CHAPTER III

SOME POSTERIOR DISTRIBUTIONS CONCERNING NORMAL SAMPLES
WITH APPLICATIONS TO ANALYSIS OF VARIANCE MODEL PROBLEMS

1. Introduction

In analysis of variance model - I problems, we have independent normal variables having the same variance. The problems of special interest in this set-up are inferences concerning the effects - the treatment effects, the variance effects, etc. In this chapter, we obtain posterior distributions useful for making inferences concerning some of the commonly used functions of the effects. We discuss situations where initially no knowledge is available about the parameters, and also situations where certain prior distributions of the parameters are known.

We consider that we are given samples of size \( n_i \) from normal populations \( N(\mu_i, \sigma^2) \), \( i = 1, 2, \ldots, p \). The probability law of the sample observations \( x \) is given by

\[
1(x \mid \theta) = \prod_i (\sqrt{2\pi\sigma^2})^{-1} \exp \left\{ -\frac{1}{2\sigma^2} \left( x_i - \mu_i \right)^2 \right\}. \quad (1.1)
\]

In Sections 2, 3 and 4, we obtain posterior distributions.
of some functions of the means \( \mu_1 \) and \( \sigma^2 \). In Section 5, we obtain the distributions which are of interest in analysis of variance model I problems.

2. No initial knowledge of the parameters

2.1. In this section, we suppose that initially we have no knowledge about the parameters \( \mu_1, \mu_2, \ldots, \mu_p \) and \( \sigma \). We therefore assume that, a priori, \( \mu_j, \) \( j = 1, 2, \ldots, p \) and \( \log \sigma \) are independent and uniformly distributed. See Jeffreys (1961), Lindley et al (1960), Box and Tiao (1962, 1966) etc. The posterior joint density of \( \mu \sim (\mu_1, \mu_2, \ldots, \mu_p) \) and \( \sigma \) with respect to the uniform prior distribution is given by

\[
p(\mu, \sigma | x) \propto \sigma^{-(\sum n_j + 1)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ s^2 + \sum n_j (\bar{x}_j - \mu_j)^2 \right\} \right] (2.1)
\]

where

\[
\bar{x}_j = \frac{1}{n_j} \sum x_{ij}, \quad s^2 = \sum s_1^2 \quad \text{and} \quad s_1^2 = \sum \left( x_{ij} - \bar{x}_j \right)^2.
\]

The posterior marginal density of \( \mu \) is

\[
p(\mu | x) \propto \left\{ s^2 + \sum n_j (\bar{x}_j - \mu_j)^2 \right\}^{-(v+p)/2}
\]

(2.3)
where
\[ v = \Sigma (n_i - 1) \] ... (2.4)
denotes the usual degrees of freedom of \( S^2 \).

The posterior marginal density \( \sigma \) is given by
\[
p(\sigma | x) \propto \sigma^{-(v+1)/2} \exp \left\{ - \frac{S^2}{2\sigma^2} \right\}. \] ... (2.5)

2.2. The posterior distribution of \( \frac{\nu}{\sigma} \)

We obtain the posterior distribution of
\[ \zeta = (\nu_1, \ldots, \nu_p) \] where
\[ \nu_1 = \sqrt{n_1 \nu_1 / \sigma} , \quad i = 1, 2, \ldots, p \] ... (2.6)

From (2.1), we have the posterior joint density of \( \nu, \sigma \)
\[
p(\zeta, \sigma | x) \propto \sigma^{-(v+1)/2} \exp \left[ - \frac{1}{2} \left\{ \frac{S^2 + \Sigma n_i \nu_i^2}{\sigma^2} \right\} + \Sigma \nu_i^2 - 2(\Sigma n_i \nu_i \nu_i / \sigma) \right]. \] ... (2.7)

On integrating over \( \sigma \) and simplifying, we obtain
\[ p(\zeta | x) \] the posterior density of \( \nu \).
\begin{align*}
p(\tau | y) &= p(\tau | t) \\
&= \left\{ \frac{p}{2} \right\}^{(v-2)/2} \left( \frac{v-2}{2} \right) \exp \left[ -\frac{1}{2} \left( \sum \tau \tau_1^2 - \frac{\left( \sum t_1 \tau_1 \right)^2}{v + \sum t_1^2} \right) \frac{v}{v + \sum t_1^2} \right] \\
\sqrt{2\pi}^{(v-1)} I_{v-1} \left( -\sum t_1 \tau_1 / (v + \sum t_1^2) \right)^{1/2} & \quad (2.3)
\end{align*}

where \( t = (t_1, t_2, \ldots, t_p) \), \( t_i \)'s being the usual Student \( t \)-variables

\begin{align*}
t_i &= \frac{\sqrt{n_i x_i}}{(s^2/v)^{1/2}} , \quad i = 1, 2, \ldots, p \quad (2.9)
\end{align*}

and \( I_v(x) \) is the function defined by Fisher (1931).

\begin{align*}
I_v(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi} v} u^v \exp \left\{ -\frac{1}{2} (u - x)^2 \right\} du \quad (2.10)
\end{align*}

