}, \frac{f_2}{2}\right)} I_x \left(\frac{f_2}{2}, \frac{1}{2}\right) \]

\[ = h_3(f_1, f_2, v_2) \text{ say,} \quad \ldots \quad (5.3) \]

and

\[ \int_{V_1} v_1^{1/2} p(v_1, v_2^2, s_2^2) d v_1 = h_2(f_2, f_1, v_2^{-1}) \]

\[ \ldots \quad (5.4) \]

(here \( f_2, v_2 \), and \( I_x \) are as defined earlier), the Bayes rule is

Accept \( d_0 \) if \( P_2 > P_2^{(o)} \)

Accept \( d_1 \) if \( P_2 < P_2^{(o)} \) \ldots \quad (5.5)

where \( P_2^{(o)} \) is the solution of

\[ g_2(P_2) = \frac{h_3(f_1, f_2, v_2)}{h_2(f_2, f_1, v_2^{-1})} = k \quad \ldots \quad (5.6) \]

It easily follows that the Bayes rule of the three-decision problem when losses are
Example 5. Given $S_1^2$ and $S_2^2$ of example 1, use the Bayes rule for choosing between $d_{-1}$, $d_0$, and $d_1$ when the loss function if (5.7) with $k = 2$.

We have

$$g_3(F_2) = \frac{f_2^{-1}}{f_1^{-1}} = \frac{f_2 f_1}{f_1 f_2} = \frac{f_2^{-1} f_1^{-1}}{f_2^{-1} f_1^{-1}} = 1/70.9.$$ 

Since $1/k < g_3(F_2) < k$, we accept decision $d_0$.

(b) Second type of loss function

In this case we consider that the loss function of the two-decision problem is

$$L(d_{-1}) = \begin{cases} 0 & v_1 \leq 1 \\ (k_0 + k_1) v_1^{1/2} & v_1 > 1 \end{cases}$$

$$L(d_0) = \begin{cases} k_0 v_1^{1/2} & v_1 < 1 \\ 0 & v_1 = 1 \\ k_0 v_1^{1/2} & v_1 > 1 \end{cases}$$

$$L(d_1) = \begin{cases} (k_0 + k_1) v_1^{1/2} & v_1 < 1 \\ 0 & v_1 \geq 1 \end{cases}$$

with a specified $k = k_1/k_0 > 1$ as given in (4.11) with $p(0)$ and $p(\infty)$ being respectively the solutions of $g_3(F_2) = k$ and $g_3(F_2) = 1/k$. 
\[ L(d_0) = \begin{cases} 0 & \text{if } V_1 \leq 1 \\ k_0(V_1^{1/2} - 1) & \text{if } V_1 > 1 \end{cases} \quad \ldots \quad (5.8) \]

and
\[ L(d_1) = \begin{cases} 0 & \text{if } V_1 \geq 1 \\ k_1(V_1^{1/2} - 1) & \text{if } V_1 < 1 \end{cases} \quad \ldots \quad (5.9) \]

where \( k = k_1/k_0 \) is specified.

Let
\[
\begin{align*}
E_4(f_1, f_2, f_2') &= \frac{f_2 - 1}{f_2} \cdot \frac{f_1 + 1}{f_1} \\
&= \left( \frac{f_2}{f_2} \right)^{1/2} \frac{B\left(\frac{f_2 - 1}{2}, \frac{f_1 + 1}{2}\right)}{B\left(\frac{f_2}{2}, \frac{f_1}{2}\right)} - \frac{B\left(\frac{f_2 - 1}{2}, \frac{f_1}{2}\right)}{B\left(\frac{f_2}{2}, \frac{f_1}{2}\right)} \quad \ldots \quad (5.10)
\end{align*}
\]

The Bayes rule is

Accept \( d_0 \) if \( F_2 > F_2^{(o)} \) \ldots \quad (5.11)

