CHAPTER II

QUASICONFORMALLY FLAT SPACE*

2.0 Introduction:

Yano and Sawaki [4] defined and studied a tensor-field $W$ on a Riemannian manifold $M_n$, which includes both conformal curvature tensor $C$ and concircular curvature tensor $Z$ as special cases. We call a space $M_n$, to be quasi-conformally flat if the tensorfield $W$ vanishes on $M_n$. A quasi-conformally flat space is either conformally flat or Einstein. Since an Einstein space need not be conformally flat, a quasi conformally flat space need not be conformally flat. In Section 2.3, we study a hypersurface of a quasi-conformally flat space and obtain a suitable extension of

* Part of this chapter was published in Tensor (N.S) Vol.31 (1977) pp. 194-198 under the title "On Quasi-conformally flat spaces".
a theorem of Ohen and Yano \(2\). In Section 2.4 we consider a quasi conformally flat space of space form of constant curvature of codimension one and point out that in a theorem of Ohen and Yano \(2\) the word conformally flat can be replaced by quasi conformally flat provided the constant 'a' appearing in the definition of the tensor \(W\), is not zero. In section 2.5 we study the \(w\)-recurrrent space.

2.1. Quasi conformally flat space:

Suppose \((M_n, g)\) is an \(n\)-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods \((U, x^i)\). Let \(g_{ij}, R_{kjih}\) and \(X_{ji}\) denote the local components of the metric tensor \(g\), the curvature tensor and the Ricci tensor respectively and let \(K\) be the scalar curvature. Then the \(W\)-tensor introduced by Yano and Sawaki \(4\), and concircular tensor are given by \((1.4.15)\) and \((1.4.16)\) respectively.

**Definition 2.1.1:** A Riemannian space \((M_n, g)\) \(n > 3\), is said to be quasi conformally flat if the tensor field \(W\) defined by \((1.4.15)\) is zero on \(M_n\).

**Proposition 2.1.1:** A quasi conformally flat space \(M_n\) is either conformally flat or Einstein.

**Proof:** The equation \((1.4.15)\), by using \((1.4.17)\), \((1.4.16)\) and \((1.4.10)\) can be written in the form
(2.1.1) \[ W_{kji}^h = -(n-2)b c_{kji}^h + \Gamma_{kji}^{n+(n-2)b} Z_{kji}^h. \]

Now by setting \( W_{kji}^h = 0 \) in (2.1.1), we get

(2.1.2) \[ (n-2)b c_{kji}^h - \Gamma_{kji}^{n+(n-2)b} Z_{kji}^h = 0. \]

Transvect (2.1.2) by \( g^{kh} \) and use the fact that

\[ g^{kh} c_{kji}^h = 0, \quad g^{kh} Z_{kji}^h = G_{ji} \]

to get

(2.1.3) \[ \Gamma_{kji}^{n+(n-2)b} G_{ji} = 0. \]

Thus from (2.1.3) we have two cases:

Case i : \( a+(n-2)b = 0 \) (\( G_{ji} \neq 0 \))

Case ii: \( G_{ji} = 0 \) (\( a+(n-2)b \neq 0 \)).

For the first case, from (2.1.2) we get

\[ a c_{kji}^h = 0. \]

Since \( a \neq 0 \) in general, \( c_{kji}^h = 0. \)

For the second case, the space becomes an Einstein space and hence the result.

Q.E.D.

Now we show under what condition a quasiconformally flat space is conformally flat, and obtain the condition for a quasiconformally flat space to be Einstein.
Corollary 2.1.1: A quasiconformally flat space is conformally flat, if the constant $a$ in the expression for the tensorfield $W_{kji}$ is not zero.

Proof: From proposition 2.1.1 we have the following two cases.

(i) $a + (n-2)b = 0$, (ii) $\xi_j = 0$.

If (i) holds with $a \neq 0$, then $b \neq 0$ and $a = -(n-2)b$. Thus from (2.1.1) we have

$$W_{kji} = a \xi_{kji}.$$ 

Hence in this case a quasiconformally flat space reduces to a conformally flat space.

Now if (ii) holds, then from (1.4.15) we get

$$W_{kji} = a Z_{kji}.$$ 

Hence in this case a quasiconformally flat space with $a \neq 0$ reduces to a space of constant sectional curvature which is necessarily conformally flat.

Q.E.D.

Corollary 2.1.2: A quasiconformally flat space $M_n$ ($n > 3$) is Einstein (and not necessarily of constant sectional curvature) if the constants $a, b$ in the expression for $W_{kji}$ are such that $a = 0$, $b \neq 0$. 
Proof: By setting $W_{klij} = 0$ and $a = 0$ in (1.4.15) we get

\[ (2.1.4) \quad b(e_{kh}G_{ji} - e_{ki}G_{jh} + e_{ji}G_{kh} - e_{jh}G_{ki}) = 0. \]

Transvection of (2.1.4) with $g_{kh}$ gives

\[ b(n-2)G_{ji} = 0, \]

by virtue of (1.4.17). Thus if $b \neq 0$, $G_{ji} = 0$ for $n > 3$ and hence the space $M_n$ will be Einstein.