The first and second order moments of the distribution \((2.8)\) are given by

\begin{align*}
E_1(\tau_1) &= t_1 \left\{ \left( \frac{2}{v} \right)^{1/2} \frac{(v-1)}{2} \left( \frac{v-2}{2} \right) \right\} \\
i = 1, 2, \ldots, p
\end{align*}
\[
\text{Var}_1(\gamma_i) = 1 + t_i^2 \left[ 1 - \frac{2}{v} \left\{ \frac{(\nu-1)}{(\nu-2)} \right\}^2 \right] \quad i = 1, 2, \ldots, v
\]

and
\[
\text{Cor}_1(\gamma_i, \gamma_j) = t_i t_j \left[ 1 - \frac{2}{v} \left\{ \frac{(\nu-1)}{(\nu-2)} \right\}^2 \right] \quad i, j = 1, 2, \ldots, v
\]

(We write the expectations in the posterior distribution with a suffix 1). We also obtain
\[
E_1(\gamma_i - E_1(\gamma_i))^3 = t_i^3 \left( \frac{2}{v} \right)^{3/2} \frac{(\nu-1)}{2} \left\{ \frac{(\nu-1)}{2} \left( \frac{\nu-2}{2} \right) \right\}^2 - \left( \frac{\nu-1}{2} \right)
\]

and
\[
E_1(\gamma_i - E_1(\gamma_i))^4 = 3 + 6 \left( \frac{2}{v} \right) \left\{ \frac{(\nu-1)}{2} \left( \frac{\nu-2}{2} \right) \right\}^2 t_i^2 \left[ v - \left\{ \frac{(\nu-1)}{2} \left( \frac{\nu-2}{2} \right) \right\} \right]
\]

\[
+ \left( \frac{2}{v} \right)^2 \left\{ \frac{v(v+2)}{2} \right\} - \left\{ \frac{(\nu-1)}{2} \left( \frac{\nu-2}{2} \right) \right\}^2 \left( \frac{v(v+2)}{2} \right) - \left( \frac{\nu-1}{2} \right)
\]

This distribution (2.8) can obviously be obtained from the distribution (3.13) of Chapter II. The only changes that arise are due to \( n' \) defined in Section 3 of Chapter II being equal to zero in the distribution (2.8).
When \( s^2 = (S^2/v) \), the estimate of \( \sigma^2 \), is based on large degrees of freedom, the posterior distribution of the noncentrality parameter \( \gamma_1 \) of the Student \( t \)-variable \( t_1 \) is approximately normal with mean \( t_1 \) and variance \( v \). And applying theorem 2.6.2 of Anderson (1958), we obtain -

For large \( v \), the approximate posterior distribution of \( \gamma_1 \) is multivariate normal with mean vector \( t_1 \) and variance-covariance matrix \( I \). It is clear that this approximate distribution is exact when the variance \( \sigma^2 \) is known and \( t_1 \) in this case is defined by (2.9) on replacing \( (S^2/v) \) by \( \sigma^2 \).

The posterior marginal density of \( \gamma_1 \) is given by

\[
p(\gamma_1|x) = p(\gamma_1|t_1) = \frac{(v-1)!}{2^{(v-2)/2} (v-2)!} \left( \frac{v}{v+t_1^2} \right)^{v/2} \exp \left\{ -\frac{1}{2} \frac{v t_1^2}{v+t_1^2} \right\}
\]

\[
I_{v-1} \left( \frac{t_1 \gamma_1}{(v+t_1^2)^{1/2}} \right) \tag{2.14}
\]

This distribution depends only on the variable \( t_1 \).

Consider the case where we have a sample from a single normal population \( N(\mu, \sigma^2) \). We note that the posterior density of \( \gamma = \sqrt{n} \mu / \sigma \) is identical with the distribution of \( \gamma_1 \) given by (2.14) when \( t_1 \) is replaced by \( t \).
and \( v = n - 1 \) denotes the degrees of freedom of
\[ s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}. \]

2.3. The posterior distribution of \( \sum n_i \mu_i^2 / \sigma^2 \)

A common statistical problem is that of inference about the null hypothesis, namely, the hypothesis which states that the means are equal to zero. If \( \mu = (\mu_1, \mu_2, \ldots, \mu_p) \) are the true means, then the function
\[
\lambda = \sum n_i \mu_i^2 / \sigma^2
\]

can be taken as a measure of distance of the true from the null hypothesis. In fact when \( \sigma^2 \) is known, the quantity \( \lambda \) denotes the weighted sum of squares of deviations of the true from the hypothetical (null) means, the weights being proportional to the reciprocals of variances of the means in the posterior distribution.

We now obtain the posterior distribution of \( \lambda \).

