CHAPTER VI

BAYES RULES FOR SELECTION OF THE BEST NORMAL POPULATION

Introduction

In this chapter, we obtain the Bayes solutions for selecting the best among several normal populations \( N(\mu_i, \sigma_i^2) \), \( i = 1, 2, \ldots, p \). We consider that an utility function \( U \) is associated with each of the normal populations and that the best population is defined as that which corresponds to the largest value of \( U \). The solution is obtained by the principle of maximizing the expected utility.

We consider that the experimenter's interest focuses on a special criterion \( h_1 = g(\theta_1) \) where \( g \) is a known function and we regard that \( U \) is a function of \( h \), that is \( U = U(h) \). For these selection problems we consider two types of criteria \( h \) and that \( U \) is an increasing function of \( h \). The first type is defined by

\[
h = \int_{a_1}^{a_2} f(y; \theta) \, dy = \Pr \{ a_1 < y < a_2 \} \tag{1.1}
\]

where \( f(y; \theta) \) is the probability law of an observation \( y \) from the population \( N(\mu, \sigma^2) \) —

\[
f(y; \theta) = (\sqrt{2\pi}\sigma)^{-1} \exp \left\{ -\frac{1}{2\sigma^2} (y - \mu)^2 \right\} \tag{1.2}
\]
and $a_1$ and $a_2$ are specified constants. The interval $(a_1, a_2)$ is referred to as the tolerance interval and $h$ is called the coverage of the interval. See Guttman (1961), Guttman and Tiao (1964).

The second type is defined by

$$h = g(\theta) = \int_{b_1}^{b_2} f(y_1/\sigma) \, d(y/\sigma)$$

$$= \Pr \{ b_1\sigma < y < b_2\sigma \}$$

where $b_1, b_2$ are known constants.

The functions $h$ are useful in many industrial applications. For instance, it is of importance for the manufacturer of a product to know that an item produced meets fairly the requirement. Also when several alternative processes of production are available, the manufacturer would wish to select the process which gives largest coverage to a certain specification interval. In some cases the intervals may be given in terms of standard deviation of the observations. In such applications the observations are deviations of the character from the value the processes are required to yield.

Let $X$ denote the set of vectors $(x_1, x_2, \ldots, x_p)$, where $x_1$ is a sample of size $n_1$ from the normal population.
In this section, we consider that the criterion \( h \) is given by (1.1). Thus

\[
h_i = \int_{a_i}^{s_2} f(y_i | \theta) \, dy_i \quad i = 1, 2, \ldots, p.
\]  

The Bayes solution of the selection problem is obtained by comparing the posterior expectations of \( h_i, i = 1, 2, \ldots, p \).

We assume that the population parameters \( \theta_i, i = 1, 2, \ldots, p \) are locally independent a priori. This means that in the region in which the likelihood is appreciable, the prior joint density of \( \theta = (\theta_1, \ldots, \theta_p) \) is \( p(\theta) = p(\theta_1) p(\theta_2) \ldots p(\theta_p) \). This assumption together with the assumption of independence of samples implies that \( h_i \)'s are independent a posteriori. We shall therefore obtain the posterior expectation of \( h \) for a typical normal population \( N(\mu, \sigma^2) \).

Let \( x = (x_1, \ldots, x_n) \) denote a sample of size \( n \) from the population \( N(\mu, \sigma^2) \). The posterior expectation of \( h \) is

\[
E(h | x) = \int h \, p(\theta | x) \, d\theta
= \int \left\{ \int_{a_i}^{s_2} f(y_i | \theta) \, dy \right\} p(\theta | x) \, d\theta.
\]

where \( f(y_i | \theta) \) is given by (1.2). It is clear that this expectation can be written as

\[
E(h | x) = \int_{a_i}^{s_2} p(y_i | x) \, dy
\]
where $p(y|\mathbf{x})$ is the posterior distribution of a future observation. This is so called as the predictive density of $y$.

As regards the prior knowledge about the parameters $\theta$, we consider three situations: (i) nothing is known initially about the parameters, (ii) the available initial knowledge about the parameters is an incompletely specified prior distribution, and (iii) it is known that a priori the parameters follow a conjugate prior distribution. (See Reiff and Schleifer (1961)).

(1) When nothing is known initially about the parameters $\mu, \sigma$, following Jeffreys (1961) we may assume that, a priori, $\mu$ and $\log \sigma$ are independent and uniformly distributed. Thus we consider that the prior distribution of $\mu, \sigma$ is

$$p(\mu, \sigma) \propto \sigma^{-1} \quad \ldots \quad (2.4)$$

The posterior density $p(y|x)$ is then given by

$$p(y|x) \propto \left\{ s^2 + \frac{n}{1+n} (y - \bar{x})^2 \right\}^{- (v+1)/2} \quad \ldots \quad (2.6)$$

where

$$s^2 = \sum (x_i - \bar{x})^2 = v s^2, \; v = n-1. \quad \ldots \quad (2.6)$$

See Jeffreys (1961).

