CHAPTER V

KAHLERIAN AND CONTACT MANIFOLDS AND THEIR SUBMANIFOLDS

5.0  Introduction: In this Chapter in the Section 5.1, we study some recurrent properties of Kahlerian manifolds. K.Yano and T.Imai \( \footnote{2} \) have introduced the notion of semisymmetric metric \( F \)-connection in a Kahlerian manifold. In section 5.2 we consider a Sasakian hypersurface of a Kahlerian manifold admitting semisymmetric metric \( F \)-connection. Under the condition that the vectorfield \( q^h \) used in the expression of semisymmetric metric \( F \)-connection
\( \footnote{See (1.6.6)} \) is unit normal vector to the hypersurface. In section 5.3 we study \( C \)-fubinian and \( C \)-umbilical hypersurfaces of a Kahlerian manifold.
5.1. Recurrent and Symmetric Kahlerian Manifolds:

The H-projective, H-conformal (Bochner) and H-conharmonic curvature tensors are defined in Section 1.6. In this section we find out some relations amongst the H-projective recurrent, the H-conformal recurrent and the H-conharmonic recurrent Kahlerian spaces $M_{n=2m}$.

For this purpose we use a tensorfield $S$ of type $(0,3)$ given in (5.1.1):

$$S_{kji} = (\nabla^K_{kji} - \nabla^K_{jki}) + \frac{1}{n}\varepsilon_{kl} \nabla^t_{k} - \delta_{j1} \nabla^t_{k} \varepsilon_{kl} \frac{1}{n+2} \varepsilon_{kl} \nabla^t_{S_{j1}}$$

By using (1.6.3) we can find the following relation.

$$\nabla^h_{kji} = (\frac{n}{n+4})S_{kji},$$

where $\nabla^h_{kji} = g^{lh} \nabla^h_{kjil}$ is defined by (1.6.5).

As in Chapter II, we use the definition that the covariant derivative of Ricci tensor is symmetric if $\nabla^h_{kji} - \nabla^h_{jki} = 0$.

Proposition 5.1.1: In a Kahlerian manifold $M_{n}$ the covariant derivative of Ricci tensor is symmetric if and only if $S_{kji} = 0$.

Proof: Suppose $S_{kji} = 0$. Then from (5.1.1) we get
(5.1.3) \( \nabla_{j}^{K} - \nabla_{i}^{K} = - \frac{1}{n} \sum_{k} g_{k1} \nabla_{j}^{K} - \delta_{j}^{k} \nabla_{i}^{K} + p_{k1} \nabla_{j}^{S} + 2p_{kj} \nabla_{i}^{S} - R_{j}^{k1} \nabla_{j}^{S} + 2p_{kj} \nabla_{i}^{S} \).

Transvect (5.1.3) with \( g^{ji} \) and use (1.6.3) and the fact that \( \nabla_{j}^{K} = \frac{1}{n} \nabla_{j}^{K}, \ p_{j}^{t} = - \delta_{j}^{1} \) and \( g^{ji} p_{ji} = 0 \), to obtain

\( \nabla_{j}^{K} = 0. \)

Hence (5.1.3) gives \( \nabla_{j}^{K} - \nabla_{i}^{K} = 0. \)

Conversely if \( \nabla_{j}^{K} - \nabla_{i}^{K} = 0 \), then clearly \( \nabla_{j}^{K} = 0. \)

Hence from (5.1.1) we get \( S_{kji} = 0. \)

Q.E.D.

Now onwards we denote a Kahlerian manifold \( M_{n} \) with \( S_{kji} = 0 \) by \( M_{n}^{*}. \)

Proposition 5.1.2: In a Kahlerian manifold \( M_{n}^{*} \),

\( \nabla_{j}^{K} - \nabla_{i}^{K} = 0 \) and \( \nabla_{j}^{K} - \nabla_{i}^{K} = 0. \)

Proof: By using (1.6.3) we have

(5.1.4) \( F_{j}^{h}(\nabla_{j}^{K} - \nabla_{j}^{K}) = (\nabla_{j}^{K} - \nabla_{j}^{K}) + \nabla_{j}^{K}. \)

But by proposition 5.1.1, \( \nabla_{j}^{K} - \nabla_{j}^{K} = 0, \) in \( M_{n}^{*}. \) So, by the transvection of (5.1.4) with \( F_{j}^{h} \) we get:

\( \nabla_{j}^{K} = 0, \)

by virtue of (1.6.3).
Now substitute $\nabla K_{ji} = 0$ in (5.1.4) to get:

$\nabla \xi S_{ji} = 0.$

Q.E.D.