Accept \( d_1 \) if \( F_2 < F_2^{(o)} \)

where \( F_2^{(o)} \) is the solution of
\[
E_4(F_2) = \frac{h_4(f_1, f_2, F_2)}{h_4(f_2, f_1, F_2^{-1})} = k. \quad \ldots \quad (5.12)
\]

It also follows that the Bayes rule of the three-decision problem with losses
with a specified $k = k_1/k_0 > 1$ is as given in (4.11), with $F_2(\alpha)$ and $F_2(\infty)$ being respectively the solutions of $g_4^{(F_2)} = k$ and $g_4^{(F_2)} = 1/k$.

Example 6. Given $S_1^2$ and $S_2^2$ of example 1, use the Bayes rule for choosing between $d_{-1}$, $d_0$ and $d_1$ when the loss function is given by (5.13) with $k = 2$.

We have

\[
g_4^{(F_2)} = \frac{f_2^{1/2} f_2^{-1} f_1 + 1}{\beta^{-2}} \frac{f_2^{1/2} f_2^{-1} f_1 + 1}{\beta^{-2}} = \frac{f_1 f_2}{\beta^{-2}} I_1 - \frac{f_1 f_2}{\beta^{-2}} I_2
\]

\[
= \frac{2.1485 > k}{2.1485 > k}
\]

and therefore we accept the decision $d_1$. 

The multiple comparisons problem is stated thus: Given samples of size $n$ each from $p$ normal populations $N(\mu, \sigma_i^2)$, $i = 1, 2, \ldots, p$, it is required to choose one of the three decisions

\[ d_1(ij) : \ \text{decision that} \ \frac{\sigma_j^2}{\sigma_i^2} < \frac{1}{1 + \delta} \]

\[ d_0(ij) : \ \text{decision that} \ \frac{1}{1 + \delta} \leq \frac{\sigma_j^2}{\sigma_i^2} \leq 1 + \delta \ldots \quad (6.1) \]

\[ d_1(ij) : \ \text{decision that} \ \frac{\sigma_j^2}{\sigma_i^2} > 1 + \delta \]

(where $\delta$ is some unspecified positive boundary value) simultaneously for all $ij$ belonging to the set of pairs $N = \{12, 13, \ldots, p-1p\}$.

In the space of $\sigma^2 = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2)$ let

\[ w_1(ij) : \ \frac{\sigma_j^2}{\sigma_i^2} < \frac{1}{1 + \delta} \]

\[ w_0(ij) : \ \frac{1}{1 + \delta} \leq \frac{\sigma_j^2}{\sigma_i^2} \leq 1 + \delta \ldots \quad (6.2) \]

\[ w_1(ij) : \ \frac{\sigma_j^2}{\sigma_i^2} > 1 + \delta \]

denote the three mutually exclusive and exhaustive regions.
corresponding to the decisions \( d_{ij}, d_0(ij) \) and \( d^*_1(ij) \) respectively.

The regions \( W_1, W_2, \ldots, W_M \) of the multiple comparisons problem are the nonempty intersections

\[
W_i = \bigcap_{i \in N} T_{ij}^{(1)}
\]

and the decision system consists of the \( M \) corresponding decisions \( d_1, d_2, \ldots, d_M \), i.e.,

\[
d_i : \text{decision that } \sigma_i^2 \text{ in } l = 1, 2, \ldots, M. \quad (C.6)
\]

This multiple decision system for variances and its regions are similar to those considered by Duncan (1961, 1965) for means; see also Winer and Freund (1967).

We suppose that the losses of \( d_1, d_2, \ldots, d_M \) are the sum of the losses of the component decisions, i.e.,

\[
L_i(\sigma^2_j) = L(\sigma^2_j, d_i) = \sum_{i \in N} L_{ij}(V_{ij})
\]

\[
t_{ij}^{(1)} = -1, 0 \text{ or } 1 \quad \text{for } l = 1, 2, \ldots, M,
\]

where \( L_{ij}(V_{ij}) \) are the losses defined for \( V_{ij} = \sigma_j^2/\sigma_i^2 \) by the three-decision loss function of Sections 3, 4 or 5 with \( k = k_1/k_0 > 1 \).