In this case the posterior expectation of $h$ is

$$E(h|x) = F_v(t_2) - F_v(t_1) \quad \ldots \quad (2.7)$$

where $t_1 = \left( n/(1+n) \right)^{1/2} (s_1 - \bar{x})$, $t_2 = \left( n/(1+n) \right)^{1/2} (s_2 - \bar{x})$.
$F_v(t)$ is the distribution function of the Student $t$-variable with $v$ degrees of freedom. This solution was given by Guttman and Tiao(1964).

(ii) We assume that the prior distribution of $\mu$ is normal with mean zero and variance $\sigma_\mu^2$ where $\sigma_\mu^2$ is not known but the ratio $\sigma_\mu^2/\sigma^2 = 1/n'$ is known. (See Duncan(1061)).

In this case we consider the following prior distribution $p(\mu, \sigma)$ for $\mu, \sigma$:

$$p(\mu, \sigma) = p_\sigma(\mu) p(\sigma) \quad \ldots \quad (2.8)$$

where $p_\sigma(\mu)$ is the given prior density of $\mu$:

$$p_\sigma(\mu) \propto \sigma^{-1} \exp \left\{ -\frac{n'}{2\sigma^2} \mu^2 \right\} \ldots \quad (2.9)$$

For the prior distribution of $\sigma$ we adopt

$$p(\sigma) \propto \sigma^{-1} \quad \ldots \quad (2.10)$$

Thus

$$p(\mu, \sigma) \propto \sigma^{-2} \exp \left\{ -\frac{n'}{2\sigma^2} \mu^2 \right\} \ldots \quad (2.11)$$

When the prior distribution of $\mu, \sigma$ is given by (2.11), we have

$$p(y|x) \propto \left\{ s^2 + \frac{mn'}{n+n'} \frac{n}{n} + \frac{n+n'}{1+n+n'} (x - \frac{nx}{n+n'})^2 \right\} \cdot (v+c)/2 \quad (2.12)$$

We therefore obtain
\[ E(h \mid x) = \hat{p}_{v+1}(t_2) - \hat{p}_{v+1}(t_1) \]  \hspace{1cm} (2.13)

Here

\[ t_1 = 1(a_1 = \frac{n\bar{x}}{n+n'}) / s_1 \]
\[ t_2 = 1(a_2 = \frac{n\bar{x}}{n+n'}) / s_2 \]

\[ s_1^2 = \left( s^2 + \frac{mn'}{n+n'} \bar{x}^2 \right) / (v+1) \]
\[ 1 = \left( \frac{n+n'}{1+n+n'} \right)^{1/2} \]  \hspace{1cm} (2.14)

(iii) We now consider that, a priori, \((\mu, \sigma)\) have the conjugate prior distribution \(p(\mu, \sigma)\) (Raiffa and Schlaifer (1961)) –

\[ p(\mu, \sigma) \propto \sigma^{-(v'+2)} \exp \left[ -\frac{1}{2\sigma^2} \left\{ \frac{2}{v+n'}(\mu - m)^2 \right\} \right] \]  \hspace{1cm} (2.15)

When the prior distribution is given by (2.15), we have

\[ p(y \mid x) \propto \left\{ s^2 + \frac{mn'}{n+n'} (\bar{x} - m)^2 + \frac{n+n'}{1+n+n'} (y - \frac{n\bar{x}+n'm}{n+n'})^2 \right\} ^{-\frac{(v'+v'+2)}{2}} \]  \hspace{1cm} (2.16)

This gives

\[ E(h \mid x) = \hat{p}_{v+v'+1}(t_2) - \hat{p}_{v+v'+1}(t_1) \]  \hspace{1cm} (2.17)

Here

\[ t_1 = 1(a_1 = \frac{n\bar{x}+n'm}{n+n'}) / s_2 \]
\[ t_2 = 1(a_2 = \frac{n\bar{x}+n'm}{n+n'}) / s_2 \]

\[ s_2^2 = \left\{ s^2 + w^2 + \frac{mn'}{n+n'}(\bar{x} - m)^2 \right\} / (v+v'+1) \]  \hspace{1cm} (2.18)
The Bayes rule in each case selects the population which corresponds to \( \max E(h|\mathbf{x}) \).

A special case

As a special case of \( h \), we consider \( a_1 = -\infty \) and \( a_2 = \infty \). Also consider that sample sizes are equal. Then for the prior distribution in (i), the rule selects the population which corresponds to \( \max \frac{s - \bar{x}}{s_1} \). For the prior distribution in (ii) with equal \( n' \)'s, the rule selects the population which corresponds to \( \max \left( s - \frac{m}{n+n'} \right) / s_1 \). And for the prior distribution in (iii) with equal \( n' \)'s \( m \) and \( n' \), the rule selects the population with \( \max \left\{ \left( s - \frac{n'\bar{x} + n\bar{z}}{(n+n')} \right) / s(2) \right\} \).