**Theorem 5.1.1:** (i) In Kahlerian manifold $M_n$, the tensorfields $P_{kjih}$, $Q_{kjih}$ and $R_{kjih}$ satisfy the second Bianchi identity.

(ii) In recurrent Kahlerian manifold $M_n$, the tensorfields $P_{kjih} = Q_{kjih} = R_{kjih} = K_{kjih}$ and $K_{ji} = 0$, $K = 0$.

Proof: (i) Find the covariant differentiations of (1.6.1), (1.6.4), (1.6.5) and then use the proposition 5.1.2 to obtain

(5.1.5) $\nabla_l P_{kjih} = \nabla_l Q_{kjih} = \nabla_l R_{kjih} = \nabla_l K_{kjih},$

which proves the result (i).

(ii) Suppose $M_n$ is recurrent with $k_1 \neq 0$ as recurrent vectorfield. Then it can be easily seen that $M_n$ is H-projective, H-conformal, H-conhormonic recurrent with the same vectorfield $k_1$ as recurrent vectorfield. If we use $M_n$ instead of $M_n$ then by using (5.1.5), we have

(5.1.6) $P_{kjih} = Q_{kjih} = R_{kjih} = K_{kjih}.$

Hence from (1.6.1), by using $P_{kjih} = K_{kjih}$ we get

(5.1.7) $g_{ji} X_{kl} - g_{kji} X_{il} + P_{jhi} S_{ki} - P_{kji} S_{hi} + 2S_{kj} P_{hi} = 0.$
Transvect (5.1.7) by $g^{jh}$ and use (1.6.3) to get

$$X_{ji} = 0$$

and hence $K = 0$.

Q.E.D.

Theorem 5.1.2: A $H$-projective (H-conformal, $H$-conharmonic) recurrent Kahlerian manifold $M_n$ is $H$-projective (H-conformal, $H$-conharmonic) symmetric and hence symmetric in the sense of Cartan.

Proof:- In $M_n$, by (5.1.5), we have

$$\nabla^P_{kjih} = \nabla^E_{kjih}$$

Suppose $M_n$ is $H$-projective recurrent with $k_1$ as recurrent vectorfield, then:

(5.1.8) \( k^P_{kjih} = \nabla^E_{kjih} \)

Now by using Rianchi second identity (1.3.4), we have

(5.1.9) \( k^P_{kjih} + k^P_{jkl} + k^P_{jli} = 0 \),

contraction of which with respect $h$ and $l$ gives

(5.1.10) \( k^h_{kjih} = 0 \);

where we have used the results $P_{tji}^t = P_{kti}^t = 0$.

Transvect (5.1.9) with $k^h$ to obtain .
\( (5.1.11) \quad (k^k_j^l)^{P}_{kji} h + k^k_j^l p^h_{jli} - k^k_j^l p^h_{kli} = 0. \)

But \( k^k_j^l p^h_{jli} = k^l (V^j_k p^h_{ilj}) = k^l (V^j_k p^h_{ihj}) = k^l p^h_{kji}. \)

Hence

\( (5.1.12) \quad k^l p^h_{jli} = 0, \)

by virtue of \((5.1.10).\)

Use of \((5.1.12)\) in \((5.1.11)\) gives

\( (5.1.13) \quad (k^k_j^l)^{P}_{kji} h = 0. \)

Thus we get \( k^l = 0, \) which from \((5.1.8)\) proves that \( \mathbb{M}_n \) is symmetric.

On the same lines we can prove the result for

H-conformal curvature tensor. But in case of H-oonharmonic curvature tensor, \((5.1.10)\) holds, directly by virtue of

\((5.1.2)\) and the remaining calculations are same as in case of H-projective curvature tensor.

Q.E.D.

We generalise the following theorems proved by

Lajapat Rai Ahuja and Ram Behari \(^6\).

**Theorem A:** A Kähler space of constant holomorphic sectional curvature is H-projective symmetric.
Theorem B: A H-projective recurrent Kahler Einstein space is symmetric.

For the generalisation we need the following propositions.