We have the integral
\[
\int (2\pi)^{-p/2} \exp \left[ -\frac{1}{2} \sum \left( \frac{x_i - \bar{x}}{\sigma} \right)^2 \right] dx
\]
\[
\lambda = \frac{1}{2} d\lambda < \lambda < \lambda + \frac{1}{2} d\lambda
\]
\[ p(\lambda | x) = \int p(\lambda, \sigma | x) d\sigma d\lambda \]

\[= \frac{1}{2} d\lambda < \lambda < \lambda + \frac{1}{2} d\lambda \]

\[= (2/2)^{-1/2} \frac{\lambda^{v/2}}{v + p} \exp(-\lambda/2) \]

\[= \frac{\beta(\lambda)}{\beta(\lambda + 1/2)} \frac{\Gamma^r}{\Gamma^{r+1/2}} \lambda^r \frac{1}{(2r+1)} \]

Thus from (2.1), we obtain

\[p(\lambda | x) = p(\lambda | F)\]

\[= \frac{\beta(\lambda)}{\beta(\lambda + 1/2)} \frac{\Gamma^r}{\Gamma^{r+1/2}} \lambda^r \frac{1}{(2r+1)} \]

where \(F\) is defined by the usual \(F\)-ratio

\[F = \frac{z\delta}{\eta} / p\]
where \( f(x^2) \) is the density function of a chi-square variable with \((p+2r)\) degrees of freedom, and

\[

v_r = \left( \frac{p}{v + pF} \right)^{v/2} \frac{v^{v/2}}{(v + pF)^{v/2 + r - 1}} \left( \frac{pF}{v + pF} \right)^r.
\]

From (2.19), we readily obtain

\[

E_1(\lambda) = p + pF
\]

and

\[

\text{Var}_1(\lambda) = 2p + \frac{1}{3}pF + 2p^2 \frac{2}{v}.
\]

The approximate posterior distribution of \( \lambda \) is that of a type III variable of the form \( C \chi^2_f \), where \( C = \frac{\text{Var}_1(\lambda)}{2E_1(\lambda)} \) and \( f' = \frac{2E_1(\lambda)}{\text{Var}_1(\lambda)} \).

2.4. The posterior distribution of \( \frac{\sum x_i^2}{\mu_1} \)

We now obtain the posterior distribution of

\[

f = \sum x_i^2 = \lambda \sigma^2.
\]

This function \( f \) can also be taken as a measure of distance of the true from the null hypothesis, when
the null hypothesis states that the means are zero.

From (2.1) and (2.16), we obtain

\[
p(\theta, \sigma^2 | x) = \text{const.} \exp \left\{ -\frac{1}{2\sigma^2} \left( \sigma^2 + \frac{\sum n_i x_i^2}{2} + \theta \right) \right\}
\]

\[
\sigma^{-1}(\sum n_i + 1) \sigma(p-2)/2 \sum_n \sigma^{-1} \left( \frac{\sum n_i x_i^2}{2} \right)^r \frac{c^r}{(2^r)!} \cdot B \left( \frac{p-1}{2}, \frac{r+1}{2} \right).
\]  

(2.24)

On integrating over \( \sigma \), we get

\[
p(\theta | x) = p(\sigma | s^2, \sum n_i x_i^2)
\]

\[
= \frac{(s^2)^{\nu/2}}{(\nu-2)/2} \sum_{r=0}^{\infty} \frac{\left( \sum n_i x_i^2 \right)^r \sigma^r}{(s^2 + \sum n_i x_i^2)^{\nu/2 + 2r}} \left( \frac{\nu/2 + 2r - 1)!}{r!} \right)
\]

\[
= \frac{(s^2)^{\nu/2}}{(\nu-2)/2} \sum_{r=0}^{\infty} \frac{\left( \sum n_i x_i^2 \right)^r \sigma^r}{(s^2 + \sum n_i x_i^2)^{\nu/2 + 2r}} \left( \frac{\nu/2 + 2r - 1)!}{r!} \right)
\]

\[
\]  

From (2.26), we observe that the posterior density

of \( \theta = \theta / (s^2 + \sum n_i x_i^2) \) is expressible as a weighted sum of density functions of Beta variables of the second kind, i.e.,

\[
p(\theta | x) = \sum_{r=0}^{\infty} w_r \frac{\Gamma(p/2 + r) \Gamma(v/2 + r)}{\Gamma(p + v/2 + r)}
\]

where

\[
f(\theta | m_1, m_2) = \frac{1}{B(m_1, m_2)} \frac{b^{m_1-1}}{(1+b)^{m_1+m_2}}
\]

\[
\]  

(2.26)
and \( v_r \) are as defined in (2.20).

From (2.26), we obtain

\[
E_1(\theta) = \sum v_r \frac{(P + r)}{(\frac{v}{2} + r - 1)}
\]

and

\[
E_1(\theta^2) = \sum v_r \frac{(P + r)(2 + r + 1)}{(\frac{v}{2} + r - 1)(\frac{v}{2} + r - 2)}
\]

On simplifying we get

\[
E_1(\theta) = \frac{P^F}{(v+p^F)} + \frac{P}{(v-2)} \frac{v}{v + p^F}
\]

and

\[
\text{Var}_1(\theta) = \frac{c_p^F}{(v+p^F)(v-2)} + \left( \frac{v}{v+p^F} \right)^2 \left\{ \frac{p(p+2)}{(v-2)(v-4)} - \frac{p^2}{(v-2)^2} \right\}
\]

2.5. The case of equal sample sizes

We consider that \( n_i = n, i = 1, 2, \ldots, p \) and obtain the posterior distributions of orthogonal contrasts in the means \( \mu_1, \mu_2, \ldots, \mu_p \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_p \) be a set of \( p' (p' < p) \) orthogonal contrasts in \( \mu_j \)'s: \( \gamma_1 = \sum_l \gamma_{lj}/\mu_j \) where

\[
\sum_l \gamma_{lj} = 0, \sum_l^2 \gamma_{lj} = 1 \quad \text{and} \quad \sum_l^2 \gamma_{lj} \gamma_{l'j} = 0, \; i \neq i'.
\]

The posterior density of \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p) \) is given by
\[ p(\lambda^* | F^*) = \sum_{r=0}^{\infty} w_r^* \Gamma(\gamma_{p' + 2r}) \quad \cdots \quad (3.33) \]

where \( F^* = \sum t_1^* / p' \), and \( w_r^* \) are obtained from \( w_r \).
(2.20) on replacing $p$ and $F$ by $p'$ and $F'$ respectively.