3. Selection Procedures = II

In this section, we consider that the criterion is given by (1.3) -

\[
h_1 = g(\theta_1) = \Pr \left\{ b_1 < \frac{y_1}{\sigma_1} < b_2 \right\} \quad \cdots \quad (3.1)
\]

Let \( \mathbf{x} = (x_1, x_2, \ldots, x_p) \) where \( x_i \) denotes the sample from the \( i \)th population. The Bayes rule selects the population which corresponds to \( \max E(h_1|\mathbf{x}) \), \( i = 1, 2, \ldots, p \). Here we have

\[
E(h_1|\mathbf{x}) = \int_{b_1}^{b_2} p(x_1|\mathbf{x}) \, dz_1 \quad \cdots \quad (3.2)
\]
where \( p(\theta | x) \) is the posterior density of \( \theta \), given \( x \).

As in Section 2, we assume that \( \theta_1, \theta_2, \ldots, \theta_p \) are independent a priori. This assumption gives that \( h_1 \)'s are independent a posteriori. We therefore obtain the posterior expectation of \( h \) for a typical population \( N(\mu, \sigma^2) \).

(1) When we adopt (2.4) as the prior distribution of \( (\mu, \sigma^2) \), we obtain the posterior density of \( z - \)

\[
p(z | x) = p(z | xt) = \frac{(\frac{n}{1+n})^{1/2}}{\pi} \cdot \frac{(v-1)!}{2(v-2)/2, \frac{v-2}{2}} \cdot \left\{ \frac{v}{v + \frac{t^2}{1+n}} \right\}^{v/2} \cdot \exp \left\{ -\frac{1}{2} \frac{s^2}{1+n} \cdot \frac{1}{(1+n)^2} \cdot \frac{t^2}{1+n} \right\} \cdot I_{v-1} \left( -\frac{\sqrt{n} t s}{(1+n)(v+\frac{t^2}{1+n})} \right)^{1/2} \]

where \( t \) is the Student \( t \)-variable

\[ t = \sqrt{n} x / s \]

and \( I_v(x) \) is the function defined by Fisher (1931)

\[ I_v(x) = \int_0^\infty (\sqrt{2\pi} v!)^{-1} u^{-v} \exp \left\{ -\frac{1}{2} (u + x)^2 \right\} du \]
The mean and variance of the posterior distribution of $z$ are

$$E(z|x) = t \left( \frac{2}{v} \right)^{1/2} \left( \frac{v-1}{2} \right)^{1/2}$$

$$\text{Var}(z|x) = \frac{1+n}{n} + \frac{t^2}{n} \left( \frac{2}{v} \right) \left[ \frac{v}{2} = \left\{ \frac{(v-1)}{2} \right\}^{2} \right]$$

(ii) When we adopt (2.11) as the prior distribution of $(\mu, \sigma)$, we obtain the posterior density of $z$ as

$$p(z|x) = \frac{p(z|\mu, \sigma)}{p(\mu, \sigma)}$$

$$= \left( \frac{n+n'}{1+n+n'} \right)^{1/2} \frac{v}{2^{(v-1)/2} (\frac{v-1}{2})^{1/2}} \left\{ \frac{v+n't^2}{n+n'} \left( \frac{v}{2} + \frac{(1+n')t^2}{1+n+n'} \right) \right\}$$

$$= \exp \left\{ \frac{1}{2} v \left( \frac{n+n'}{1+n+n'} + \frac{n}{1+n+n'} \right) \right\}$$

$$I_v \left( \frac{\sqrt{n} t^2}{(1+n+n')(v+\frac{(1+n')t^2}{1+n+n'})} \right)$$

The mean and variance of this distribution are

$$E(z|x) = t \left( \frac{2}{v} \right)^{1/2} \left( \frac{v-1}{2} \right)^{1/2}$$

$$\text{Var}(z|x) = \frac{1+n+n'}{n+n'} + \frac{n't^2}{(n+n')(v+n't^2)} \left[ \frac{v+n't^2}{(n+n')} \right]^2$$
(iii) When the prior distribution of \((\mu, \sigma^2)\) is
the conjugate prior distribution given in (2.15), we obtain

\[
p(z|\mathbf{x}) = \frac{1}{2} \sqrt{\frac{1+\tau n}{v+v^\prime}} \frac{2^{(v+v^\prime)/2}}{(\text{det } \Sigma)^{1/2}} e^{-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \Sigma^{-1} (\mathbf{x} - \mathbf{m})} \quad \frac{1}{2} \sqrt{\frac{1+\tau n}{v+v^\prime}}
\]

\[
\exp \left[ -\frac{1}{2} \frac{2}{v+v^\prime} \left\{ \frac{n+1}{1+n} \frac{(n\bar{x} + n'm)^2}{(1+n+n') \left( \frac{1}{2} s_n^2 + \frac{(n\bar{x} + n'm)^2}{(1+n+n')(n+n') \right) \right\} \right]
\]

\[
I_{v+v^\prime} \left( -\frac{(n\bar{x} + n'm)^2}{(1+n+n')(\frac{1}{2} s_n^2 + \frac{(n\bar{x} + n'm)^2}{(1+n+n')(n+n') \right) \right) \right)
\]

\[
(3.9)
\]

where

\[
s_n^2 = s^2 + \sum (x_i - m)^2 / (n+n')
\]

\[
(3.10)
\]

