A SET OF MINIMUM NUMBER OF INDEPENDENT POSTULATES FOR A BOOLEAN ALGEBRA.

It is known that a Boolean Algebra can be defined by widely varying postulate systems. For example, a set of postulates for an Algebra $A$ which includes Boolean Algebra was given by Newman as follows.

- $P_1$  \[ a(b + c) = ab + ac; (a+b)c = ac + bc \] for all $a, b$ and $c \in A$
- $P_2$  There exists $1$ in the algebra such that $a1 = a$ for all $a \in A$
- $P_3$  There exists $0$ in the algebra with $a + 0 = a = 0 + a$
- $P_4$  To each $a$ there corresponds at least one $\bar{a}$ such that $a\bar{a} = 0$ and $a + \bar{a} = I$.

Another set of independent postulates for a Boolean Algebra was given by E.V. Huntington $[5]$ as follows.

- $P'_1$  \[ a + b = b + a; ab = ba \]
- $P'_2$  \[ a + b.c = (a + b)(a + c) \]
- $P'_3$  \[ a(b + c) = ab + ac \]

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There exist 0 and 1 in the algebra such that $a + 0 = a$ and $a \cdot 1 = a$ for any $a$.

To each $a$ there corresponds $\overline{a}$ such that $a + \overline{a} = 1$; $a \cdot \overline{a} = 0$.

We attempt here to define a Boolean Algebra by a set of independent postulates which do not assume idempotence, associativity of either operation (+, Boolean sum and $\cdot$, Boolean product) and one of the distributive laws namely, $x + yz = (x + y)(x + z)$.

We start with a system $(B, +, \cdot, \overline{\cdot})$ where $B$ is a non-empty set of elements closed under the two binary operations $+$, $\cdot$, and a unary operation $\overline{\cdot}$ (Boolean complement), which satisfy the following postulates:

I. $x + y = y + x$; $x \cdot y = y \cdot x$ for all $x, y \in B$.

II. $x(y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in B$.

There exist two distinct elements 0 and 1 in $B$ so that for all $x \in B$,

III. $1 + x = 1$; $1 \cdot x = x$.

IV. $x + \overline{x} = 1$; $x \cdot \overline{x} = 0$.

The main result we prove here is the following:

Theorem.— A system $B$ satisfying I-IV is a Boolean Algebra. That is, as a consequence of I-IV above, we shall have

V. $x + 0 = x$; $x \cdot 0 = 0$.

VI. $\overline{\overline{x}} = x$; $\overline{0} = 1$; $\overline{1} = 0$. 
VII. \( x + x = x; x.x = x; x(x + y) = x; \)
\( x + x.y = x. \)

VIII. \( x + y.z = (x + y)(x + z). \)

IX. \( x + (y + z) = (x + y) + z; \)
\( x(yz) = (xy)z. \)

The proof of the above theorem is embodied in the following results 1-23:

1. \( x + 0 = 0 + x = x. \)
   
   Proof.— \( x + 0 = 0 + x \) follows by 1.

Now,

\[ x + 0 = x + x.\bar{x} = x(1 + \bar{x}) = x.1 = x \quad (IV,III,II). \]

2. \( x.0 = 0. \)
   
   Proof.— \( x.0 = x.0 + 0 = x.0 + x.\bar{x} = x(0 + \bar{x}) = x.\bar{x} = 0 \quad (I, IV, II). \)

3. \( x.x = x. \)
   
   Proof.— \( x.x = x.x + 0 = x.x + x.\bar{x} = x(x + \bar{x}) = x.1 = x \quad (I, IV, II, III). \)

4. \( \bar{x} = x. \)
   
   Proof.— \( \bar{x} = 1.x = (x + \bar{x})\bar{x} = x.\bar{x} + \bar{x}.x = x.\bar{x} + 0 \)
   
   \[ = x.\bar{x} + x.\bar{x} = x(\bar{x} + \bar{x}) = x.1 = x \quad (III,IV,II,1). \]

5. \( 0 = 1; \bar{1} = 0 . \)

These follow from IV and 1, 2 when \( x = 0 \) and \( x = 1 \) respectively.