When $p' = (p'-1)$, the function $\lambda^* = \frac{\sum (\mu_i - \bar{\mu})^2}{\sigma^2}$ measures the distance of the true hypothesis from the null hypothesis, when the null hypothesis states that all means are equal.

(See Bulmer (1958)).

The posterior density of $\theta^* = \frac{\lambda^* \sigma^{-2}}{(s^2, \Sigma n_i y_i^2)}$ is expressible as

$$
p(\theta^* | x) = \sum v_{r^*} f\left( b, \frac{x}{2 + r^* \frac{v}{2 + r}} \right) \ldots \ldots \ldots (2.34)
$$

where $v_{r^*}$ are defined as in (2.33).

3. Incompletely specified prior distributions

3.1. We suppose it is known that $\mu_i$, $i = 1, 2, \ldots, p$

have apriori independent normal densities with means $\mu_i$

and variances $\sigma^2 / \mu_i$, where $m_i$ are known, $\sigma^2$ are not

known but the ratios $\sigma^2 / \sigma^2 = 1 / n_i$ are known. Instead

of this incompletely specified prior distribution, we adopt a

completely specified prior distribution obtained by using

the method suggested in Chapter I.

We are given that the prior density of $\mu$ is

$$
ge_0(\mu) \propto \sigma^{-p} \exp \left\{ -\frac{1}{2 \sigma^2} \sum n_i \left( \frac{\mu_i - m_i}{\sigma} \right)^2 \right\} \ldots \ldots \ldots (2.4)$$
Since $\sigma$ is a scale parameter, we adopt the following prior distribution for $(\mu, \sigma)$:

$$g(\mu, \sigma) \propto \sigma^{-1} e^{\sigma \frac{1}{\sigma}} \quad \ldots \quad (3.2)$$

The posterior joint density of $(\mu, \sigma)$ with respect to the prior distribution (3.2) is given by

$$p(\mu, \sigma | x) \propto \sigma^{-\left(\sum n_i + p + 1\right)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ \frac{s^2 + \sum n_i (\bar{x}_i - \mu)^2}{\sigma^2} \right\} \right] \quad \ldots \quad (3.3)$$

We have therefore the posterior marginal densities

$$p(\mu | x) \propto \left\{ \frac{s^2}{\sigma^2} + \sum (n_i + n'_i) \left( \mu - \frac{n_i \bar{x}_i + n'_i \bar{x}'_i}{n_i + n'_i} \right)^2 \right\}$$

and

$$p(\sigma | x) \propto \sigma^{-(\nu + p)/2} \quad \ldots \quad (3.4)$$

where

$$\nu = \nu + p.$$ 

We now obtain the posterior distribution of $\mu / \sigma$.

Let $\gamma_i = \sqrt{n_i / \mu} / \sigma$, $\gamma_i^2 = n_i / n'_i$ and $\beta_i^2 = 1 + 1 / \gamma_i^2 = (n_i + n'_i) / n_i$.

$i = 1, 2, \ldots, p$. We shall define $\eta_i = \beta_i \gamma_i$. 

The posterior density of \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p) \) is given by

\[
p(\gamma \mid x) = \left\{ \frac{1}{(2\pi)^{p/2}} \frac{1}{\begin{vmatrix} \mathbf{v}(1) \end{vmatrix}^{2}} \right\}
\cdot \left\{ \frac{\gamma_i^{2}}{s(1)^{2} + \sum_{i=1}^{p} \frac{1}{n_i - n_i'}} \right\}^{\gamma_i/2} \exp\left\{ -\frac{1}{2} (\gamma_i - z_i)^2 \right\}
\cdot \left\{ \sqrt{2\pi} \left( \mathbf{v}(1)^{-1} \right) \right\}^{-1} (-z_i) \quad \text{(3.7)}
\]

where

\[
z = \frac{\left( n_i \overline{x}_i + n_i' m_i \right) \gamma_i / (n_i + n_i')^{1/2}}{\left\{ s(1)^{2} + \sum_{i=1}^{p} \frac{1}{(n_i \overline{x}_i + n_i' m_i)^2 / (n_i + n_i')} \right\}^{1/2}} \quad \text{(3.8)}
\]