The mean and the variance of the distribution (3.9)

are

\[
E(z|\mathbf{x}) = \frac{n\bar{x} + n'm}{(n+n')} \left( \frac{2}{v+v^\prime} \right) \quad \frac{1}{2} \sqrt{\frac{1+\tau n}{v+v^\prime}}
\]

\[
\frac{2}{v+v^\prime} \left( \frac{v+v^\prime-1}{2} \right)
\]

\[
\frac{1}{n+n'} \left( \frac{n\bar{x} + n'm}{(n+n') \right)^2 \left( \frac{2}{s_n^2} \right) \left( \frac{v+v^\prime-1}{2} \right)
\]

\[
\frac{1}{n+n'} \left( \frac{n\bar{x} + n'm}{(n+n') \right)^2 \left( \frac{2}{s_n^2} \right) \left( \frac{v+v^\prime-1}{2} \right)
\]

\[
(3.11)
\]
The expected values of $E(h|y)$ for each of the three prior distributions are obtained by integrating $p(z|x)$ over the range $b_1$ to $b_2$. The values can be computed by using the Tables of $I_v(x)$ functions. When $n$ is large, a good approximation to the expected values is obtained by means of $t$-distribution tables. Approximation by using $t$-distribution: when the prior distribution is given by (2.4), as $n$ increases, $\sigma$ will tend to $\sigma_{\text{posteriori}}$, we therefore put

$$E(h|y) = \Pr \{ b_1 < y/\sigma < b_2 \} = \Pr \{ b_1 < y/s < b_2 \} .$$

Now from (2.5) it follows that

$$\Pr \{ b_1 < y/s < b_2 \} = F_v(t_2) - F_v(t_1) \quad \cdots \quad (3.12)$$

where

$$t_2 = \left( \frac{n}{1+n} \right)^{1/2} (b_1 - \bar{x}) \quad \cdots \quad (3.13)$$

$$t_2 = \left( \frac{n}{1+n} \right)^{1/2} (b_2 - \bar{x}) \quad \cdots \quad (3.13)$$

When the prior distribution is given by (2.11), using a similar approximation we obtain

$$E(h|y) = F_{v+1}(t_2) - F_{v-1}(t_1) \quad \cdots \quad (3.14)$$

where

$$t_1 = 1(b_1 - \frac{n\bar{x}}{(n+n')^2}) \quad \quad t_2 = 1(b_2 - \frac{n\bar{x}}{(n+n')^2}) \quad \cdots \quad (3.15)$$
Similarly for the conjugate prior distribution (2.15) we obtain

\[ E(h|z) = f_{v+v',+1}(t_2) - f_{v+v',+1}(t_1) \]  \hspace{1cm} (3.16)

where

\[ t_1 = 1(b_1 = \frac{n\bar{x} + n'm}{(n-n')s^2_2}) \]  \hspace{1cm} (3.17)

\[ t_2 = 1(b_2 = \frac{n\bar{x} + n'm}{(n+n')s^2_2}) \]

We may also use the normal approximation to the posterior distribution of \( z \) by means of the first two moments and obtain \( E(h|z) \) approximately.

A special case

Let us consider the special case of \( h \) where \( b_2 = -b_1 = b \). Also consider that sample sizes are equal. Then irrespective of the value of \( b \), we have the following solutions:

For the prior distribution in (i), the rule selects the population with minimum \( |t| \). Also for the prior distribution in (ii) with equal \( n' \)'s, this rule selects the population with \( \min |t| \). And for the prior distribution in (iii) with equal \( n' \)'s, \( v' \), the rule selects the population with minimum \( 1((n\bar{x}+n'm)/(n+n')s^2_2) \).
In Sections 2 and 3 we have dealt with situations where we have no initial knowledge about the parameters involved in the distribution of the sample observations apart from the prior distributions. But there exist situations where we may have some additional knowledge — for example, it may be known that the variances of the populations are equal. We therefore consider (i) three situations where the variances are known, (ii) three situations where the variances are equal but unknown, and (iii) three situations where the variances are (completely) unknown. The three situations in each case arise from (a) the means are known, (b) the means are equal but unknown, and (c) the means are (completely) unknown.