**Proposition 5.1.3:** In an Einstein Kahlerian manifold $M_n(n > 3)$, $S_{kji} = 0$ but the converse is not true in general.

**Proof:** Since $M_n$ is Einstein we have

$$K_j^i = \frac{K}{n} \delta_j^i. \tag{5.1.14}$$

Differentiate (5.1.14) with respect to $\nabla_i$ to obtain

$$\nabla_i K_j^i = \frac{1}{n} \nabla_j K. \tag{5.1.15}$$

But by using $\nabla_i K_j^i = \frac{1}{n} \nabla_j K$ and (5.1.15) we find

$$\nabla_j K = 0, \tag{5.1.16}$$

and hence from (5.1.16),

$$\nabla_i K_j^i = 0. \tag{5.1.17}$$

Substitute (5.1.14), (5.1.16) and (5.1.17) in (5.1.1) to obtain $S_{kji} = 0$. Q.E.D.

**Proposition 5.1.4:** In a Kahlerian manifold $M_n(n > 3)$ of constant holomorphic sectional curvature, $S_{kji} = 0$. 
Proof: Since a Kahlerian space $\mathbb{M}_n$ of constant holomorphic sectional curvature is Einstein, the result follows from Proposition 5.1.4.

Q.E.D.

From above Propositions 5.1.4 and 5.1.5, we conclude that a Kahlerian manifold $\mathbb{M}_n (n > 3)$ which is Einstein or space of constant holomorphic sectional curvature is $\mathbb{N}_n$.

Remark 5.1.1: (i) If we take the Kahlerian manifold $\mathbb{M}_n$ as a space of constant holomorphic sectional curvature then $\mathbb{M}_n$ is regarded as $\mathbb{N}_n$ and

$$
\nabla^K_{\mathbb{M}_n} K_{kji} = \frac{\mathbb{K}}{\mathbb{M}_n} \nabla^K_{\mathbb{M}_n} (\mathbb{E}_{kh} \mathbb{E}_{ji} - \mathbb{E}_{jh} \mathbb{E}_{ki}) + (\mathbb{F}_{kh} \mathbb{F}_{ji} - \mathbb{F}_{jh} \mathbb{F}_{ki} - 2\mathbb{F}_{kj} \mathbb{F}_{ih}) \frac{\mathbb{J}}{\mathbb{M}_n},
$$

since for a space of constant holomorphic sectional curvature $\mathbb{K}$

$$
K_{kji} = \frac{\mathbb{K}}{\mathbb{M}_n} (\mathbb{E}_{kh} \mathbb{E}_{ji} - \mathbb{E}_{jh} \mathbb{E}_{ki}) + (\mathbb{F}_{kh} \mathbb{F}_{ji} - \mathbb{F}_{jh} \mathbb{F}_{ki} - 2\mathbb{F}_{kj} \mathbb{F}_{ih}) \frac{\mathbb{J}}{\mathbb{M}_n}. 
$$

But $\nabla^K_{\mathbb{M}_n} K = 0$ in $\mathbb{M}_n$ and hence we get $\nabla^K_{\mathbb{M}_n} K_{kji} = 0$. So, the theorem 5.1.1 gives the result of theorem A.

(ii) Since an Einstein Kahlerian manifold $\mathbb{M}_n$ is regarded as $\mathbb{N}_n$ the theorem 5.1.2 gives the result of theorem B.

Now we prove some results regarding the divergency of $\mathbb{F}_{kjih}$, $\mathbb{O}_{kjih}$ and $\mathbb{B}_{kjih}$ defined in a Kahlerian manifold $\mathbb{M}_n$. 
Proposition 5.1.5: In Kahlerian manifold \( M_n \)

(i) \( \nabla_h P_{kji}^h = 0 \) i.e. \( \text{div} P_{kji}^h = 0 \),

(ii) \( \nabla_h C_{kji}^h = 0 \), (iii) \( \nabla_h R_{kji}^h = 0 \).

Conversely if any one of the above three results hold in \( M_n \), then the Kahlerian manifold \( M_n \) reduces to \( \hat{M}_n \).