The posterior marginal density of \( \gamma_1 \) is given by

\[
p(\gamma_1 \mid x) \propto \frac{\gamma_1^{(v_1 - 1)/2}}{\left( \frac{1}{v_1} - \frac{1}{2} \right)^{(v_1 - 2)/2}} \cdot \left\{ \frac{s(1)^{2}}{2 + \frac{(n_1 \overline{x}_1 + n_1' m_1)^2}{n_1 + n_1'}} \right\}^{v_1/2} \cdot \exp\left\{ -\frac{1}{2} (\gamma_1 - z_1)^2 \right\} \quad \text{(3.9)}
\]

where

\[
z_1 = \frac{(n_1 \overline{x}_1 + n_1' m_1) \gamma_1 / (n_1 + n_1')^{1/2}}{\left\{ s(1)^{2} + \frac{(n_1 \overline{x}_1 + n_1' m_1)^2 / (n_1 + n_1')} \right\}^{1/2}} \quad \text{(3.10)}
\]
3.2 Incompletely specified prior distribution-II

We suppose it is given that $\mu_1$'s have priori independent normal distributions with equal means $\mu_1$ and equal variances $\sigma_{\mu}^2$, where $\mu_1$ and $\sigma^2$ are not known, but the ratio $\sigma_{\mu}^2 / \sigma^2 = 1/n^1$ is known. See Duncan (1961, 1965).

Let $\xi_1 = (\xi_1, \xi_2, \ldots, \xi_{p-1})$ be a set of $(p-1)$ mutually orthogonal contrasts in $\mu_1$, $j = 1, 2, \ldots, p$. We are given that the prior density of $\xi_1, \Sigma_{\mu_1}$ is

$$g_0 (\xi_1, \Sigma_{\mu_1}) = g_0 (\xi_1) g_0 (\Sigma_{\mu_1}) \quad \ldots \quad (3.11)$$

where

$$g_0 (\xi_1) = \sigma^{-(p-1)} \exp \left\{ - \frac{n^1}{2 \sigma^2} \xi_1^2 \right\} \quad \ldots \quad (3.12)$$

and

$$g_0 (\Sigma_{\mu_1}) = \sigma^{-1} \exp \left\{ - \frac{n^1}{2 p \sigma^2 \sigma^2} (\Sigma_{\mu_1} - \mu_{\mu_1}^2 \Sigma_{\mu_1}^2) \right\} \quad \ldots \quad (3.13)$$

Since $m$ is not known, we adopt a uniform prior distribution for $\Sigma_{\mu_1}$. Also since $\sigma$ is a scale parameter, we adopt the following prior distribution for $(\xi_1, \Sigma_{\mu_1}, \sigma)$

$$g (\xi_1, \Sigma_{\mu_1}, \sigma) = \sigma^{-p} \exp \left\{ - \frac{n^1}{2 \sigma^2} \Sigma_{1=1}^{j-1} \xi_1^2 \right\} \quad \ldots \quad (3.14)$$
When the sample sizes are equal, i.e.,

\[ n_i = n, \quad i = 1, 2, \ldots, p, \]

the posterior joint density of \((\mu, \sigma)\) is given by

\[
p(\mu, \sigma | \mathbf{x}) \propto \sigma^{-(n_p + p)} \exp\left[ -\frac{1}{2\sigma^2} \left\{ s_{(2)}^2 + (n+n') \sum_{i=1}^{n} (x_{i} - \frac{n}{n+n'} \bar{x}_{i})^2 \right\} \right]
\]

where

\[
s_{(2)}^2 = \sum_{i=1}^{n} s_{i}^2 + \frac{mn'}{n+n'} \sum_{i=1}^{n} y_{i}^2
\]

The posterior densities of \(\mu\) and of \(\sigma\) are given by

\[
p(\mu | \mathbf{x}) \propto \left\{ s_{(2)}^2 + (n+n') \sum_{i=1}^{n} \left( 1 - \frac{n}{n+n'} y_{i} \right)^2 \right\}^{\frac{v(2)+p-1}{2}}
\]

and

\[
p(\sigma | \mathbf{x}) \propto \sigma^{-(v(2)+1)} \exp \left\{ -\frac{1}{2\sigma^2} s_{(2)}^2 \right\}
\]

where

\[ v(2) = v(1) - 1 \]

4. Conjugate Prior Distribution

We suppose that a priori \((\mu, \sigma)\) have the conjugate prior distribution (see Raiffa and Schlaifer [1961]).
The posterior marginal densities of \( \mu \) and of \( \sigma \) are given by (3.4) and (3.5) respectively after replacing \( s_1^2 \) by \( s_1^2(3) \) and \( v_1 \) by \( v_3 \) where

\[
s_1^2(3) = s_1^2(1) + v
\]

and

\[
v_3 = v(1) + v'
\]

It is clear that the posterior densities of other functions are obtained similarly by effecting these changes.