Q.E.D

Now onwards we use $\nabla$ as the operator of covariant derivative on $M_n$. The covariant derivative of a tensorfield $T_{ji}$ of type $(0,2)$ is said to be symmetric if the following condition is satisfied:

\[ \nabla_k T_{ji} = \nabla_j T_{ki}. \]

The tensorfield $A_{ji}$ of type $(0,2)$ defined in (1.4.20) can be written in the form $[3]$

\[ (2.1.5) \quad A_{ji} = bK_{ji} - \frac{K}{n}(b + \frac{a}{2(n-1)}) e_{ji}. \]

It is known $[3]$ that the covariant derivative of the tensorfield $A_{ji}$ is symmetric if the covariant derivative of Ricci tensor is symmetric, that is, if

\[ \nabla_k K_{ji} = \nabla_j K_{ki}. \]
Proposition 2.1.2: In a quasiconformally flat space $M_n (n > 3)$ the covariant derivative of the tensorfield $A_j^i$ is symmetric.

Proof: If $M_n$ is a quasi conformally flat space, then by proposition 2.1.1 we have two cases:

(i) $a+(n-2)b = 0$ ; (ii) $g_{ji} = 0$.

In first case $M_n$ is conformally flat and in second case it is Einstein and hence the scalar curvature $k$ of $M_n$ is constant.

For case (i), we get from (2.1.5),

\[(2.1.6) \quad A_j^i = sG_j^i,\]

where $G_j^i$ are given in (1.4.11). It is known that in a conformally flat space $M_n (n > 3)$,

\[\nabla_k G_{ji} = \nabla_j G_{ki}.\]

Hence from (2.1.6) we have

\[\nabla_k A_j^i = \nabla_j A_k^i.\]

For case (ii), we use the following formula obtained by K. Amur and P. Desai:

\[(2.1.7) \quad \nabla_h W_{kji}^h = (a+b)\left( \nabla_k X_{j}^i - \nabla_j X_{ki}^i \right) + \frac{1}{n-2}\left( (n-1)(n-4)b - 2a \right) g_{ji} \nabla_k (X_{k}^i - \kappa_{ki} \nabla_j X^i).\]
Also, as for the case (in) $K$ is constant, by setting $w_{kji} = 0$ and $K = \text{constant}$ in (2.1.7) we get

$$(2.1.8) \quad \nabla_{kji} = \nabla_{jki}.$$ 

Differentiation of (2.1.5) with respect to $\nabla_i$, gives

$$(2.1.9) \quad \nabla_{kji} = b \nabla_{kji},$$

where again we have used the fact $K = \text{constant}$. Interchanging the indices $k$ and $j$, and subtract the resulting expression from (2.1.9) to get,

$$(2.1.10) \quad \nabla_{kji} - \nabla_{jki} = b(\nabla_{kji} - \nabla_{jki}).$$

Since $b \neq 0$, by using (2.1.8) we get the required result.

Q.E.D.

2.2 Quasiconformally flat space $M_n$ with $n=3$

In the last section 2.1 we have considered a quasi conformally flat space $M_n$ for $n > 3$. In the present section we consider a space $M_n$ with $n=3$, that is, $M_3$ and study the quasiconformal flatness for the same.

We know [5], that for $M_3$ the conformal curvature tensor $\mathcal{Q}_{kji}^h$ defined by (1.4.10) vanishes identically. Thus by setting $n=3$ in (2.1.1) we get
(2.2.1) \( W_{kji}^h = (a+b)Z_{kji}^h \).

We set,

(2.2.2) \( W_{kji} = \nabla_k A_{j1} - \nabla_j A_{ki} \),

where \( A_{ji} \) are given by (2.1.5). It is clear from the Proposition 2.1.2 that for a quasiconformally flat \( M_n(n > 3) \), \( W_{kji} = 0 \). Also for \( M_3 \) by setting \( n = 3 \) in (1.4.11) and (2.1.5) we get respectively,

(2.2.3) \( \delta_{j1} = -K_{j1} + \frac{K}{4} \delta_{j1} \),

and

(2.2.4) \( A_{j1} = b K_{j1} - \frac{K}{12} (a+b)g_{j1} \).

But we write (2.2.4) in the form:

(2.2.5) \( A_{j1} = -b \delta_{j1} - \frac{(a+b)K}{12} \delta_{j1} \).

Differentiating (2.2.5) with respect to \( \nabla \), to get,

(2.2.6) \( \nabla_k A_{j1} = -b \nabla_k \delta_{j1} - \frac{(a+b)K}{12} (\nabla_k K) \delta_{j1} \).