Proof: Operate \( \nabla^h \) to (1.6.1) and use the fact that

\[ \nabla_t^h K_{kji} = \nabla^h K_{kji} - \nabla^h K_{kli} \]

along with the Proposition 5.1.2 to obtain the result (i). In the similar way we can obtain the other results (ii) and (iii). Conversely suppose

\[ \nabla_h P_{kji}^h = 0, \]

that is,

\[ (\nabla K_{kji} - \nabla K_{kli}) + \frac{1}{n+2} \sum (\nabla K_{kji} - \nabla K_{kli}) + \nabla f^h F_j^h v_h s_{kli} - F_k^h v_h s_{kji} + 2 F_1^h v_h s_{kji} = 0, \]

which by using (1.6.3)(i) reduces to

\[ (\frac{n-2}{n+2}) (\nabla K_{kji} - \nabla K_{kli}) = 0. \]

Hence for \( n > 3 \), covariant derivative of Ricci tensor will be symmetric. If we use (ii), then result follows directly from (5.1.2) and Proposition 5.1.1.
Now suppose the result (iii) holds, that is, \( V_h B_{kji}^h = 0 \).

But \( V_h B_{kji}^h = 0 = V_h C_{kji}^h - \frac{\sqrt{n}}{n+2} \sum_{jkl} \varepsilon_{jkl} \varepsilon_{kji} F_{jl}^f j_{hi} + F_{jl}^f j_{hi} - F_{ij}^f j_{hi} + 2 F_{ij}^f j_{hi} \).

Thus by using (5.1.2) and (5.1.1) in the above expression we get:

\[
(5.1.18) \quad n( V_j K_{ki} - V_j K_{ki} ) = \frac{1}{n+2} \left( V_h K_{ki} - \delta_j^h \varepsilon_{kji} - \delta_k^h \varepsilon_{jli} \right) + F_{j}^h K_{kli} - F_{j}^h F_{ki}^l + 2 F_{j}^h F_{ki}^l \]

\[
- i \sqrt{n} \left( \sum_{jkl} \varepsilon_{jkl} V_{j} K_{ki} - \varepsilon_{jli} V_{k} K_{ji} + F_{ki}^l t F_{j}^t \right) \]

\[
- F_{ji}^l t V_{j} K_{ki} + 2 F_{ji}^l t V_{j} K_{ki} \]

Transvect (5.1.18) with \( g_{kli} \) to get

\[
(5.1.19) \quad \left( \frac{n-1}{2} \right) V_{j} K = 0,
\]

which for \( n > 3 \) gives \( \nabla_{j} K = 0 \). By substituting this fact in (5.1.18) we see that the covariant derivative of Ricci tensor is symmetric and hence in all the cases \( M_n \) reduces to \( M_n \).

Q.E.D.

**Theorem 5.1.5**: A Kahlerian manifold \( M_n \) is H-conformal symmetric if it is H-conformal symmetric

**Proof**: If \( M_n \) is H-conformal symmetric then from (5.1.2) we get

\( \nabla_{j} K = 0 \).
Thus \( \nabla^B_{kji} = \nabla^4_{kji} - \frac{1}{n+4} \nabla^K_{kji} \sum g_{kh} g_{ki} - g_{kh} g_{ji} \)
\[ + P_{jhi}P_{ki} - P_{khi}P_{ji} + 2P_{kjih}J, \quad \]
gives,
\[ \nabla^B_{kji} = 0. \quad \text{Q.E.D.} \]

5.2. **Sasakian hypersurface of a Kahlerian manifold admitting semisymmetric metric connection:**

Let \( M_{n-1} \) be a Sasakian hypersurface of a Kahlerian manifold \( M_n (n=2m) \) with complex structure \( F \). With respect to the coordinate system \( (\nu, \nu^a) (a, b, c, \ldots = 1, 2, 3, \ldots, n) \) in \( M_n \) we have:

\[ (5.2.1) \quad F_{b}^a = F_{ba}g^{ca}, \quad F_{bc} = -F_{cb}, \]
\( g_{cb} \) being the components of Hermitian metric \( g \)
and
\[ (5.2.2) \quad \nabla^a_{b}F_{ba} = 0. \]

The induced structure \( (\partial^h, \nu_j^h, g^h, \xi^h) \) of \( M_{n-1} \) satisfies the condition \( (4.2.1) \).