In this section, we consider that the criterion $h$ is given by (1.1), i.e., $h = \Pr \{ a_1 < y < a_2 \} \cdot$

(1) Variances are known

(a) Means are known.

In this case

$$E(h_1 | y) = \Phi \left( \frac{a_2 - \mu_1}{\sigma_2} \right) - \Phi \left( \frac{a_1 - \mu_1}{\sigma_1} \right).$$

for $i = 1, 2, \ldots, p$. The function $\Phi(x)$ is the distribution function of a standard normal variable.
(b) Means are equal but unknown.

When the prior distribution of $\mu_j$ is given by

$$p(\mu) \propto (\sigma_j^{-1})^\text{1/2} \exp \left\{ -\frac{1}{2\sigma_j^2} (\mu - m_j)^2 \right\} \quad \ldots \quad (4.2)$$

we obtain

$$E(h_1 | X) = \mathcal{F} \left\{ \left( \frac{1}{\sigma_j^2} \frac{\sum_j x_j + v^i}{\sum_j x_j + v^i + \frac{1}{\sigma_j^2}} \right)^{1/2} \left( s_1 - \frac{\sum_j x_j + v^i}{\sum_j x_j + v^i} \right) \right\} \quad (4.3)$$

where $w_j = \frac{n_j}{\sigma_j^2}$, $v^i = \frac{1}{\sigma_j^2}$.

If priori $\mu$ is uniformly distributed, the expectation $E(h_1 | X)$ is obtained from (4.3) by putting $m = 0$ and $w^i = 0$.

(c) Means are not known.

If we assume that, a priori, $\mu_j$'s are independent and have the distribution

$$p(\mu_j) \propto (\sigma_j^{-1})^\text{1/2} \exp \left\{ -\frac{n_j}{2\sigma_j^2} (\mu_j - m_j)^2 \right\} \quad \ldots \quad (4.4)$$

we get
\[
E(h_1 | x) = \mathcal{F}\left\{ \frac{n_1 + n'_1}{1 + n_1 + n'_1} \left( s_2 - \frac{n_1 \bar{x}_1 + n'_1 \bar{x}_1'}{n_1 + n'_1} \right) / \sigma \right\}
\]

\[
- \mathcal{F}\left\{ \frac{n_1 + n'_1}{1 + n_1 + n'_1} \left( s_1 - \frac{n_1 \bar{x}_1 + n'_1 \bar{x}_1'}{n_1 + n'_1} \right) / \sigma \right\} \ldots (4.5)
\]

If a priori the \( \mu \)'s are independent and uniformly distributed, the expectations \( E(h_1 | x) \) are obtained from (4.5) by putting \( n'_1 \) equal to zero.

(ii) Variance are equal but unknown

(a) Means are known.

Let us assume that the common parameter \( \sigma \)

\[ p(\sigma) = \sigma^{-1} \ldots (4.6) \]

We then have

\[
E(h_1 | X) = \mathcal{F}_N(t_2) - \mathcal{F}_N(t_1) \ldots (3.7)
\]

where

\[
t_1 = (s_1 - \mu_1) / \left( \Sigma i j (x_{ij} - \mu_1)^2 / (p) \right)^{1/2}
\]

\[
t_2 = (s_2 - \mu_1) / \left( \Sigma i j (x_{ij} - \mu_1)^2 / (p) \right)^{1/2} \ldots (3.8)
\]

\[
\Sigma i j (x_{ij} - \mu_1)^2
\]

\[
N = \Sigma n_1
\]
When the prior distribution of the common parameter \( \sigma \)

is

\[
p(\sigma) \propto \sigma^{-(v' + 1)} \exp\left\{-\frac{v^2}{2\sigma^2}\right\} \quad \ldots \quad (4.9)
\]

we get

\[
E(h_1) = F_{v' + v'}(t_2) - F_{v' + v'}(t_1) \quad \ldots \quad (4.10)
\]

where

\[
t_1 = \left( s_1 - \mu_1 \right) / \left\{ \left( \frac{s_1^2}{v} + \frac{v^2}{v'} \right) / (v' + v) \right\}^{1/2} \quad \ldots \quad (4.11)
\]

\[
t_2 = \left( s_2 - \mu_1 \right) / \left\{ \left( \frac{s_2^2}{v} + \frac{v^2}{v'} \right) / (v' + v) \right\}^{1/2}
\]

(b) Means are known to be equal

In this case the populations are identical and the selection problem does not arise

(c) Means are unknown.