Interchanging the indices \( k \) and \( i \), and subtract the resulting expression from (2.2.6) to obtain

(2.2.7) \( W_{kji} = -b \delta_{kji} - \frac{(a+b)\nabla_k K}{12} \delta_{j1} - (\nabla_j K) \delta_{ki} \),

where we have used (1.4.14) and (2.2.2).
Thus for $M_2$ (2.2.2) reduces to (2.2.7).

When $W_{kji}=0$ identically (which is in fact the case if $a+b=0$) we call $M_2$ to be quasioonformally flat if for $M_2$

$W_{kji}=0$.

**Proposition 2.2.1:** (a) Suppose $a+b\neq 0$. $M_2$ is quasioonformally flat if and only if it is of constant sectional curvature. Further for $M_2$ of constant sectional curvature $W_{kji}=0$. (b) Suppose $a+b=0$; $a,b \neq 0$. $M_2$ is quasi conformally flat if and only if it is conformally flat.

**Proof:** (a) If $a+b\neq 0$, then from (2.2.1),

$W_{kji}^h=0$ if and only if $Z_{kji}^h=0$,

which proves the first part. Further in the second part the condition that $M_2$ is a space of constant sectional curvature implies that it is an Einstein space and so the scalar curvature $\kappa$ of $M_2$ is constant. Using this fact in (2.2.3) and substituting the resulting expression of $\sigma_{ji}$ in (1.4.14), we get

$(2.2.8)$

$\sigma_{kji} = -\frac{1}{12} \left[ (\nabla^2)_{ijkl} - (\nabla_j^2)_{ijkl} \right] = 0$.

Finally using (2.2.8) in (2.2.7) we have the required result,

$W_{kji} = -\frac{a}{12} \left[ (\nabla^2)_{ijkl} - (\nabla_j^2)_{ijkl} \right] = 0$. 
(b) In this case, (2.2.7) reduces to

\[(2.2.9) \quad W_{kji} = -b C_{kji}.\]

This leads to the desired result.

Q.E.D.

Remark 2.2.1: From the proposition 2.2.1, it is clear that, when \( a = 0 \), \( b \neq 0 \) a quasi conformally flat space \( M \) is of constant sectional curvature. This is to be expected since an Einstein space \( M \) is necessarily of constant sectional curvature.

2.3. Hypersurface of a quasi-conformally flat space:

Let \( M \) be an \( n \)-dimensional hypersurface of a Riemannian space \( M \) and let these spaces \( M \) and \( M \) be covered by the systems of coordinate neighbourhoods \((U, x^i)\) and \((V, y^a)\) respectively, where the indices \( i, h, j, \ldots = 1, 2, 3, \ldots n \), and \( a, b, c \ldots = 1, 2, 3, \ldots n+1 \).

Suppose \( g_{ij} \) and \( g_{ab} \) denote the components of metric tensors in \( M \) and \( M \), respectively and \( \sigma \) be unit normal vector-field to \( M \) in \( M \). Then they satisfy (1.7.2).

Let \( \nabla_a \) and \( \nabla_i \) be the covariant derivative operators in \( M \) and \( M \) respectively. The curvature tensors \( K_{kjih} \) and \( R_{abcd} \) of \( \nabla_i \) and \( \nabla_a \) are related by the equation of Gauss (1.7.12). We shall denote the
Ricci tensor of $M_{n+1}$ by $R_{ba}$ and its scalar curvature by $R$.

We write the following tensorfields in $M_{n+1}$ which are similar to (1.4.11) and (2.1.5) defined on $M_n$.

(2.3.1) \[ c_{ba} = - \frac{R_{ba}}{n-1} + \frac{R}{2n(n-1)} \delta_{ba}, \]

(2.3.2) \[ A_{ba} = \beta R_{ba} - \frac{R}{n+1} (\beta + \frac{a}{2n}) \delta_{ba}, \]

(2.3.3) \[ W_{dcb} := a R_{dcb} + A_{bcd}, \]

where

\[ A_{dcb} = \delta^a_d A_{cb} - \delta^a_c A_{db} + \delta^a_b \delta_{cb} - A^a \delta_{db}, \]

with $A^a_b = A_{bc} \delta^{ca}$.

Suppose the hypersurface $M_n$ is quasi umbilical then by (1.7.9) it follows that there exist two functions $\mu$ and $\gamma$ such that

(2.3.4) \[ H_{ji} = \mu \delta_{ji} + \gamma \nu_i \nu_j. \]

We use the tensorfields $c_{ba}$ and $A_{ba}$ to define

(2.3.5) (a) \[ M_{ji} = c_{ba} B_j^b B_i^a, \]

(b) \[ M_{ji} = A_{ba} B_j^b B_i^a. \]

If $M_n$ is quasiumbilical hypersurface and satisfies the condition
for some constant \( k \), then \( M_n \) is said to be quasiumbilical hypersurface of type \( k \).