We now obtain the Bayes solution of this multiple
decision problem when a priori $a^j$, \( i = 1, 2, \ldots, p \) are independent and each is distributed as in (2.4).

Let $s^1_s^2, \ldots, s^2_p$ be as defined in (2.5). Let

\[ g_1 = g_1(s^1, \ldots, s^2_p) = 0 \text{ or } 1 \]

denote the not making or making the decision $d^1$, $l = 1, 2, \ldots, N$. Let $g_1(ij)$ be defined similarly for the decisions $d^1(ij)$, $t^1_{ij} = -1, 0$ and $1$ and $ij \in D$.

From the additive loss assumptions, it can be proved that the average risk of the decision rule $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_N)$ is expressible as the sum of the average risks of the $p(p-1)/2$ component three-decision rules $\mathcal{G}(ij) = (\mathcal{G}_{1}(ij), \mathcal{G}_{2}(ij), \mathcal{G}_{3}(ij))$ provided that the components are compatible. The compatibility condition may be written as

\[ \prod_{ij \in D} g_1(ij) = 0 \quad t^1_{ij} = -1, 0 \text{ or } 1 \quad \ldots \quad (G_0) \]

for all products leading to incompatible decisions not included in $d_1, d_2, \ldots, d_N$.

When the compatibility condition holds, the Bayes rule $\mathcal{G} = (\mathcal{G}_1, \ldots, \mathcal{G}_N)$ is obtained from the products

\[ \mathcal{G}_i = \prod_{ij \in D} g_1(ij) \quad t^1_{ij} = -1, 0 \text{ or } 1 \quad \ldots \quad (G_1) \]

of the elements of the three-decision Bayes rules $\mathcal{G}(ij)$.
Since the constant k involved in the loss function is greater than one, the condition of compatibility holds, and the Bayes solution is obtained by the simultaneous application of the following three-decision Bayes rules for all $ij \in \mathbb{N}$:

$$
\theta(ij) = (\theta_{-1}(ij), \theta_0(ij), \theta_1(ij))
$$

$$
= \begin{cases} 
(1 \, 0 \, 0) & \text{if } \frac{s_i^2}{s_j^2} > \frac{p(oo)}{p(o)} \\
(0 \, 1 \, 0) & \text{if } \frac{s_i^2}{s_j^2} < \frac{p(oo)}{p(o)} \\
(0 \, 0 \, 1) & \text{if } \frac{s_i^2}{s_j^2} < \frac{p(oo)}{p(o)} 
\end{cases}
$$

(6.8)

where $p(o)$ and $p(oo)$ are given by the corresponding three
decision rules.

Example 7. Analysis of experiments conducted at six
experimental stations in Rajasthan gives $s_1^2 = 5776$, $s_2^2 = 3628$, $s_3^2 = 4518$, $s_4^2 = 3562$, $s_5^2 = 7056$ and $s_6^2 = 5526$ as unbiased esti-
mates of $\sigma_i^2$, $i = 1, 2, \ldots, 6$, respectively, each having 20
degrees of freedom. Find the Bayes solution of the multiplo
decision problem stated above when the loss function of the
component three-decision problems is given by (3.5) with $k = 2$.