5. Bayesian comparisons of effects in Two-way Classification Analysis of Variance Model I set up.

Consider that we are given observations \( x_{ij} \) from a two-way classification in analysis of variance model I set-up: \( x_{ij} \) is the observation in the \((i,j)\) cell, \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, q \). The \( x_{ij} \) satisfy the model

\[
x_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij}
\]

where

\[
\epsilon_{ij} \sim N(0, \sigma^2)
\]

\[
E(\epsilon_{ij, \epsilon_{i'j'}}) = 0, \quad (i,j) \neq (i'j'), \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q
\]

(The notation (e.g., \( \alpha \)'s, \( \beta \)'s etc.) is different from that of previous Sections).
The probability law of the observations $x_{ij}$ is given by

$$1(x_1|\sigma) = \prod_{i,j} \left( \frac{1}{\sqrt{2\pi} \sigma} \right)^{-1} \exp \left\{ -\frac{1}{2\sigma^2} (x_{ij} - \mu - \tau_i - \beta_j)^2 \right\} \quad (5.3)$$

Let $\tau_i$'s denote the treatment effects and $\beta_j$'s denote the varietal effects. The parameters of interest in this set-up are the contrasts in $\tau_i$'s and the contrasts in $\beta_j$'s. These contrasts, $\mu = \mu + \bar{\tau} + \bar{\beta}$

$$\bar{\tau} = \frac{1}{p} \sum \tau_i, \quad \bar{\beta} = \frac{1}{q} \sum \beta_j$$

and their linear functions constitute the totality of estimable functions in this set up.

(See Sceffé (1959)).

Let us denote $x_1 = \frac{1}{q} \sum x_{1j}$, $x_j = \frac{1}{p} \sum x_{ij}$

and $x_{..} = \frac{1}{pq} \sum_{i,j} x_{ij}$. Let $s^2 = \sum (x_{ij} - x_{i.} - x_{.j} - x_{..})^2$

be the usual sum of squares due to error with $v_e = (p-1)(q-1)$ the usual error degrees of freedom.

(1) Let us suppose that we have initially no knowledge about the parameters $\tau$, $\beta$, $\mu$, $\sigma$.

Let $\mathcal{I} = (i_1, i_2, \ldots, i_{p-1})$ be a set of $(p-1)$ mutually orthogonal contrasts in $\tau_k : i_k = \sum_{k \leq k} \tau_k$.

Similarly let $\hat{\delta} = (\delta_1, \delta_2, \ldots, \delta_{q-1})$ be a set of $(q-1)$ mutually orthogonal contrasts in $\beta_k : \delta_j = \sum_{k \leq k} \beta_k$.
The likelihood \( l(\mathbf{x} | \theta) \) of the observations \( \mathbf{x} \) given in (5.3) can also be written as

\[
l(\mathbf{x} | \mu, \tau, \gamma, \sigma) \propto \sigma^{-(pq)} \exp \left[ -\frac{1}{2 \sigma^2} \left\{ s_e^2 + pq (x_e - \mu)^2 \right\} \right.
\]
\[
+ \frac{q}{2} (\tau_1 - \hat{\tau}_1)^2 + p \Sigma (\gamma_j - \hat{\gamma}_j)^2 \left. \right]\tag{5.3}
\]

where \( \hat{\tau}_1 \) are obtained from \( \tau_1 \) on replacing \( \hat{\tau}_k \)'s by \( \hat{\tau}_k \)'s, and \( \hat{\gamma}_j \) are obtained from \( \gamma_j \) on replacing \( \hat{\gamma}_k \)'s by \( \hat{\gamma}_k \)'s.

We shall assume that, apriori, \( \mu, \tau_1 \)'s, \( \gamma_j \)'s and \( \log \sigma \) are independent and uniformly distributed.

The posterior density of \( \hat{\tau}_1 \) is given by

\[
p(\hat{\tau}_1 | \mathbf{x}) \propto \left\{ s_e^2 + q \Sigma (\hat{\tau}_1 - \hat{\tau}_1)^2 \right\}^{-(v_0 + p - 1)/2}
\tag{5.5}
\]

Similarly the posterior density of \( \hat{\gamma}_j \) is given by

\[
p(\hat{\gamma}_j | \mathbf{x}) \propto \left\{ s_e^2 + p \Sigma (\hat{\gamma}_j - \hat{\gamma}_j)^2 \right\}^{-(v_0 + q - 1)/2}
\tag{5.6}
\]

The posterior density of \( \bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{p-1}) \),

\( \bar{\alpha} = \sqrt{n} \hat{\tau}_1 / \sigma \) is given by

\[
p(\bar{\alpha} | \mathbf{x}) \propto \exp \left[ -\frac{1}{2} \left\{ \Sigma \alpha_2^2 - \frac{(\Sigma \alpha_t t_1)^2}{v_0 + \Sigma t_1^2} \right\} \right]
\]
\[
I_{v_0} \left( -\frac{\Sigma t_1 \bar{\alpha}_1}{(v_0 + \Sigma t_1^2)^{1/2}} \right)
\tag{5.7}
\]
where 
\[ t_1 = \frac{\sqrt{n} \hat{\beta}_1}{S_2/n} \sqrt{\frac{1}{v_0}} \]  
for \( i = 1, 2, \ldots, p-1 \ldots \)

The posterior density of \( (\sqrt{n} \hat{\beta}_1 / \sigma, \ldots, \sqrt{n} \hat{\beta}_p / \sigma) \) is given in a similar way. These distributions are similar to those in Section 2.