Similar to the above definition of quasiumbility of type \( k \) involving the tensorfield \( N_{ji} \) defined in (2.3.5)(a), we give the following definition which is more general than the previous one.

**Definition 2.3.1:** If \( M_n \) is quasiumbilical hypersurface and satisfies the following condition:

\[
(2.3.7) \quad M_{ji} = \alpha N_{ji}, \quad (\alpha \neq 0)
\]

then \( M_n \) is said to be quasiumbilical hypersurface of type \( k^* \).

Using (2.3.4) we write (2.3.2) in the form,

\[
(2.3.8) \quad A_{ba} = -\beta (n-1)\xi_{ba} + R \left[ \frac{\alpha + (n-1)\beta}{2n(n+1)} \right] g_{ba}.
\]

Transvecting (2.3.8) with \( E_{j1}^{ba} \) and using (2.3.5)(a) and (2.3.5)(b), we have

\[
(2.3.9) \quad M_{ji} = -\beta (n-1)N_{ji} - R \left[ \frac{\alpha + (n-1)\beta}{2n(n+1)} \right] g_{ji}.
\]

Now we prove

**Theorem 2.3.1:** A quasiumbilical hypersurface \( M_n(n > 3) \) of a quasiconformally flat Riemannian manifold \( M_{n+1} \) is of
constant sectional curvature if and only if it is quasi-umbilical of type $k^*$.

Proof: Since $M^*$ is quasiconformally flat, we have from (1.4.18) and (1.4.19),

$$0 = \mathcal{W}^a_{db} = \gamma^a_{db} + \delta^a_{d} \delta^b_{ob} - \delta^a_{o} \delta^b_{db} + \delta^a_{c} A^b_{cb}$$

For convenience we set,

$$E^k_a = \mathcal{E}_{ad} E^a_{bj}$$

Transvers (2.3.10) with $E^d_k E^c_j E^b_i E^h_a$ and use the Gauss equation (1.7.12) to get,

$$\left(2.3.11\right) \alpha^k_{jji} = \left\{\delta^a_{d} \delta^b_{ob} - \delta^a_{o} \delta^b_{db} + \delta^a_{c} A^b_{cb}\right\} E^d_k E^c_j E^b_i E^h_a$$

Using (2.3.4) and (2.3.5)(b) in (2.3.11), we get.

$$\left(2.3.12\right) \alpha^k_{jji} = -\left\{\delta^h_{k} \delta^h_{j} - \delta^h_{j} \delta^h_{k} + M^h_{k} \delta^h_{j} - M^h_{j} \delta^h_{k}\right\}$$

$$+ \alpha \sum \mu^2 \left(\delta^h_{j} \delta^h_{j} - \delta^h_{j} \delta^h_{k}\right) + \mu \left(\delta^h_{k} \delta^h_{j} \delta^h_{i} - \delta^h_{j} \delta^h_{k} \delta^h_{i}\right)$$

$$- \delta^h_{j} \delta^h_{k} \delta^h_{i} + \epsilon^h_{j} \delta^h_{k} \delta^h_{i}$$
Suppose $M_n$ is of constant sectional curvature, then by (1.3.5) we have

\[(2.3.13) \quad k_{kji}^h = k(\delta_k^h g_{ji} - \delta_j^h g_{ki});\]

where $k$ is a constant. Substitute (2.3.13) in (2.3.12) to reduce it to the following form

\[(2.3.14) \quad \delta_k^h Q_{ji} - \delta_j^h Q_{ki} + Q_k^h g_{ji} - Q_j^h g_{ki} = 0,\]

where we have set,

\[(2.3.15) \quad Q_{ji} = M_{ji} - \alpha \left\{ \mu \gamma v_j v_i - \frac{1}{4}(k-\mu^2)g_{ji} \right\},\]

and $Q_j^h = Q_{ji} g^{ih}.$

Contraction of (2.3.14) with respect to $k$ and $h$ yields

\[(2.3.16) \quad (n-2)Q_{ji} + Q_t^t g_{ji} = 0.\]

But transvection of (2.3.16) with $g_{ji}$ gives

\[(2.3.17) \quad (n-1) Q_t^t = 0,\]

so that for $n > 3$, $Q_t^t = 0$. Using this fact in (2.3.16) to obtain,

\[Q_{ji} = 0.\]

Thus, from (2.3.15), we have

\[(2.3.18) \quad M_{ji} = \alpha \left\{ \mu \gamma v_j v_i - \frac{1}{4}(k-\mu^2)g_{ji} \right\}.\]
But by the transvection of (2.3.4) with $\mu$ we have

\[(2.3.19) \quad \mu v_j v_i = \mu H^{i}_{j} - \mu^2 g_{ij}.
\]

So, use of (2.3.19) in (2.3.18) gives (2.3.7), in virtue of (2.3.6). Hence $M_n$ is a quasiumbilical of type $k^*$. Conversely, suppose $M_n$ is a quasiumbilical hypersurface of type $k^*$ of a quasiconformally flat Riemannian manifold $M_{n+1}$. Using (2.3.7) and (2.3.4) in (2.3.12) we have

\[a F_{kji} = k (g_{kji} - g_{kl} g_{lji}),
\]

which shows that $M_n$ is of constant sectional curvature.

