CHAPTER X

FLOW OF AN ELASTICO-VISCOS FLUID PAST AN OSCILLATING FOROUS PLATE WITH SUCTION
10.1 Introduction

Stokes studied the flow problem of an incompressible fluid by the action of viscosity when an infinite flat plate immersed in the fluid is set into linear harmonic oscillation parallel to itself (Stokes second problem). Panton gave the transient solution of Stokes second problem for incompressible fluid. Because of the wide applications in the technological field, the study of flow behaviour of non-Newtonian fluids is receiving the attention of research workers. One such fluid, those constitutive equations are characterised by Walters (liquid B') does help us to explain the flow behaviour of a certain class of non-Newtonian fluids with short memories. The well known problems of a flow past an impulsively started infinite plate, and the flow past an oscillating infinite plate, first studied by Stokes, were studied in the case of Walters liquid B' by Soundalgekar. In the present chapter, the flow of Walters liquids past an infinite oscillating porous plate is studied when the plate is subjected to uniform suction. In the physiology of saliva, blood and for certain polymers having short memories, such a study will be found applicable.

10.2 Mathematical analysis and results

The constitutive equation characterising the elastico-
viscous liquid $B'$ are given by equation (1.3.1) and (1.3.2), where $\lambda_1$, $\mu$'s, $\psi$'s are taken equal to zero. Thus the constitutive equation characterising this Walters liquid $B'$ are

$$\begin{align*}
\rho_1 &= \tau d_{ik} + \rho_{ik}^1 \\
\rho_{ik}^1 &= 2\eta_0 \varepsilon_{ik} - 2k_0 \frac{\partial}{\partial t} \varepsilon_{ik}
\end{align*}$$

(10.2.1)

where $\eta_0$ is the limiting viscosity at small rates of shear $k_0$ and $\partial/dt$ denotes the convected differentiation of a tensor quantity.

In the present problem, we take the coordinate origin $O$ at an arbitrary point on the oscillating porous plate. The $x$-axis is taken in the direction of oscillation and the $y$-axis normal to the plate. With this choice of axes the equation governing the flow of an incompressible elasto-viscous liquid $B'$ with well-known Boussinesq approximation is (as the plate is assumed infinite, all the physical variables are functions of $y$ and $t$ only),

$$p \frac{\partial u'}{\partial t} = v_o \frac{\partial u'}{\partial y} = \eta_o \frac{\partial^2 u'}{\partial y^2} + k_o \frac{\partial^3 u'}{\partial y^3} + v_o k_o \frac{\partial^3 u'}{\partial y^3}$$

(10.2.3)

Here the limiting viscosity is $\eta_0 = \int_0^\infty \bar{H}(\tau) d\tau$

at small rates of shear $k_0 = \int_0^\infty \bar{H}(\tau) d\tau$
where \( N(\tau) \) is the distribution function of relaxation times \( \tau \).

The boundary conditions are:

\[
\begin{align*}
    u' &= u_o \cos \omega t \quad \text{at } y = 0 \\
    u' &= 0 \quad \text{as } y \to \infty
\end{align*}
\]

(10.2.4)

with the following non-dimensional quantities,

\[
\begin{align*}
    u &= u'/u_0 = y/\sqrt{\tau}, \quad t_2 = t/\tau, \quad k = \frac{k_o}{\eta_0 \tau} \\
    \eta &= \eta_0 \sqrt{\frac{v}{v_o}}, \quad \nu = \frac{v_o \tau}{v}, \quad w = w/\tau, \quad \tau_1 = \eta_0 \sqrt{v_o/2} \\
    \eta_1 &= y_1 \sqrt{w/2}
\end{align*}
\]

(10.2.5)

the equations (10.2.3) and (10.2.4) become,

\[
\begin{align*}
    \frac{\partial u}{\partial t_2} + \nu \frac{\partial u}{\partial y_1} &= \frac{\partial^2 u}{\partial y_1^2} - k \left( \frac{\partial^3 u}{\partial y_1^3} + \nu \frac{\partial^3 u}{\partial y_2^3} \right) \\
    u &= \cos \omega t_1 \text{ at } y_1 = 0, \quad u = 0 \text{ as } y_1 \to \infty
\end{align*}
\]

(10.2.6)

(10.2.7)

To solve equation (10.2.6) we follow Beard and Walters and assume the solution in the form

\[
u = u_o + k u_1
\]