Consider that the prior distribution of \((\mu_1, \mu_2, \ldots, \mu_p, \sigma)\)

is

\[
p(\mu_1, \ldots, \mu_p, \sigma) \propto \sigma^{-1} \quad \ldots \quad (4.12)
\]

We then get

\[
E(h_1) = F_{v(p)}(t_2) - F_{v(p)}(t_1) \quad \ldots \quad (4.13)
\]

where

\[
t_2 = \left( \frac{n_1}{1 + n_1} \right)^{1/2} \left( \frac{s_1 - \bar{x}_1}{s(p)} \right)
\]

\[
t_2 = \left( \frac{n_1}{1 + n_1} \right)^{1/2} \left( \frac{s_2 - \bar{x}_1}{s(p)} \right)
\]
\[
s^2(p) = \frac{s^2(p)}{v(p)} \\
S^2(p) = \sum \sum (x_{ij} - \bar{x}_1)^2 \\
v(p) = \sum (n_i - 1)
\]

If the prior distribution of \( \mu_1, \mu_2, \ldots, \mu_p, \sigma \) is

given by
\[
p(\mu_1, \ldots, \mu_p, \sigma) \propto \sigma^{-(p+1)} \exp \left\{ -\frac{1}{2\sigma^2} \sum n_i (\mu_i - \bar{x}_1)^2 \right\} \quad (4.18)
\]

we obtain
\[
E_n(h|\bar{x}) = F_{v_1(p)}(t_2) - F_{v_1(p)}(t_1) \quad \ldots \quad (4.19)
\]

where
\[
t_1 = \frac{1}{2} \left( s_1 - n_1 \bar{x}_1 / (n_1 + n'_1) \right) / s_1(p) \\
t_2 = \frac{1}{2} \left( s_2 - n_1 \bar{x}_1 / (n_1 + n'_1) \right) / s_1(p) \\
s^2_1(p) = \frac{s^2_1(p)}{v_1(p)} \quad \ldots \quad (4.17)
\]
\[
S_1^2(p) = \sum \sum (x_{ij} - \bar{x}_1)^2 + \frac{n_1 n'_1}{n_1 + n'_1} \bar{x}_1^2 \\
l_1 = \frac{n_1 + n'_1}{1 + n_1 + n'_1}, \quad v_1(p) = v(p) + p
\]

If the prior distribution of \( \mu_1, \mu_2, \ldots, \mu_p, \sigma \) is

the conjugate prior distribution
\[ p(\nu_1, \nu_2, \ldots, \nu_p, \sigma) \propto \sigma^{-(v^* + p + 1)} \exp \left[ -\frac{1}{2\sigma^2} \left( \nu^* \right)^2 + \Sigma n_i' (\nu_i - m_i)^2 \right] \]  

we obtain

\[ x(h_{1|X}) = \frac{1/2}{v^*} \left( t_2 - \frac{1/2}{v^*} v_2(p) t_1 \right) \]  

where

\[ t_1 = \frac{1/2}{n_1} \left( a_1 - \frac{n_1 \bar{x}_1 + n_1' m_1}{n_1 + n_1'} \right) / s_2(p) \]

\[ t_2 = \frac{1/2}{n_2} \left( a_2 - \frac{n_2 \bar{x}_2 + n_2' m_2}{n_2 + n_2'} \right) / s_2(p) \]

\[ s_2^2 = s_2^2 / v_2(p) \]

\[ s_2^2 = \Sigma \Sigma \left( x_{ij} - \bar{x}_i \right)^2 + \frac{2}{v^*} + \Sigma \frac{n_1 n_1'}{n_1 + n_1'} \left( \bar{x}_i - m_i \right)^2 \]

\[ v_2(p) = v_1(p) + v' \]

(iii) Variances are unknown

(a) Means are known.

We assume that, \textit{a priori}, \( \sigma_i, i=1,2, \ldots \), are independently distributed. The variables \( h_i \) have in this case independent posterior distributions. We give the posterior expectations of \( h \) for a typical population \( \mathcal{N}(\mu, \sigma^2) \).
When \( \log \sigma \) has an uniform prior distribution

\[
E(h \mid x) = F_n \left( \frac{a_2 - \mu}{s'} \right) - F_n \left( \frac{a_1 - \mu}{s'} \right) \quad \ldots \quad (4.21)
\]

where

\[
s'^2 = s^2/n, \quad s^2 = \frac{n}{\sum (x_j - \mu)^2} \quad \ldots \quad (4.22)
\]

When the prior distribution of \( \sigma \) is the conjugate prior distribution

\[
p(\sigma) \propto \sigma^{-(v' + 1)} \exp \left\{ -\frac{1}{2} \frac{y^2}{\sigma^2} \right\} \quad \ldots \quad (4.23)
\]

we obtain

\[
E(h \mid x) = F_{n+v'}(t_2) - F_{n+v'}(t_1) \quad \ldots \quad (4.24)
\]

where

\[
t_1 = \left( a_1 - \mu \right) / \left\{ (s'^2 + y^2) / (n + v') \right\}^{1/2}
\]
\[
t_2 = \left( a_2 - \mu \right) / \left\{ (s'^2 + y^2) / (n + v') \right\}^{1/2} \quad \ldots \quad (4.25)
\]

(b) Means are equal but unknown.