(iii) Suppose it is known that, a priori, \( \mu_1, \mu_2, \ldots, \mu_p \) and \( \beta_1, \beta_2, \ldots, \beta_q \) have independent normal distributions, \( \mu_1 \)'s having equal means \( m(1) \) and equal variances \( \sigma_{\mu}^2 \), and \( \beta_j \)'s having equal means \( m(2) \) and equal variances \( \sigma_{\beta}^2 \). Suppose that \( m(1), m(2), \sigma_{\mu}^2, \sigma_{\beta}^2 \) are not known, but the ratios \( \sigma_{\mu}^2 / \sigma^2 = 1/q(1) \) and \( \sigma_{\beta}^2 / \sigma^2 = 1/p(1) \) are known.

This prior knowledge about the parameters can be summarised in the prior density

\[ g_\sigma(\beta, \mu) \propto \sigma^{-(p+q-2)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ q(1) \Sigma \hat{\beta}_1^2 + p(1) \Sigma \hat{\mu}_1^2 \right\} \right] \]  

Following the argument of Section 6 of Chapter I, we adopt as the prior distribution

\[ g(\mu, \beta, \hat{\beta}, \sigma) \propto \sigma^{-1} g_\sigma(\beta, \mu) \]

\[ \propto \sigma^{-(p+q-1)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ q(1) \Sigma \hat{\beta}_1^2 + p(1) \Sigma \hat{\mu}_1^2 \right\} \right] \]  

\[ \cdot \cdot \cdot \]  

\( \cdot \cdot \cdot \)  

\( (5.10) \)
The posterior joint density of \((\mu^*, \tau_1^*, \beta, \sigma)\) with respect to the prior distribution (5.10) is given by

\[ p(\mu^*, \tau_1^*, \beta, \sigma | x) \propto 1 \left( \frac{1}{2} \right) \| (\mu^*, \tau_1^*, \beta, \sigma) - (\mu, \tau_1, \beta, \sigma) \|^2 \| \mu, \tau_1, \beta, \sigma \|^{-\frac{1}{2}} \]  

(5.11)

The posterior density of \(\tau_1^*\) is given by

\[ p(\tau_1^* | x) \propto \left\{ \frac{s^2}{1} + (q + q(1)) \Sigma (\hat{\tau}_1^* - \frac{q}{q + q(1)} \tau_1^* )^2 \right\}^{-\frac{(v_1 + p - 1)}{2}} \]  

(5.12)

where

\[ s^2 = s_0^2 + \frac{qq(1)}{q + q(1)} \Sigma \hat{\tau}_1^* + \frac{pp(1)}{p + p(1)} \Sigma \hat{\delta}_j^2 \]  

(5.13)

and

\[ v_1 = v_0 + p + q - 2. \]  

(5.14)

The posterior density of \(\beta\) is given by

\[ p(\beta | x) \propto \left\{ \frac{s^2}{1} + (p + p(1)) \Sigma \left( \hat{\beta}_j - \frac{p}{p + p(1)} \delta_j^* \right)^2 \right\}^{-\frac{(v_1 + p - 1)}{2}} \]  

(5.15)

Let us define \(\alpha_i^* = (q + q(1))^{1/2} \tau_i^*/\sigma, i = 1, 2, \ldots, p-1\).

The posterior density of \(\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_{p-1}^*)\) is given by

\[ p(\alpha^* | x) \propto \exp \left\{ -\frac{1}{2} \left( \Sigma \alpha_i^* \delta_i^2 - z_i^2 \right) \right\} I_{v_1-1}(-2_i) \]  

(5.16)
where

\[ z_1 = \left( \frac{q}{q+q(1)} \right)^{1/2} \frac{\sum t_i q_i^*}{(v_e + \sum t_i^2 + \frac{p(1)}{p+p(1)} \sum t_j^2)^{1/2}} \]  

and

\[ t_j' = \sqrt{p \hat{\sigma}_j / (s_e^2/v_e)^{1/2}} \]  

The posterior density of \( \lambda = \sum q_i^* \) is expressible as

\[ p(\lambda | x) = \sum_{r=0}^w \nu_r f\left( \chi^2(p-1) + 2r \right) \]  

where \( \nu_r \)'s are defined by

\[ \nu_r = (1-d) \frac{v(1)/2 \left( \frac{v(1)}{2} + r-1 \right)!}{r! \left( \frac{v(1)}{2} - 1 \right)!} \]  

and

\[ d = \frac{\frac{q}{q+q(1)} \sum t_i^2}{v_e + \sum t_i^2 + \frac{p(1)}{p+p(1)} \sum t_j^2} \]  

The posterior density of \( q = \lambda \sigma^2/(s(1) + q^2 \sum q_i^2) \) is obtained as

\[ p(q | x) = \sum_{r=0}^w \nu_r f\left( \frac{b_{p-1}}{2} + r, \frac{v(1)}{2} + r \right) \]  

where \( \nu_r \) are as defined in (5.20).
Suppose it is known that, a priori, \( \tau_1 \)'s have independent normal distributions with equal means \( m(1) \) and equal variances \( \sigma^2 \), where \( m(1) \) and \( \sigma^2 \) are not known, but the ratio \( \sigma^2/\sigma'^2 = 1/q(1) \) is known. Suppose that in addition to this, nothing is known initially about the parameters.