The values of $P(o)$ and $P(oo)$ for $f = 20$ are $P(o) = 0.323$ and
$P(oo) = (P(o))^{-1} = 1.216$. We write $s_i^2$, $i = 1, 2, \ldots, 6$ in the
ascending order of magnitude as

$$
s_3 < s_6 < s_1 < s_5 < s_2 < s_4
$$

and compute the ratios $F_{ij} = \frac{s_i^2}{s_j^2}$, $F_{36} = 1.225$, $F_{61} = 1.044$. 

\[
P_{15} = 1.222, \ P_{52} = 1.135, \ P_{24} = 1.187 \text{ and } P_{54} = 1.350. \text{ On applying the rules given in (6.8), we accept the decision}
\]
\[
\frac{\sigma_3^2 \sigma_6^2 \sigma_1^2 \sigma_5^2 \sigma_2^2 \sigma_4^2}{\sigma_3^2}.
\]

In this presentation any two \(\sigma^2\)'s underlined by a common line do not differ significantly; when not underlined by a common line the one on the right is significantly greater than the one on the left.

7. Decision procedures for a different prior distribution

In this section we consider a different prior distribution of the variances. This distribution results from the conjugate prior density procedure (see Haffie and Schlaifer (1961)).

We suppose that the prior density of \((1/\sigma_i^2)\) is

\[
p\left(\frac{1}{\sigma_i^2}\right) \propto \left(\frac{1}{\sigma_i^2}\right)^{h_i/2-1} \exp\left\{-\frac{1}{2} \frac{v_i^2}{\sigma_i^2}\right\} \quad i = 1, 2, \ldots (7.1)
\]

and that \(\sigma_i\) are independent a priori. When the means of the normal populations are not known, we assume that these means are a priori uniformly distributed and are independent of the variances. Thus given samples of size \(n_1\) and \(n_2\) from two normal populations \(N(\mu_1, \sigma_1^2)\) and \(N(\mu_2, \sigma_2^2)\), we have the posterior
The joint density of $\sigma_1^2, \sigma_2^2$

$$p(\sigma_1^2, \sigma_2^2 | s_1^2, s_2^2) \propto \prod_{i=1}^{2} \left( \frac{1}{\sigma_i^2} \right)^{-(f_1+h_1)/2+1} \exp \left\{ -\frac{1}{2} \frac{s_i^2 + w_i^2}{\sigma_i^2} \right\}$$

where $s_i^2, w_i^2$ are as defined in Section 2.

Making the transformation $V_1 = \sigma_2^2/\sigma_1^2$ and integrating out $\sigma_1^2$, we obtain the posterior density of $V_1$

$$p(V_1 | s_1^2, s_2^2) \propto V_1^{-(f_1+h_1)/2-1} \left( 1 + \frac{s_2^2 + w_2^2}{s_1^2 + w_1^2} V_1 \right)^{-1}$$

Thus the posterior distribution of the variable

$$y' = V_1 \left\{ \frac{s_1^2 + w_1^2}{(f_1+h_1)} \right\} / \left\{ \frac{s_2^2 + w_2^2}{(f_2+h_2)} \right\}$$

is an $F$-distribution with $f_1 + h_1$ and $f_2 + h_2$ degrees of freedom.

We shall now give briefly the Bayes solutions of the decision problems considered in Sections 2 to 6 with respect to this prior distribution.

The Bayes solution of the two-decision problem of Section 2 is
Accept $d_0$ if $F_2(W) = \frac{(s_1^2 + u_1^2)/(f_1 + h_1)}{(s_2^2 + u_2^2)/(f_2 + h_2)} > F_2'(o)$

Accept $d_1$ if $P_2(W) < F_2'(o)$

where $F_2'(o)$ is obtained from

$$P_2(F_1 + h_1, f_2 + h_2) > F_2'(o) = \frac{k}{1+k}$$

In a similar way the Bayes solutions of the decision problems of Sections 3, 4 and 5 with respect to this prior distribution are obtained from the solutions given therein after replacing $P_2$, $F_2'(o)$ and $F_2'(oo)$ by $F_2(W)$, $F_2'(o)$ and $F_2'(oo)$ respectively. The values of $F_2'(o)$ and $F_2'(oo)$ are obtained from equations corresponding to $F_2(O)$ and $F_2(oo)$ after replacing $f_1$ by $f_1 + h_1$ and $f_2$ by $f_2 + h_2$.