6. If \( x + y = 1 \) and \( x.y = 0 \), then \( y = \bar{x} . \)
For,
\[ y = l.y = (x + \bar{x}).y = x.y + \bar{x}.y = 0 + \bar{x}.y = x.\bar{x} \]
+ \bar{x}.y = \bar{x}(x + y) = \bar{x}.l = \bar{x}.

7. \( x + (x.y) = x \).

Proof. — \( x + x.y = x.l + x.y = x.y = x(1 + y) = x.l = x \).

8. \( x(x + y) = x \).

For,
\[ x(x + y) = x.x + x.y = x + x.y = x.\bar{x} \quad (3, 7). \]

9. \( x(\bar{x}.y) = 0 ; \bar{x}(x.y) = 0 \).

Proof. — \( x(\bar{x}.y) = x(\bar{x}.y) + x.\bar{x} = x[\bar{x}.y] + \bar{x} \]
= \( x.\bar{x} = 0. \) by 7.

10. \( x(x.y) = x.y \).

For,
\[ x(x.y) = x(x.y) + \bar{x}(x.y) = (x + \bar{x})(x.y) = x.y (9). \]

11. \( (x.y)(y + z) = x.y \).

Proof. — L.H.S. = (x.y).y + (x.y)z = (x.y) + (x.y)z by 1
= x.y. by 7.

It is to be noted that 9, 10 and 11 are special cases of the associative law \( x(yz) = (xy)z \) when (i) \( \bar{x} = y \); (ii) \( x = y \); (iii) \( z = y + \text{some element} \).

We now prove
12. \( y + (\bar{y}.x) = y + x \).

This is a special case of the distributive law
\[ a + b.c = (a + b) (a + c) \].
Proof. — \( y + (\bar{y} \cdot x) = y + (y \cdot \bar{y} + \bar{y} \cdot x) = y + \bar{y} (y + x) \)
\[ = y(y + x) + \bar{y}(y + x) \] (8)
\[ = (y + \bar{y})(y + x) = 1.(y + x) = (y + x). \]

We now prove the distributive law

13. \( (x + y)(x + z) = x + y \cdot z \).

Proof. — \( (x + y)(x + z) = x(x + z) + y(x + z) \) (II)
\[ = x + y(x + z) \] (8)
\[ = x + \bar{x}[y(x + z)] \] (12)
\[ = x + \bar{x}[(yx) + (yz)] \]
\[ = x + [0 + \bar{x}(yz)] \] (9)
\[ = x + \bar{x}(yz) \]
\[ = x + y \cdot z. \] (12)

Now we prove

14. \( x + x = x \).

For,
\[ (x + x) = (x + x) \cdot 1 = (x + x \cdot \bar{x})(x + \bar{x}) = x + x \cdot \bar{x} = x + 0 \]
\[ = x. \] (13)

We now notice that all the "duals" of the postulates I-IV, with which we have started, are true. For, the two parts of 1 are the duals of each other, so also the two parts of IV. The result 13 is the dual of II and the results 1 and 2 are the duals of the two parts of III. Hence any result gives rise to its dual.

The following three results are the "duals" of 9, 10 and 11 respectively:—
15. \( x + (\overline{x} + y) = 1 \).
16. \( x + (x + y) = x + y \).
17. \( (x + y) + (yz) = x + y \).

We next prove

18. \( (x + y)^- = \overline{x} \overline{y} \).

Proof. — \( (x + y) + \overline{x} \overline{y} = (x + y) + \overline{x} \cdot (x + y) + \overline{y} = 1.1 \) by 13, 15.

Also

\((x + y) \cdot (\overline{x} \overline{y}) = x(\overline{x} \overline{y}) + y(\overline{x} \overline{y}) = 0 \) by 9.

Now the result follows by 6.