**Remark 2.3.1:** If we choose the constants $\alpha$ and $\beta$ appearing in the definition of the tensor $W_{\alpha \beta \gamma}^a$ such that $\alpha = 1$, $\beta = -\frac{1}{n-1}$, then $W_{\alpha \beta \gamma}^a = 0$. This implies that the quasi conformally flat space $M_{n+1}$ is conformally flat. Also $M_j^i$ reduces to $N_j^i$. Theorem 2.3.1 is then that due to Chen and Yano $\{2.7\}$.

### 2.4. Quasi conformally flat hypersurface of a space of constant sectional curvature

A conformally flat space $M_n$ is said to be $k$-special if there exist three functions $\alpha$, $\beta$, $\gamma$ on $M_n$ such that $\omega$ is a 1-form on the open set $U = \{ p \in M_n : \beta \neq 0 \}$ with $d\alpha = \beta \omega$ on $U$ and...
\[ (2.4.1) \quad \sigma = - \frac{1}{2}(k + \alpha^2) g - \alpha \beta \omega \otimes \omega, \]
for some constant \( k \).

In terms of local components, (2.4.1) can be written as

\[ (2.4.2) \quad \sigma_j = - \frac{1}{2}(k + \alpha^2) \sigma_{j1} - \alpha \beta v_i v_j, \]

\( v_i \) being the local components of \( \omega \).

Chen and Yano have proved the following

**Theorem A**: Every conformally flat space of a space form of constant curvature \( k \) of co-dimension one is \( k \)-special. Conversely every simply connected \( k \)-special conformally flat space can be isometrically immersed in a space form of constant curvature \( k \) as hypersurface.

In view of corollary 2.1.1 the word 'conformally flat' in the theorem A can be replaced by 'quasiconformally flat' provided the tensor field given by (1.4.15) is such that \( a \neq 0 \).

2.5. **Quasi conformally flat space of codimension two in an Euclidean space, which is umbilical with respect to a normal direction**:

Chen and Yano obtained a condition for a submanifold \( M^q \) of codimension two of \( \mathbb{E}^{n+2} \) to be conformally
flat. We extend this study and obtain conditions for \( M_n \) to be quasi conformally flat.

The immersion of \( M_n \) in \( \mathbb{R}^{n+2} \) is represented by

\[
X = X(x^h),
\]

where \( X \) is the position vector from the origin of \( \mathbb{R}^{n+2} \) to a point of the submanifold \( M_n \). We set \( X_i = \alpha_i X \). The components of metric tensor on \( M_n \) are given by \( g_{ij} = X_j \cdot X_i \).

If \( C \) and \( D \) denote the unit normal vectorfields to \( M_n \), then the Weingarten equations are given by

\[
(2.5.1) \quad (a) \quad \nabla_j C = H^i_j X_i + L_j D
\]
\[
(b) \quad \nabla_j D = -M^i_j X_i - L_j C,
\]

where \( H^i_j = H^i_{jh} \), \( M^i_j \) are second fundamental forms and \( L_j \) third fundamental forms. If \( L_j = 0 \), then \( C \) (respectively \( D \)) is parallel in the normal direction of \( D \) (respectively \( C \)).

The equations of Gauss, Codazzi and Ricci take the form

\[
(2.5.2) \quad K^h_{kji} = H^h_k - H^h_{ji} - H^h_{j} + M^h_k M^h_{ji} - M^h_j M^h_{ki}
\]
By the use of the equations (2.5.2), (2.5.3) and (2.5.4), B.Y. Chen and K. Yano [5] have proved that, if

\begin{equation}
H_{ji} = \alpha g_{ji}, \quad L_j \neq 0,
\end{equation}

then

\begin{equation}
M_{ji} = \lambda g_{ji} + \mu \Lambda_j L_j;
\end{equation}

where \( \lambda = \frac{q_{L_j}^t}{L_j^2} \); \( \mu = \frac{(M_j^t - n \lambda)}{L_j^2} \).