In a Kahlerian manifold \( M_n \) with Hermitian metric tensor \( g_{cb} \) and complex structure \( F_{cb} \), a semisymmetric metric \( F \)-connection \( F^a_{cb} \) is given by \( (1.6.6) \) satisfying the conditions \( (1.6.7) \).
The curvature tensors $\tilde{R}^{\alpha}_{\beta\gamma\delta}$ of $\tilde{R}^{\alpha}_{\beta}$ and $R_{\alpha\beta}$ of $\{\alpha\}$ are related by (1.6.8), which we write in the following convenient form

\[(5.2.3) \quad \tilde{R}^{\alpha}_{\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \varepsilon_{\alpha\beta\gamma\delta} - \varepsilon_{\alpha\beta\gamma\delta} + \varepsilon_{\alpha\beta\gamma\delta} - \varepsilon_{\alpha\beta\gamma\delta},\]

where

\[(5.2.4) \quad \varepsilon_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} - F_{\alpha\beta\gamma\delta} + F_{\alpha\beta\gamma\delta} - F_{\alpha\beta\gamma\delta} + \gamma_{\alpha\beta\gamma\delta} - \gamma_{\alpha\beta\gamma\delta} + \gamma_{\alpha\beta\gamma\delta} - \gamma_{\alpha\beta\gamma\delta},\]

with

\[(5.2.5) \quad \varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta} + \varepsilon_{\alpha\beta} + \varepsilon_{\alpha\beta},\]

and

\[(5.2.6) \quad \varepsilon_{\alpha\beta} = -(\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}),\]

\[\beta_{\alpha\beta} = 2(\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}).\]

In the Sasakian manifold the metric semisymmetric connection $\tilde{\gamma}_{\alpha\beta}$ is introduced by A. Sharfuddin and S.I. Husain (1.1.1), by using the 1-form $\eta_{\alpha}$ instead of $p_{\alpha}$ given in (1.6.11).

The curvature tensors $\tilde{K}_{\alpha\beta\gamma\delta}$ of $\tilde{K}_{\alpha\beta}$ and $K_{\alpha\beta}$ of $\{\alpha\}$ are related by the relation (1.4.23). The tensorfield $\alpha$ of type $(0,2)$ given by (1.4.24) is written in the form

\[(5.2.7) \quad \alpha_{\alpha\beta} = \phi_{\alpha\beta} - \eta_{\alpha} \eta_{\beta} + \frac{1}{2} \phi_{\alpha\beta},\]

by using (1.4.24) with $p_{\alpha} = \eta_{\alpha}$ and (1.5.12).
The corresponding symbols and formulas of $M_n$ (respectively of $M_{n-1}$) as regarded with respect to the connection $\{a_{cb}\}$ (respectively $\{h_{ji}\}$) are also written with respect to the connection $f^a_{cb}$ (respectively $f^h_{ji}$) in the same way but with a '$^\prime$' on the top of each symbol.

**Proposition 5.2.1** Suppose $M_{n-1}$ is a Sasakian hypersurface of a Kählerian manifold $M_n$ which admits the semisymmetric metric $\mathcal{F}$-connection and the vectorfield $q^a$ used in its expression is normal to $M_{n-1}$, then $M_{n-1}$ admits a metric semisymmetric connection.

**Proof** - Suppose $f^h_{ji}$ is the connection in $M_{n-1}$ induced by the semisymmetric metric $\mathcal{F}$-connection $f^a_{cb}$ of $M_n$. Then by (1.7.3) we have

\[(5.2.8) \quad f^h_{ji} = \left( \delta^a_{j} b^b_{i} + f^a_{cb} b^c_{j} b^b_{i} \right) b^h_{a}.
\]

Now by using (1.6.6) we get

\[(5.2.9) \quad f^h_{ji} = \delta^a_{j} b^b_{i} + f^a_{cb} b^c_{j} b^b_{i} \cdot x b^h_{a}.
\]

By using the fact that $q^a$ is unit normal to $M_{n-1}$ and the results (1.6.7), (4.2.1) and (1.7.3) we have

\[(5.2.10) \quad f^h_{ji} = \left( \delta^h_{j} q^i + \delta^h_{i} q^j - \delta^h_{j} q^i - \delta^h_{i} q^j \right).
\]
where \( \varphi_h = p \cdot a^a_h \). Hence the result is proved.