19. "Dually" \( (x \cdot y)^- = \overline{x} + \overline{y} \).

20. \([x + (y + z)] + \overline{y} = 1\).

Proof. — L.H.S. \( = [x + (y + z)] + \overline{y} \cdot [y + \overline{y}] \)
\( = [y(x + (y + z)) + \overline{y}] \) (13)
\( = [(xy + y) + \overline{y}] \) by II and 8
\( = y + \overline{y} = 1 \) by 7.

21. \([x \cdot (yz)] \cdot \overline{y} = 0\)

This is the dual of 20.

We now prove the associative laws as a consequence of the above results.

22. \( (x + y) + z = x + (y + z) \).

Proof. — Put
\( x + (y + z) = r \).
Then
\[ r + \bar{x} = 1 \text{ by 15} . \]
\[ r + \bar{y} = 1 \text{ by 20} . \]

Similarly
\[ r + \bar{z} = 1 . \]

So,
\[ r + (\bar{x}y) = (r + \bar{x}) (r + \bar{y}) (r + \bar{z}) = 1 \text{ (13)}. \]

Also
\[
[r[(\bar{x}y)z] = x[(\bar{x}y)z] + (y + z)[(\bar{x}y)z] \text{ by II}
\]
\[ = (y + z)[(\bar{x}y)z] \text{ by 21.}
\]
\[ = y[(\bar{x}y)z] + z[(\bar{x}y)z]
\]
\[ = 0 \text{ by 21 and 9} . \]

Now, since
\[ r + (\bar{x}y)z = 1 \text{ and } r[(\bar{x}y)z] = 0 . \]

We have, by 6 and 4
\[ r = [(\bar{x}y)z] = (x + y) + z . \]

23. "Dually" \( xyz = x(yz) \).

Independence of the postulates.

We now prove that the postulates are independent. That is, none of them is dispensable and none of them is a consequence of others. To prove this we give examples of sets whose elements satisfy all but one of the axioms and all the consequences of all but this one axiom.
1. **Independence of** \( a + b = b + a \)

We consider an algebra with just two elements 0, 1 each the complement of the other and the operations +, defined by the tables

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

Here \( a + b \neq b + a \) for all \( a \) and \( b \) for, \( 1 + 0 = 1 \) and \( 0 + 1 = 0 \). Also we can easily verify that this algebra satisfies all the other axioms as follows.

1. \( 0 \cdot 0 = 0 \cdot 1 \) (\( ab = ba \))
2. \( 1 + 0 = 1 ; 1 + 1 = 1 \) (\( 1 + a = 1 \))
3. \( 1 \cdot 0 = 0 ; 1 \cdot 1 = 1 \) (\( 1 \cdot a = 1 \))

\( 1(1 + 0) = 1 \cdot 1 = 1 \) and \( 1 \cdot 1 + 1 \cdot 0 = 1 + 0 = 1 \)

Similarly for other combinations of values for \( a, b \) and \( c \) where \( a, b, c \) are 0 or 1, we can verify \( a(b + c) = ab + ac \);

\( 1 + 0 = 1 ; 1 \cdot 0 = 0 \); (\( a + \bar{a} = 1 \); \( a\bar{a} = 0 \))

Thus all the axioms are satisfied except \( a + b = b + a \).

2. **Independence of** \( ab = ba \)

As before we consider an algebra with just 0 and 1 where the operations +, \( \cdot \) are defined by
Here $ab \neq ba$ for all $a$ and $b$, for $1.0 = 0$ and $0.1 = 1$. But the other axioms are satisfied as follows.

$$1 + 0 = 1 = 0 + 1 \quad (a + b = b + a)$$
$$1(1 + 0) = 1.1 = 1 \text{ and } 1.1 + 1.0 = 1 + 0 = 1.$$  
Similarly for other combinations of values for $a$, $b$ and $c$ we can verify the axiom $a(b + c) = ab + ac$

$$1 + 0 = 1; \quad 1 + 1 = 1 \quad (1 + a = 1)$$
$$1.0 = 0; \quad 1.1 = 1 \quad (1.a = a)$$
$$1 + 0 = 1; \quad 1.0 = 0; \quad (a + \overline{a} = 1; \quad a\overline{a} = 0).$$