Also in the same paper \( q_k \) is expressed as

\begin{equation}
q_k = \lambda L_k.
\end{equation}

Substituting (2.5.5) and (2.5.6) in (2.5.2), we get

\begin{equation}
K_{kji} = (\alpha^2 + \lambda^2)(\delta^h_{kji} - \delta^h_{jki}) + \lambda^2 \left( \sigma^h_{kL_j} - \delta^h_{jL_k} \right) L_j +
\left( L_k g_{ji} - L_j g_{ki} \right) L^h j.
\end{equation}

Contract (2.5.8) with respect to \( h \) and \( k \) and then transvect the resulting expression by \( g^{ji} \) to obtain
Lemma 2.5.1: Suppose $M_n$ is a submanifold of an Euclidean space $E_{n+2}$. Let $M_n$ be umbilical with respect to a non-parallel direction $G$, then $M_n$ is Einstein if $L_{ji} = 0$.

Proof: Substitute (2.5.9), (2.5.10) in (1.4.17), then we obtain the following equation in virtue of (2.5.11)

\[(2.5.12) \quad G_{ji} = \lambda \mu (n-2)L_{ji},\]

which proves the Lemma 2.5.1.

Q.E.D.

By the use of (2.5.11), (1.4.21) and (2.5.9), (2.5.10) we have

\[\mathcal{W}_{kjih} = \lambda \mu^{a+(n-2)b} \mathcal{J}(g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki}).\]

If $a+(n-2)b = 0$, then $\mathcal{W}_{kjih} = \mathcal{S}_{kjih} = 0$. So, $M_n$ is conformally flat. This result is in conformity with the result proved by Chen and Yano [5].
Now suppose $a+(n-2)b \neq 0$, $\lambda \neq 0$, $\mu \neq 0$. Then if $L_{ji} = 0$, $W_{klij} = 0$. By Lemma 2.5.1 it follows that the space $M_n$ is Einstein, and also quasi conformally flat. However, in view of the result of Chen and Yano \cite{5}, $M_n$ is both conformally flat and Einstein which implies that $M_n$ is of constant sectional curvature. Hence we have the following

**Theorem 2.5.1:** Suppose $M_n (n > 3)$ is a submanifold of an Euclidean space $E^{n+2}$. Let $M_n$ be umbilical with respect to a non-parallel direction $\sigma$. Then $M_n$ is quasi-conformally flat if (i) the constants $a, b$ used in the expression of the tensorfield $W$ are such that $a+(n-2)b = 0$, or if (ii) $a+(n-2)b \neq 0$ but $L_{ji} = 0$.

2.6. **Conformal transformation in $W$-recurrent Spaces:**

Tyusi Adati and Seiichi Yamaguchi \cite{6} have studied conformal transformation in a recurrent space and found that such a transformation is necessarily homothetic if the space is not conformally flat and on the other hand if the transformation is not homothetic then the space must be conformally flat.

We recall that $W_n$ is a Riemannian space $(M_n, g)$ in which the tensorfield $W$ defined by (1.4-15'), is recurrent.
In this section we consider the conformal transformation in \( W \). Suppose \( W \) admits a conformal transformation defined by a vector field \( \mathcal{V} \). Then

\[
\varepsilon_{y} e_{ji} = \nabla_{j} \mathcal{V}_{i} + \mathcal{V}_{i} \nabla_{j} = 2 \mathcal{O} e_{ji},
\]

where \( \mathcal{O} = \frac{1}{n} \nabla_{i} \mathcal{V}^{i} \).

From (2.6.1) we also have

\[
\varepsilon_{V} \left\{ \frac{h}{j_{1}} \right\} = \delta_{j}^{h} \mathcal{V}_{i} + \delta_{i}^{h} \mathcal{V}_{j} - \delta_{j}^{h} e_{ji},
\]

(2.6.3)

\[
\varepsilon_{V} \kappa_{h}^{k} = - \delta_{k}^{h} \nabla_{j} \mathcal{V}_{i} + \delta_{j}^{h} \nabla_{k} \mathcal{V}_{i} - \nabla_{k} \delta_{j}^{h} e_{ji} + \nabla_{j} \delta_{h}^{k} e_{ki},
\]

(2.6.4)

\[
\varepsilon_{V} \alpha_{h}^{k} = 0,
\]

where \( \alpha_{h}^{k} \) is conformal curvature tensor given by (1.4.10).