In this case we merely give the approximate solution obtained by substituting an estimate \( \hat{\mu} \) for the unknown common mean.

When the prior distribution of \( (\mu, \sigma_1, \sigma_2, \ldots, \sigma_p) \) is

\[
p(\mu, \sigma_1, \sigma_2, \ldots, \sigma_p) \propto \frac{1}{\sigma_1} (\sigma_1^{-1}) \quad \ldots \quad (4.26)
\]
the estimate \( \hat{\mu} \) is taken as

\[
\hat{\mu} = \frac{\sum n_i \bar{x}_i / s_i^2}{\sum n_i / s_i^2}
\]  
(4.27)

and the expectation \( E(h(x)) \) is obtained approximately from (4.21) by using \( \hat{\mu} \) in place of \( \mu \).

When the prior distribution of \( \sigma_1, \sigma_2, \ldots, \sigma_p, \mu \) is

\[
p(\sigma_1, \sigma_2, \ldots, \sigma_p, \mu) \propto \left\{ \prod_{i=1}^{p} \sigma_i^{-1} \right\} \exp \left\{ -\frac{1}{2 \sigma^2} (\mu - m)^2 \right\}
\]

we take the estimate of \( \mu \),

\[
\hat{\mu} = \frac{\sum \frac{n_i \bar{x}_i}{s_i^2} + m}{\sum \frac{n_i}{s_i^2} + \sigma^2}
\]
(4.23)

and obtain \( E(h(x)) \) using (4.21).

When the prior distribution of \( \Phi, \sigma_1, \sigma_2, \ldots, \sigma_p \) is given by

\[
p(\Phi, \sigma_1, \sigma_2, \ldots, \sigma_p) \propto \sigma_\mu^{-1} \exp \left\{ -\frac{1}{2 \sigma^2} (\mu - m)^2 \right\}
\]

\[
p(\Phi, \sigma_1, \sigma_2, \ldots, \sigma_p) \propto \prod_{i=1}^{p} \sigma_i^{-1} \exp \left\{ -\frac{\sum_{i=1}^{p} (X_i - \mu_i)^2}{2 \sigma_i^2} \right\}
\]
(4.30)
We may use the estimate of \( \mu \),

\[
\hat{\mu} = \frac{\sum_{1}^{n_{1}} \frac{n_{1} \overline{X}_{i} + m}{s_{i}^{2}} + \frac{1}{\sigma_{\mu}^{2}}}{\sum_{1}^{n_{1}} \frac{n_{1}}{s_{i}^{2}} + \frac{1}{\sigma_{\mu}^{2}}}
\]

(4.31)

where

\[
s_{i}^{2} = (s_{i}^{2} + v_{i}^{2})/(n_{i} + v_{i}^{1} - 1)
\]

(4.32)

The expectation \( E(b|\mathbf{X}) \) is then obtained approximately using (4.24).

(c) When the means are unknown, we have the situation discussed in Section 2.

5. **Selection Procedures - IV**

In this section, we consider the case when the criterion \( h \) is given by (1.3), i.e., \( h = \Pr \{ b_{1} < y_{i}/\sigma_{1} < b_{2} \} \).

We give briefly the solutions for the different situations discussed in the last section.

- **(1) Variances are known.**

The expectations \( E(b_{1}|\mathbf{X}) \) in the various situations are obtained from those given for the corresponding situations in Section 4 by putting \( s_{1} = b_{1}\sigma_{1} \), \( s_{2} = b_{2}\sigma_{1} \).
(ii) Variances are equal but unknown.

(a) Means are known.

The expected values $E(h_1|X)$ are obtained from the posterior density of $z_i = \gamma_i / \sigma$.

When the common parameter $\sigma$ has the prior distribution given in (4.6), we obtain

we obtain

$$p(z_i | X) \propto \exp \left\{ -\frac{1}{2} z_i^2 \left( \frac{s_1^2}{(p)} \right) \right\}$$

$$I_N \left( -\frac{\mu_1 \bar{z}_i}{(s_1^2 + \psi^2)^{1/2}} \right)$$

where $s_1^2$ is defined in (4.3).

When the prior distribution of the parameter $\sigma$ is given in (4.9), we obtain

we obtain

$$p(z_i | X) \propto \exp \left\{ -\frac{1}{2} z_i^2 \left( \frac{s_1^2 + \psi^2}{(p)} \right) \right\}$$

$$I_{N+v} \left( -\frac{\mu_1 \bar{z}_i}{(s_1^2 + \psi^2)^{1/2}} \right)$$

(b) Means are known to be equal.