**Q.E.D.**

**Proposition 5.2.2:** Suppose \( M_{n-1} \) is a Sasakian hypersurface of a Kahlerian manifold \( M_n \) admitting a semisymmetric metric \( \nabla \)-connection with the vectorfield \( q^a \) in its expression, as normal to the hypersurface \( M_{n-1} \), then the vector vectorfield \( p^a \) is tangent to \( M_{n-1} \).

**Proof:** By using (1.6.7) we write:

\[ q_a p^a = p_b a_{bc} c^b \]

\[ = -\varepsilon_{bc} c^b, \]

hence,

\[ q_a p^a = 0. \quad \text{Q.E.D.} \]

In view of Proposition 5.2.2 we can write

(5.2.11) \( p^a = \varphi_{h-a}^a_h. \)

**Proposition 5.2.3:** Suppose \( M_{n-1} \) is a hypersurface of a Kahlerian manifold \( M_n \) admitting a semisymmetric metric \( \nabla \)-connection with \( q^a \) as normal vector to \( M_{n-1} \) then the second fundamental forms \( \vec{S}_{ji} \) and \( H_{ji} \) are related by

(5.2.12) \( \vec{S}_{ji} = H_{ji} - \phi_{ji}. \)

**Proof:** In view of Proposition 5.2.1; \( M_{n-1} \) admits a metric semisymmetric connection. Then the Gauss and Weingarten
equations (1.7.6) and (1.7.7), with respect to semisymmetric metric connection are given by.

\begin{align}
(5.2.13) \quad \tilde{\nabla}_j a^b &= \tilde{\nabla}_j q^a \\
\intertext{and}
(5.2.14) \quad \tilde{\nabla}_j q^b &= -\tilde{\nabla}_j B_i^a.
\end{align}

Now transvect (5.2.13) with \( q_a = q^b g_{ba} \) to obtain

\begin{align}
(5.2.15) \quad \tilde{\nabla}_j q_a &= q_a \tilde{\nabla}_j B_i^a.
\end{align}

Substitute (1.7.4) written with respect to \( q_j \), in (5.2.15) and use (1.6.6), (1.6.7) to obtain

\begin{align}
(5.2.16) \quad \tilde{\nabla}_j q_a &= \sum b B_i^a + \left\{ a \right\} + \delta_a^c b_b - g_{cb} a^b + r_c a^b + r_b a^b \\
&\quad - r_{cb} a^b B_j^a - \left\{ b \right\} \eta^h_j = g_{jh} \eta^h a^b - \left\{ b \right\} \eta^h a^b.
\end{align}

Again by using (1.7.4) and (5.2.15) for \( \left\{ h \right\} \), \( \left\{ a \right\} \), \( \left\{ \eta \right\} \), \( \left\{ b \right\} \), and the fact that \( q^a \) is unit normal to \( M_{n-1} \) and (4.2.1) we get the required result (5.2.12).

Q.E.D.

5.3. C-Fubinian Manifolds:

A Sasakian manifold \( M_{2n+1} \) with the structure

\( (\phi_j, \eta_j, \eta^i, g_{ji}) \) is said to be locally C-fubinian manifold \( \left[ 4 \right] \) if its curvature tensor is given by:

\[ \quad \]
A Sasaki an manifold $M_{2n+1}$ is said to be $C$-Einstein if the Ricci tensor is of the form.

\[(5.3.2) \quad K_{ji} = p \, \varepsilon_{ji} + q \, \eta_{j} \eta_{i}, \]

$p, q$ being the constants.

We give the following well known result in the form of

**Lemma 5.3.1**: For a $C$-Einstein Sasaki an manifold $M_{2n+1}$

\[p + q = 2n,\]

where $p, q$ are constants given in (5.3.2).

**Proof**: Transverse the result $K_{ji} = K_{sjji}^s$ by $\eta^i$ and use the property,

\[K_{kji}^h \eta^i = (\delta_k^h \eta_j - \delta_j^h \eta_k),\]

satisfied by $M_{2n+1}$ to obtain

\[(5.3.3) \quad K_{ji}^i = 2n \, \eta_j.\]
Now by using (5.3.2), (5.3.3) and (1.5.12) we get the required result.

Q.E.D.

Proposition 5.3.1: For a $C^\infty$-rubinian manifold $M_{2n+1}$ the contact Bochner curvature tensor is twice the curvature tensor.