By taking Lie derivative of (2.1.1) we get the following result, by the use of (2.6.4).

\[
\varepsilon_{V} \tilde{W}_{h}^{k} = \int a + (n-2)b \int \varepsilon_{V} \tilde{Z}_{h}^{k},
\]

where

\[
\varepsilon_{V} \tilde{Z}_{h}^{k} = - \delta_{k}^{h} \nabla_{j} \mathcal{V}_{i} + \delta_{j}^{h} \nabla_{k} \mathcal{V}_{i} - \nabla_{k} \delta_{j}^{h} e_{ji} + \nabla_{j} \delta_{h}^{k} e_{ki}.
\]

\[
+ \frac{2}{n} \Delta \mathcal{O} \left( \delta_{k}^{h} e_{ji} - \delta_{j}^{h} e_{ki} \right),
\]

\[
\Delta \mathcal{O} = \varepsilon_{ji} \nabla_{j} \mathcal{V}_{i} \mathcal{O},
\]
\( \Delta \) - being the Laplacian operator. Since \((M_n, g)\) is \(W\)-recurrent we have 

\[
\nabla_1 \hat{W}_{kji}^h = k_1 \hat{W}_{kji}^h ,
\]

\( k_1 \neq 0 \) being a recurrent vectorfield. By taking Lie derivative of (2.6.6), we get 

\[
(2.6.7) \quad \varepsilon_v (\nabla_1 \hat{W}_{kji}^h) = (\varepsilon_v \varepsilon_{k_1}) \hat{W}_{kji}^h + \sum a + (n-2) b \varepsilon_{k_1} \hat{Z}_{kji}^h ,
\]

where we have used (2.6.6) in the formula 

\[
\varepsilon_v (\nabla_m \hat{W}_{kji}^h) - \nabla_m (\varepsilon_v \hat{W}_{kji}^h) = -(\varepsilon_v \{m \, k\}) \hat{W}_{lj1}^h - (\varepsilon_v \{m \, l\}) \hat{W}_{k1l}^h - (\varepsilon_v \{m \, i\}) \hat{W}_{kji}^1
\]

\[
+ (\varepsilon_v \{m \, h\}) \hat{W}_{kjh}^1
\]

to get 

\[
(2.6.8) \quad (\varepsilon_v \varepsilon_{k_m}) \hat{W}_{kji}^h + \sum a + (n-2) b \{k_m \varepsilon_v \hat{Z}_{kji}^h - \nabla_m (\varepsilon_v \hat{Z}_{kji}^h)\} = \\
= -\sum \delta^l_m \varepsilon_{k}^l + \delta^l_k \varepsilon_{m}^l - \varepsilon_{mk} \varepsilon^l \nabla_{lji}^h - \\
-\sum \delta^l_m \varepsilon_{j}^l + \delta^l_j \varepsilon_{m}^l - \varepsilon_{mj} \varepsilon^l \nabla_{k1l}^h - \\
-\sum \delta^l_m \varepsilon_{i}^l + \delta^l_i \varepsilon_{m}^l - \varepsilon_{mi} \varepsilon^l \nabla_{kjl}^h + \\
+ \sum \delta^h_m \varepsilon_{k}^h + \delta^h_k \varepsilon_{m}^h - \varepsilon_{mh} \varepsilon^h \nabla_{kji}^h .
\]

For the sake of convenience we set
(2.6.9) \[ \sum_{a+(n-2)b} \{ V_m^i (\varepsilon Z_{kji}^h) - k_m (\varepsilon Z_{kji}^h) \} = M_{mkji}^h \]

which reduces (2.6.8) to the form

(2.6.10) \( \varepsilon_m^{k+2} \varepsilon_m^{k} W_{kji}^h = g_m^{kl} W_{kji}^{l-1} - g_m^{kmj} W_{kmi}^h - g_m^{kmj} W_{kmj}^h - g_m^{kmj} W_{kmj}^h - g_m^{kmj} W_{kmj}^h + g_m^{kmj} W_{kmj}^h + g_m^{kmj} W_{kmj}^h + g_m^{kmj} W_{kmj}^h. \)

Contraction of (2.6.10) with respect to \( h \) and \( m \) gives

(2.6.11) \( \varepsilon_r^{k} W_{kji}^r = (n-3) \varepsilon_r^{k} W_{kji}^r + M_{rkji}^r, \)

where we have used the properties

\[ W_{kjih} = - W_{jkhi}; \quad W_{kjih} = - W_{kjhi} \]

and

\[ W_{kjih} = W_{ihkj}; \quad W_{kjih}^* + W_{jih} + W_{ikjh} = 0. \]

Transvect (2.6.10) with \( \varepsilon_m^{k} \) and \( \varepsilon_h^{k} \), then we get respectively the following two equations

(2.6.12) \( \varepsilon_m^{k} W_{kji}^h = \varepsilon_m^{k} M_{mkji}^h, \)

and

(2.6.13) \( \varepsilon_t^l \{ (\varepsilon_k^{m} + \varepsilon_m^{k}) W_{kji}^{t+k} + \varepsilon_m^{k} W_{kmi}^{t+k} + \varepsilon_j^{k} W_{kmi}^{t+k} + \varepsilon_1^{k} W_{kjm}^{t+k} -
- \varepsilon_1^l \{ g_m^{kji} W_{kji} + g_m^{kji} W_{kli} + g_m^{kji} W_{klj} \} \} =
- \varepsilon_t^t W_{kjm} + \varepsilon_t^t M_{mkji}^t. \)
Substitute (2.6.11) in (2.6.13) to obtain

\begin{equation}
\sum_{k} \left( \sum_{m} \left( c_{k} + c_{m} \right) e_{k}^{m} \right) w_{kji} + \sum_{m} e_{k}^{m} m_{j} + \sum_{i} e_{i}^{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} - \sum_{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} \nonumber
\end{equation}