In the multiple decision problem of Section 6 of comparing several variances, we assume a priori that $\sigma_i^2$, $i = 1, 2, \ldots, p$ are independent and their distributions have the same pair of parameters $W$ and $h$, i.e.,

$$p\left(\frac{1}{\sigma_i^2}\right) \propto \left(\frac{1}{\sigma_i^2}\right)^{\frac{h/2-1}{2}} \exp\left\{ -\frac{1}{2\sigma_i^2} \right\} i = 1, 2, \ldots, p. \quad (7.7)$$

Thus given $\sigma_1^2$ independent $\sigma_i^2$, $i = 1, 2, \ldots, p$ variables, $i = 1, 2, \ldots, p$. 


the Bayes solution is obtained from that in Section 6 after replacing $s_i^2$ by $s_i^2 + w^2$, $i = 1, 2, \ldots, p$, and $p_2(c)$ and $p_2(\infty)$ by $p_2'(c)$ and $p_2'(\infty)$ respectively.

8. Decision problems concerning scale parameters of exponential distributions

The probability density of an exponential population with origin $\gamma$ has the form

$$p(x|\gamma, \theta) = \frac{1}{\theta} \exp \left\{ - \frac{(x-\gamma)}{\theta} \right\}, \quad x \geq \gamma$$

$$0, \quad \text{otherwise},$$

where $\gamma$ can be regarded as location parameter and $\theta$ the unknown scale parameter.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sample of size $n$ from $p(x|\gamma, \theta)$. When $\gamma$ is known, we assume that $\log \theta$ has a uniform prior distribution. Then the posterior density of $\theta$ given $\mathbf{x}$ is

$$p(\theta | \mathbf{x}) = p(\theta | \mathbf{z}) \propto \theta^{-(p+1)} \exp \left\{ - \frac{\mathbf{z}}{\theta} \right\}$$

where

$$\mathbf{z} = \Sigma (x_i - \gamma) \quad \text{and} \quad p = n.$$
When $\gamma$ is not known, we assume that a priori $\gamma$ and $\log \xi$ are independent and uniformly distributed. The posterior marginal distribution of $\xi$ is then given by (3.2) where

$$Z = x(x_1 - x^*)$$

and $x^*$ is the smallest observation in the particular sample. (See Guttman and Tiao (1964)).

Now consider the following decision problem: Given independent samples of sizes $n_1$ and $n_2$ from two exponential populations $p(x_1|\gamma_1, \theta_1)$ and $p(x_1|\gamma_2, \theta_2)$, we have to choose between two decisions $d_0: V_1 = (\beta_2/\beta_1) \leq 1$ and $d_1: V_1 > 1$ when the loss function is given by (1.8) and (1.3).

Let us assume that a priori the parameters of the two populations are independent and have distributions stated above. The posterior distribution of

$$y = \frac{\beta_2 z_1/\rho_1}{\beta_1 z_2/\rho_2}$$

given the sample observations is $F$ with $f_1 = 2\rho_1$ and $f_2 = 2\rho_2$ degrees of freedom. Here $z_1, \rho_1$ and $z_2, \rho_2$ refer to the two samples and are defined according to (3.3) or (3.4).

The Bayesian solution of the two-decision problem is as given in (1.11) where

$$f_2 = \frac{z_2 \rho_1}{z_2 \rho_2}$$

and $f_1 = 2\rho_1$ and $f_2 = 2\rho_2$. (3.6)
The Bayes solutions for the decision problems concerning the ratios of $\beta$'s similar to those concerning ratios of $\sigma^2$'s discussed in Sections 3 to 6 are given by defining $\tilde{F}$'s, $\tilde{F}_1$, $\tilde{F}_2$, etc., according to (3.6) in the corresponding solutions for variances.