In this case we shall adopt the following prior distribution:

\[
\tilde{E}(\mu, \Sigma, \sigma) \propto \sigma^{-p} \exp \left\{ -\frac{1}{2} \frac{q(1)}{\sigma^2} \Sigma \tilde{t}_1^2 \right\} \quad (5.23)
\]

Using (5.23) for the prior distribution, we get the posterior densities

\[
p(\tilde{t}_1 | x) \propto \left\{ s_2^2 + \left( q + q(1) \right) \Sigma (\tilde{t}_1 - \frac{q(1)}{q + q(1)} \tilde{t}_1^2) \right\}^{-\left( v(2) + q - 1 \right)/2} \quad (5.24)
\]

\[
p(\tilde{t}_2 | x) \propto \left\{ s_2^2 + p \Sigma (\tilde{t}_2 - \tilde{t}_2^2) \right\}^{-\left( v(2) + q - 1 \right)/2} \quad (5.25)
\]

\[
p(q^2 | x) \propto \exp \left\{ -\frac{1}{2} (2q_1^2 + \tilde{z}_2^2) \right\} I_{v(2)}^{-1} \left( -\tilde{z}_2 \right) \quad (5.26)
\]

where

\[
s_2^2 = s_2^2 + \frac{q(1)}{q + q(1)} \Sigma t_1^2 \quad (5.27)
\]

\[
v(2) = v_e + (p - 1) \quad (5.29)
\]

and
\[ z_2 = \left( \frac{\eta_2}{q+q(1)} \right) \frac{\Sigma t_1^2 a_i^*}{(\nu_0 + \Sigma t_1^2)^{1/2}} \quad \ldots \quad (5.23) \]

The posterior density of \( \lambda = \Sigma a_i^2 \) is as given by (5.19) where \( \nu_r \)'s are obtained from (5.20) on replacing \( \nu(1) \) by \( \nu(2) \) and defining \( d \) by

\[ d = \frac{q}{q+q(1)} \frac{(p-1)F}{\nu_0 + (p-1)F}, \quad F = \frac{\Sigma t_1^2}{(p-1)} \quad (5.20) \]

The posterior density of \( q' = \lambda \sigma^2 / (a^2 + \frac{q}{q+q(1)} \Sigma \tau_i^2) \)

is given by the R.H.S. of (5.22) after effecting the above changes.

(iii) Suppose that \( (\mu, \tau, \delta, \sigma) \) have a conjugate prior distribution given by

\[ p(\mu, \tau, \delta, \sigma) \propto \sigma^{-(v' + p + q - 1)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ V + a(1) | \mu - m(3) |^2 \right\} \right. \]

\[ + \left. \frac{q}{q+q(1)} \Sigma \tau_i^2 \right\} \beta(\tau) \Sigma \delta^2 ] \quad \ldots \quad (5.31) \]

We then obtain the following posterior densities

\[ p(\tau | x) \propto \left\{ a^2 + (q+q(1)) \Sigma \left( \frac{q}{q+q(1)} \tau_i^2 \right) - \frac{q}{q+q(1)} \right\}^{-(v_{q+1} + p - 1)/2} \quad (5.32) \]
\[ p(\xi | x) \propto \{s^2(3) + (p+p_1) \sum (\sigma_j - \frac{p}{p+p_1} \delta_j)^2 \}^{-(v+q-1)/2} \]  
\[ p(q^2 | x) \propto \exp\left\{ -\frac{1}{2} \left( \sum \alpha_i^2 - z^2 \right) \right\} I_{v(3)}(1) \left(-z\right) \]  
where

\[ s^2(3) = \frac{s^2 + \frac{pq n(1)}{q+q(1)} (x - n(3))^2}{k} \]
\[ + \frac{qq(1)}{q+q(1)} \sum_{i=1}^{2} \frac{P_{R(1)}}{p+p_1} \sum_{j=1}^{2} \delta_j^2 \]  
\[ v(3) = v_e + v' + p + q-1 \]  
\[ z_3 = \left( \frac{q}{q+q(1)} \right)^{1/2} \frac{\sum_{i=1}^{4} \alpha_i^4}{\left( s^2 + \frac{q^2}{q+q(1)} \sum_{i=1}^{4} \delta_i^2 \right)^{1/2}} \]  

Also we have the posterior densities

\[ p(\lambda | x) = \Xi v_{r} f(\chi^2_{p+q-2r}) \]  
and

\[ p\left( q^2 = \lambda \sigma^2(3) + \frac{q^2}{q+q(1)} \sum_{i=1}^{4} \delta_i^2 \right) | x \) = \Xi v_{r} f(b_{2r}, v(3)) \]  

where
\[
v_r = (1 - d^2) \frac{v_{(3)/2}}{\frac{2}{r}(\frac{v_{(3)} + r - 1}{2}) !} \frac{d^r}{r} \quad \text{(S.40)}
\]

and
\[
d = \left( \frac{q}{q + q_{(1)}} \right) \frac{z_{(1)}^2 + z_{(3)}^2}{s_2^2 + \frac{q}{q + q_{(1)}} z_{(1)}^2} \quad \text{(S.41)}
\]

On similar lines we can obtain posterior distributions of commonly used functions of effects in any general complete classification of the analysis of variance model I set-up.

**References**