Proof:* Transvect (5.3.1) with $g^{kh}$ and use (1.5.12) to obtain

\[(5.3.4) \quad K_{j} = p \epsilon_{j} + q \eta_{j} \eta_{1},\]

where

\[(5.3.5) \quad p = 2n(\lambda + 1) + 2\lambda \quad \text{and} \quad q = -2n\lambda - 2\lambda.\]

We note here that $p+q = 2n$ and thus from lemma 5.3.1 we can say that $M_{2n+1}$ is $C^\infty$-Einstein.

Again transvect \((5.3.4)\) with $g_{11}$ to get

\[(5.3.6) \quad K = 2n(p+1).\]

Substitute (5.3.6) in (1.5.14)(ii) and use the lemma 5.3.1 and (5.3.5) to obtain

\[(5.3.7) \quad L = -\zeta n(\lambda + 1) + 1.\]

Now by using (5.3.4), (5.3.6) and (5.3.7) in (1.5.14)(i) we get
Substitution of (5.3.8) in (1.5.14)(ni) gives

\[(5.3.9) \quad M_{ji} = \left(\frac{\lambda + 1}{2}\right) \phi_{ji} \text{ or } M_{ji} = -\left(\frac{\lambda - 1}{2}\right) \phi_{ji}.\]

Now by the substitution of (5.3.1), (5.3.8) and (5.3.9) in the expression (1.5.13) and after some simplification we get

\[(5.3.10) \quad B_{kijh} = 2\left(\lambda + 1\right)\left(\varepsilon_{kh} e_{ji} - e_{ki} e_{jh}\right) + \lambda (\phi_{kh} \phi_{ji} - \phi_{ki} \phi_{jh}) - 2\phi_{kjih} - \lambda (\varepsilon_{kh} \eta_{ij} - \varepsilon_{ki} \eta_{jh} + \varepsilon_{ji} \eta_{kh} - \varepsilon_{jh} \eta_{ki})J = 2K_{kijh}.\]

Q.E.D.

Yamaguchi has proved the following

**Theorem A:** Let \( M \) be a \( c \)-umbilical hypersurface of a Kahlerian manifold with vanishing Bochner curvature tensor. Then \( M \) is locally \( c \)-fubinian.

So, now by using theorem A and the above Proposition 5.3.1 we have the following

**Theorem 5.3.1:** Suppose \( M_{2n+1} \) is a \( c \)-umbilical hypersurface of a Kahlerian manifold \( M_{2(n+1)} \) with vanishing Bochner...
curvature tensor. Then the contact Bochner curvature tensor of $M_{2n+1}$ vanishes if and only if it is locally flat.

Now suppose $M_{2n+1}$ is a Sasakian hypersurface of a Kahlerian manifold $M_{2(n+1)}$ of constant holomorphic curvature tensor. Then the curvature tensor $R_{dcba}$ of $M_{2(n+1)}$ satisfies

$$R_{dcba} = \frac{k}{4}(g_{dc} g_{ob} - g_{db} g_{ca} + g_{da} g_{ob} - g_{db} g_{ca} - \frac{2}{3} g_{ba} g_{da} g_{ob}).$$

$k$ being a constant.

Now substitute (5.3.11) in the Gauss equation (1.7.12) to obtain,

$$K_{kjh} = \frac{k}{4}(g_{kh} g_{ji} - g_{ki} g_{jh} + g_{ji} g_{kh} - g_{kj} g_{jh} - 2g_{kj} g_{ij})$$

$$+ (g_{kh} g_{ji} - g_{ki} g_{jh}) + \frac{k}{4}(g_{kh} \eta_j \eta_l - g_{kj} \eta_j \eta_i)$$

$$+ g_{jl} \eta_k \eta_h - g_{jh} \eta_k \eta_l);$$

where we have used (4.2.1).

Now if we take the mean curvature $H=constant = \frac{1+k}{4(2n+1)}$ then (5.3.12) is written as

$$K_{kjh} = (1+ \frac{k}{4})(g_{kh} g_{ji} - g_{ki} g_{jh}) + \frac{k}{4}(g_{kj} g_{ij} - g_{ki} g_{jh}) -$$

$$- 2g_{kj} g_{ij} - \frac{k}{4}(g_{kh} \eta_j \eta_l - g_{kj} \eta_j \eta_i + g_{ji} \eta_k \eta_l - g_{jh} \eta_k \eta_l).$$
where we have used the fact that \( \mu = 2n(H-1) \) and \( H = \frac{1}{2n+1} h_{ij} g^{ji} \).