But from (2.6.10) we have the following equation

\begin{equation}
\sum_{k} \left( \sum_{m} \left( c_{k} + c_{m} \right) e_{k}^{m} \right) w_{kji} + \sum_{m} e_{k}^{m} m_{j} + \sum_{i} e_{i}^{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} - \sum_{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} \nonumber
\end{equation}

So, using (2.6.15) in (2.6.14), and (2.6.11) in the resulting expression, we get.

\begin{equation}
\sum_{k} \left( \sum_{m} \left( c_{k} + c_{m} \right) e_{k}^{m} \right) w_{kji} + \sum_{m} e_{k}^{m} m_{j} + \sum_{i} e_{i}^{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} - \sum_{m} \left( \sum_{l} \left( c_{l} + c_{m} \right) e_{l}^{m} \right) w_{kji} \nonumber
\end{equation}

Transvect (2.6.16) with \( \varepsilon \) to obtain
Change the index $i \rightarrow l \rightarrow i$ in (2.6.17) and obtain an expression for $-\sum_{m} \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l}$ and add this new equation to (2.6.17) to get:

\[ (2.6.18) \sum_{m} \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} = \sum_{m} \{ \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} \}^{t} + \{ \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} \}^{t} + \{ \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} \}^{t} + \{ \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} \}^{t} + \{ \epsilon_{v} \epsilon_{m} \ell_{m}^{l} \ell_{l}^{m} \epsilon_{w_{kjl}}^{l} \}^{t} \]

If $\sum_{a+(n-2b)} \mathcal{J} = 0$, then from (1.4.21) and (2.6.9) the above result takes the form

\[ (2.6.19) \sum_{a+(n-2b)} \mathcal{J} = 0. \]

Hence (2.6.19) proves the theorems 1, 2, 3 $\sum_{a+(n-2b)} \mathcal{J}$. But now we use the condition

\[ (2.6.20) \nu_{m}(\epsilon_{v} \sigma_{ji}) = k_{m}(\epsilon_{v} \sigma_{ji}) \]
where $\xi^j_{ij} = V^j_i$, $\xi^j_{ij}$ are those given by (1.4.11).
But then (2.6.18) reduces to.

\[(2.6.21) \quad \sum \xi^m_{ij} - (n-3) \xi^l_{ij} \xi^l = 0.\]

From (2.6.21) we prove the following

**Theorem 2.6.1:** Suppose $(M^n, g)$ admits a conformal transformation defined by a vector field $v$ and is a $W$-recurrent space with recurrent vector field $k_l \neq 0$ satisfying the condition
and $v_m \xi^m_{ij} = k_m \xi^m_{ij}$. Then either the transformation is homothetic or the space is quasi-conformally flat.

Proof: From (2.6.21) we have

\[(1) \quad \xi^j_{jkm} = 0 \quad \text{or} \quad (ii) \quad \sum \xi^m_{ij} - (n-3) \xi^l_{ij} \xi^l = 0.\]

Now if (i) holds then by transvecting (2.6.10) with $\xi^h_i$ we get $\xi^h_i \xi^h_{jkm} = 0$ and hence

\[(2.6.22) \quad \xi^h_i \xi^h_{jkm} = 0 \quad \text{or} \quad \xi^h_{jkm} = 0.\]

Also, if (ii) holds then by using (2.6.11) we have (2.6.22). So, for both the cases theorem is proved.

Q.E.D.
Remark 2.6.1: If \( \epsilon^h \rho_h \neq 0 \) then from theorem 2.6.1, \( W_{kjm} = 0 \). Hence the space \((M_n, g)\) is either conformally flat or Einstein. But the possibility of a space being conformally flat is proved by (2.6.19) without the extra condition that \( \nabla_m \xi \xi j = k_m \xi \xi j \). So, it follows that for \((M_n, g)\) to be Einstein the condition \( \nabla_m \xi \xi j = k_m \xi \xi j \) is needed.
<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s)</th>
<th>Title and Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B.Y. Chen</td>
<td>Geometry of submanifolds, Marcel Dekker Inc., New York, (1973)</td>
</tr>
<tr>
<td>6</td>
<td>Tyuzi Adati and Sciichi Yamaguchi</td>
<td>On some transformations in Riemannian Recurrent Spaces, Journal of Science, University of Tokyo (Booklets, pp.8-12).</td>
</tr>
</tbody>
</table>