3. **Independence of** $a(b + c) = ab + ac$

Again we consider an algebra with just two elements $0$ and $1$ where the operations $+$ and $\cdot$ are defined by the tables

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\quad \quad \quad
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 1 \\
\end{array}
\]
Here \( a(b + c) \neq ab + ac \) for all \( a, b, c \). For \( 0(1 + 0) = 0 \cdot 1 = 0 \) and \( 0.1 + 0.0 = 0 + 1 = 1 \). Also all the other axioms are satisfied as follows.

\[ \begin{align*}
1 + 0 &= 1 = 0 + 1 \quad (a + b = b + a) \\
0.1 &= 0 = 1.0 \quad (ab = ba)
\end{align*} \]

\[ \begin{align*}
l + 0 &= l = 1 + l \quad (1 + a = l) \\
l.0 &= 0; \quad 1.l = 1 \quad (1.a = a) \\
l + 0 &= l; \quad 1.0 &= 0 \quad (a + \bar{a} = 1; \quad \bar{a} = 0)
\end{align*} \]

4. Independence of \( l + a = 1 \) for all \( a \).

Here also we consider an algebra with \( 0 \) and \( 1 \) where the operations \( + \) and \( \cdot \) are defined by the tables

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

Here the axiom \( l + a = 1 \) for all \( a \) is not satisfied, for \( l + 1 = 0 \). But the other axioms can be easily verified as follows.

\[ \begin{align*}
l + 0 &= l = 0 + 1 \; ; \; 1.0 &= 0 = 0.l \quad (a + b = b + a; \quad ab = ba.) \\
l(1 + 0) &= l.1 = 1 \quad \text{and} \quad l.1 + 1.0 = 1 + 0 = 1.
\end{align*} \]

Similarly for the other combinations of values for \( a, b \) and \( c \) we can verify the axiom \( a(b + c) = ab + ac \).
5. Independence of \( l.a = a \) for all \( a \).

Here the operations \(+\) and \(\cdot\) are defined by the tables:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\cdot)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Here \( l.a = a \) for all \( a \) is not satisfied since \( l.1 = 0 \).

But the other axioms are satisfied as follows.

\[ l + 0 = 1 = 0 + l \; \quad l.0 = 0 = 0.l \; \quad (a + b = b + a \; \quad ab = ba) \]

\[ l(1 + 0) = 1.1 = 0 \; \quad \text{and} \; \quad 1.1 + 1.0 = 0 + 0 = 0. \]

Similarly for the other combinations of values for \( a, b \) and \( c \) we can verify \( a(b + c) = ab + ac \).

Also \( l + 0 = 1 \; \quad l + 1 = 1 \; \quad (1 + a = 1) \)

and \( l + 0 = 1 \; \quad l.0 = 0 \; \quad (a + \bar{a} = 1 \; \quad aa = 0) \).

6. Independence of \( a + \bar{a} = 1 \) and \( aa = 0 \).

Here we consider a system consisting of \( 0, 1 \) and \( a \) (not equal to either \( 0 \) or \( 1 \)) where the operations \(+\) and \(\cdot\) are defined by the tables.
It is clear from these tables that the complement of \( a \) does not exist and all the other axioms I-III are satisfied.

**Conclusion.** We summarise our results by writing down the following conclusion:

In the system of postulates \( P_1' - P_4' \) due to E.V. Huntington, by replacing \( a + 0 = a \) in postulate \( P_3' \) by \( 1 + a = 1 \) as in our postulate III, the postulate \( a + b \cdot c = (a + b)(a + c) \) in \( P_2' \) becomes unnecessary. It will follow (Theorem 13) as a consequence of our postulates I - IV.
REFERENCES