(10.2.8)

which is valid for small values of \( k \).
substituting (10.2.9) in (10.2.6) and (10.2.7) and equating the coefficients of different powers of \( k \), neglecting those of \( k^2 \), we get

\[
\frac{d^2 u_0}{dt_1^2} - v \frac{du_0}{dy_1} = \frac{d^4 u_0}{dy_1^4} \quad (10.2.9)
\]

\[
\frac{d^2 u_1}{dt_1^2} - v \frac{du_1}{dy_1} = \frac{d^4 u_1}{dy_1^4} = \frac{d^6 u_0}{dy_1^6} \quad (10.2.10)
\]

and the boundary conditions are

\[
\begin{align*}
\text{at } t_1 & \rightarrow 0; \ y_1 = 0, \ y_1 = \infty \\
\text{at } t_1 & \rightarrow 0; \ y_1 = 0, \ y_1 = \infty \\
\end{align*}
\]

To solve these coupled non-linear equations we assume the suction to be small. Hence we expand as

\[
u_0 = u_{01} + v u_{02}, \ u_1 = u_{11} + v u_{12} \quad (10.2.12)
\]

substituting (10.2.12) in (10.2.9), (10.2.10) and (10.2.11), and equating the coefficients of different powers of \( v \), neglecting the terms with \( v^2 \), we get

\[
\frac{d^2 u_{01}}{dt_1^2} = \frac{d^2 u_{01}}{dy_1^2} \quad (10.2.13)
\]
\[
\frac{d u_{02}}{dt_1} = \frac{d u_{01}}{dy_1} = \frac{d^2 u_{02}}{dy_1^2}
\]

\[
\frac{d u_{11}}{dt_1} = \frac{d^2 u_{11}}{dy_1^2} = \frac{d^4 u_{01}}{dy_1^4}
\]

\[
\frac{d u_{12}}{dt_1} = \frac{d u_{11}}{dy_1} = \frac{d^2 u_{12}}{dy_1^2} = \frac{d^4 u_{02}}{dy_1^4}
\]

\[u_{01} = \cos \omega t_1, \quad u_{02} = 0, \quad u_{11} = u_{12} = 0 \text{ at } y_1 = 0\]

\[u_{01} = 0, \quad u_{02} = 0, \quad u_{11} = 0, \quad u_{12} = 0 \text{ as } y_1 \to \infty\]

The solutions of these equations (10.2.13)-(10.2.16) are obtained with (10.2.17) and are substituted in (10.2.12). Then from (10.2.18), the solution for velocity field is given by

\[u = e^{-\eta} \cos (\omega t_1 - \eta) \left[ 1 - \frac{\omega t_1}{\sqrt{2\nu}} \right] + \frac{\omega \kappa_n}{4} e^{-\eta} \sqrt{2\nu} e^{-\eta} \sin (\omega t_1 - \eta) + \frac{\omega \kappa_n}{4} \sqrt{2\nu} e^{-\eta} \sin (\omega t_1 - \eta) \]

The velocity profile \(u\) given by (10.2.18) will represent the damped oscillation of the amplitude about \(e^{-\eta}\). Fig. (10.1) shows the velocity profiles for different \(\omega t_1\), \(\omega\) and \(\nu\). The velocity \(u\) is perpendicular to the direction of
flow and its amplitude decreases exponentially as the distance \( n \) from the plate increases. Here we find that for \( \omega t_1 = 0 \), at large values of \( v \), a point of inflexion occurs on the velocity profiles of an elastico-viscous fluid which may lead the flow to become unstable, and for \( \omega t_1 > 0 \), an increase in \( \omega \) leads to an increase in velocity in the vicinity of the oscillating plate. We observe from these curves that the effect of suction is to decrease the magnitude of the velocity.

Knowing the velocity field, we now calculate the shearing stress, which, for the case of elastico-viscous liquid \( 3^* \), is given by

\[
p_{xy} = - \left( \eta _0 \frac{\partial u}{\partial y} - k \frac{\partial^2 u}{\partial y^2} + k \omega \frac{\partial^2 u}{\partial y^2} \right)_{y=0} \quad (10.2.19)
\]

and in view of (10.2.9), (10.2.19) reduces to

\[
p_{xy_1} = p_{xy} / \omega^2 = - \left( \frac{\partial u}{\partial y_1} - k \frac{\partial^2 u}{\partial y_1^2} + k \omega \frac{\partial^2 u}{\partial y_1^2} \right)_{y_1=0}
\]

\[
\text{......} \quad (10.2.20)
\]

From (10.2.19) and (10.2.20), we get

\[
p_{xy} = - \sqrt{w} \left[ \sin (v t_1 - \pi/4) - \frac{k \omega}{2} \cos (v t_1 - \pi/4) \right] + \frac{v}{2} \left[ \cos (v t_1) + \frac{3}{2} k \omega \sin (v t_1) \right] \quad (10.2.21)
\]
From (10.2.21) we conclude that the shearing stress increases with suction and it decreases with increasing \( w \), in the case of these elastic-viscous fluids.

10.3 Conclusions

1. For large values of \( w \), for \( w t = 0 \), the flow will become unstable.

2. With increase in suction there is decrease in the magnitude of the velocity.

3. Shearing stress increases with increase in suction.