Thus (5.3.13) shows that \( M_{2n+1} \) is c-fubinian and hence from Proposition 5.3.1 we have the following.

**Theorem 5.3.2:** Suppose \( M_{2n+1} \) is a c-umbilical hypersurface with constant mean curvature \( H = \frac{1+k}{4(2n+1)} \) in a Kahlerian manifold \( M_{2(n+1)} \) of constant holomorphic sectional curvature \( k \). Then the contact Bochner curvature of \( M_{2n+1} \) vanishes if and only if it is locally flat.

### 5.4 \( W \)-recurrent Sasakian manifold:

In this section we consider a c-Einstein Sasakian manifold \( M_{2n+1} \) with recurrent \( W \)-tensor given by (1.4.15) and obtain a condition for such a space to be Einstein.

By using (5.3.2), (5.3.3), (1.4.17) and the Lemma 5.3.1 we have the following results for a c-Einstein Sasakian manifold \( M_{2n+1} \) of dimension \( 2n+1 \).

\[(5.4.1) \quad (i) \quad \kappa = 2n(p+1),
(ii) \quad g_{ji} = -\frac{q}{2n+1} g_{ji} + q \eta_j \eta_i,
(iii) \quad g_{ji} \eta_i = \frac{2nq}{2n+1} \eta_j,
\]

\( p, q \) being the constants used in (5.3.2).
For a G-Einstein Sasakian manifold it is given

\[(5.4.2) \quad K_{kji}^h \eta^i = (\delta_k^h \eta_j - \delta_j^h \eta_k).\]

**Proposition 5.4.1:** In a G-Einstein Sasakian manifold
\[C_{kji}^h \eta_i = 0, \quad C_{kji}^h \] being the conformal curvature tensor.

**Proof:** By using \((5.4.1)(i), (5.3.2)\) in \((1.4.11)\) written for \(M_{2n+1}\) we get:

\[(5.4.3) \quad \sigma_{j i} = -\frac{1}{2(2n-1)} \sum_{p-1} g_{ji} + 2q \eta_j \eta_i.\]

which by transvecting with \(\eta_i\) gives

\[(5.4.4) \quad \sigma_{j i} \eta^i = -\frac{1}{2(2n-1)} \sum_{4n-p-1} g_{ji} \eta_j.\]

Hence by the substitution of \((5.4.2), (5.4.3)\) and \((5.4.4)\) in \((1.4.10)\) written for \(M_{2n+1}\), we get the required result.

Q.E.D.

**Theorem 5.4.1:** Suppose \(M_{2n+1}\) is a \(W\)-recurrent G-Einstein Sasakian manifold with \(a + (2n-1)b \neq 0\). Then either the recurrent vectorfield \(k_1\) is normal to \(\eta_1\) or the space \(M_{2n+1}\) is Einstein.

**Proof:** Substitute \((5.4.1)(ii), (5.4.1)(iii)\) in \((1.4.21)\) and use the Proposition 5.4.1 to obtain
(5.4.5) \( W_{kji} \eta^i = \sum_{n=0}^{2n+1} \eta_j \eta_k \). Now differentiate (5.4.5) with respect to \( \eta_j \) and use (1.5.12) to get

(5.4.6) \( k_1 W_{kji} \eta^i + W_{kji} \eta^i \eta_1 = \sum_{n=0}^{2n+1} \eta_j \eta_k \). where \( W_{kji} \eta^i = k_1 W_{kji} \eta^i \).

By the transvection of (5.4.6) with \( \eta_1 \) we get

(5.4.7) \((k_1 \eta^1) W_{kji} \eta^i = 0\),

by virtue of (1.5.12). Hence we have

(i) \( k_1 \eta^1 = 0 \) or (ii) \( \sum_{n=0}^{2n+1} \eta_j \eta_k = 0 \).

For the case (i) \( k_1 \) is normal to \( \eta^1 \).

In case (ii) If we take \( \eta^1 \eta_j = 0 \) then \( \eta_j = 0 \).

Thus assuming \( \eta^1 \eta_j = \eta^1 \eta_k \neq 0 \) we have \( q = 0 \).

So, from (5.3.2) we get \( k_j = 0 \) and hence \( M_{2n+1} \) is Einstein. Q.E.D.
REFERENCES