We may also consider the prior distribution of $\beta$ which follows from the conjugate prior argument. Thus if a priori $\beta$'s are independent and have densities

$$p(\beta_i) = (\beta_i)^{-\left(\gamma_i + 1\right)} \exp\left(-\frac{\gamma_i}{\beta_i}\right), \quad i = 1, 2 \quad (3.7)$$

the posterior distribution of

$$y = \frac{\beta_1 (z_1 + z^*) / (p_1 + r_1)}{\beta_1 (z_1 + z^*) / (p_1 + r_1) \beta_2 (z_2 + z^*) / (p_2 + r_2)}$$

given sample observations is $F$ with $f_1 = 2(p_1 + r_1)$ and $f_2 = 2(p_2 + r_2)$ degrees of freedom. Using this distribution the Bayes solutions of the decision problems can be obtained by making the obvious changes.

9. Some remarks

9.1 The problem of testing the homogeneity among variances of $p$ normal populations when we have independent variables $\tilde{S}_1^2$ distributed as $\sigma^2 \chi^2_1, \quad i = 1, 2, \ldots, p$, is...
commonly solved by the Bartlett's test of homogeneity or by utilizing the distribution of $F_{\text{max}}$ (maximum $F$ ratio). These methods depend on the number of variances involved. A striking feature of the Bayes solution of the multiple comparisons problem is that the solution is independent of the number of variances involved. The test procedure by using the distribution of $F_{\text{max}}$ is to accept at significance level $\alpha$ the hypothesis $H_0$: All variances are equal, if all ratios $F_{ij} = S_j^2/S_1^2$ satisfy

$$\frac{1}{F(\alpha)} < F_{ij} < F(\alpha),$$

and to reject $H_0$ otherwise, where $F(\alpha)$ is the upper 100$\alpha$ percent point of the distribution of $F_{\text{max}}$ based on $p$ variances (see Ramachandran (1956)). For the additive loss function defined in Section 6, the Bayes solution gives: Accept $H_0$ if all $F_{ij}$ satisfy

$$F^{(0)} < F_{ij} < F^{(\infty)}$$

where $F^{(0)}$ and $F^{(\infty)}$ do not depend on the number of variances involved.

9.2 When the degrees of freedom of the several $S_j^2$ differ, the component Bayes rules of the multiple decision problem of Section 6 will not in general be compatible and therefore will not give the required solution. However, if we desire to compare
all the $p$ variances with a particular variance $\sigma_o^2$ involved in $S_0^2$, a $\sigma_o^2 \times 2$ variable, it is always possible by the application of the three-decision Bayes rules to divide the set of $p$ variances into significantly less than $\sigma_o^2$, not different from $\sigma_o^2$ and greater than $\sigma_o^2$ groups. The losses of the $3^p$ decisions should be the sum of the losses of the $p$ individual components similar to (6.5) and the component three-decision problem of comparing any $\sigma_i^2$ with $\sigma_o^2$, $i = 1, 2, \ldots, p$ may have the loss function defined in the Sections 3, 4 or 5.

Also the ratio $k = k_1/k_0$ may vary for the different comparisons. The Bayes solution is given by the following:

Define

$$F_i = \frac{S_o^2/\tau_0}{S_1^2/\tau_1} \quad i = 1, 2, \ldots, p.$$ 

Decide $\sigma_1^2$ is significantly less than $\sigma_o^2$ if $F_i > F_i(\infty)$

Decide $\sigma_1^2$ is significantly not different from $\sigma_o^2$ if $F_i < F_i(\infty)$

Decide $\sigma_1^2$ is significantly greater than $\sigma_o^2$ if $F_i < F_i(\infty)$

where $F_i(\infty)$ and $F_i(\infty)$ are essentially similar to $F_2(\infty)$ and $F_2(\infty)$ respectively.

Similar remarks will hold in the problems concerning $\beta$'s of exponential populations.
References