In this case the populations are identical.
(c) Means are unknown

When the prior distribution is given by (4.12), we obtain

\[ p(z_1|x) = p(z_1|t_1) = \exp \left\{ - \frac{1}{2} s_1^2 \left( \frac{n_1}{1+n_1} - \frac{n_1 t_1^2}{(1+n_1)(v(p) + \frac{t_1^2}{1+n_1})} \right) \right\} \]

\[ I_{v(p)} - 1 \left( \frac{\sqrt{n_1 t_1^2}}{(1+n_1)(v(p) + \frac{t_1^2}{1+n_1})} \right) \] \hspace{1cm} (5.3)

where \( t_1 = \frac{\sqrt{n_1 s_1^2}}{s(p)} \). \hspace{1cm} (5.4)

The \( s_2^2 \), \( v(p) \) are as defined in (4.14).

When the prior distribution is given by (4.15), we obtain

\[ p(z_1|x) = p(z_1|t_1) = \exp \left\{ - \frac{1}{2} s_1^2 \left( \frac{n_1 + n_1'}{1+n_1 + n_1'} - \frac{n_1 t_1^2}{(1+n_1 + n_1')(v_1(p) + \frac{(1+n_1')t_1^2}{1+n_1 + n_1'})} \right) \right\} \]

\[ I_{v_1(p)} - 1 \left( \frac{\sqrt{n_1 t_1 s_1}}{(1+n_1 + n_1')(v_1(p) + \frac{(1+n_1')s_1}{1+n_1 + n_1'})} \right) \] \hspace{1cm} (5.5)

where \( t_1 \)'s are defined by (5.4) and \( v_1(p) \) in (4.17).
When the prior distribution is given by (4.18), we get

\[ p(z_1 | x) \propto \exp \left[ -\frac{1}{2} z_1^2 \left\{ \frac{n_1 + n'_1}{1 + n_1 + n'_1} - \frac{(n_1 \bar{x}_1 + n'_1 m_1)^2}{(1 + n_1 + n'_1)(2p) + \frac{(n_1 \bar{x}_1 + n'_1 m_1)^2}{(1 + n_1 + n'_1)(n + n'_1)} } \right\} \right] \]

\[ \frac{(n_1 \bar{x}_1 + n'_1 m_1)^2}{(1 + n_1 + n'_1)(2p) + \frac{(n_1 \bar{x}_1 + n'_1 m_1)^2}{(1 + n_1 + n'_1)(n + n'_1)} } \]

The \( s^2 \) and \( v^2 \) are defined in (4.20).

The expected values \( E(h_i | x) \) are obtained by integrating \( p(z_1 | x) \) over the range \((a, b_2)\).

(iii) Variances are unknown

(a) Means are known.

We assume that, a priori, \( \sigma_i, i = 1, 2, \ldots, p \) are independently distributed. The variables \( z \) will then have independent posterior distributions. We give the posterior distribution of \( z \) for a typical population \( N(\mu, \sigma^2) \).

When log \( \sigma \) has a uniform prior distribution, we get

\[ p(z_1 | x) \propto \exp \left\{ -\frac{1}{2} z^2 \left( \frac{s^2}{s^2 + \mu^2} \right) \right\} I_{n-1} \left( \frac{\mu^2}{s^2} \right) \]  

(5.7)

The \( s^2 \) is defined in (4.22).
When the prior distribution of $\sigma$ is as given by (4.23), we get

$$p(z|x) \propto \exp \left\{ -\frac{1}{2} s^2 \left( \frac{s^2 + w^2}{s^2 + w^2 + \mu^2} \right) \right\}$$

$$I_{n+v'-1} \left( \frac{\mu^2}{s^2 + w^2 + \mu^2} \right)^{1/2}$$

We should integrate $p(z|x)$ over the range $(b_1, b_2)$ to obtain $E(h|x)$.

(b) Means are equal but unknown.

In this case we use the approximation suggested in Section 4. Thus when the prior distribution is given by (4.23), for a typical normal population $N(\mu, \sigma^2)$ we may approximate

$$E(h|x) \approx F_n\left( b_2 - \frac{\hat{\mu}}{s'} \right) = F_n\left( b_1 - \frac{\hat{\mu}}{s'} \right) \ldots \quad (5.9)$$

with $\hat{\mu}$ defined as in (4.27) and $s'$ defined by (4.22) using $\hat{\mu}$ in place of $\mu$. When the prior distribution is given by (4.23), we may use (5.9) with $\hat{\mu}$ defined by (4.29). When the prior distribution is given by (4.30), we use the estimate $\hat{\mu}$ given in (4.31) and write

$$E(h|x) \approx F_{n+v'}\left( b_2 - \frac{\hat{\mu}}{s''} \right) = F_{n+v'}\left( b_1 - \frac{\hat{\mu}}{s''} \right) \quad (5.10)$$

where

$$s''^2 = \frac{s^{''2}}{n+v'}, \quad s^{''2} = s'^2 + w^2, \quad \ldots \quad (5.11)$$
(e) The means are unknown

This case is considered in Section 3.

References


