CHAPTER V

SOLUTION TO UNSTEADY HYDROMAGNETIC FLOW PAST A POROUS PLATE

POROUS PLATE MOVING WITH TIME DEPENDENT VELOCITY
5.1 Introduction

Chapters II, III and IV present the solutions for the flow problem with a stationary flat plate. However, if we consider the motion of the plate in the main stream which induces an unstable state in the fluid, the problem would be of practical interest. Since the classical Stokes problem, a number of such studies involving the motion of the plate have been made. In the present Chapter four problems (sections A, B, C and D) of incompressible fluid flow past an infinite porous plate with uniform suction moving with time-dependent velocity when the initial distribution of velocity is an exponential form are considered. The incompressible laminar viscous fluid flow between two stationary parallel flat plates with an arbitrary time-varying pressure gradient and with an arbitrary initial distribution of velocity has been studied by Hepworth and Rice. The same problem has been considered by Prakash under the same conditions with the only difference that in his case the flow takes place in an annular space bounded by two stationary coaxial circular cylinders. The problem of viscous incompressible flow past an infinite plate moving parallel to itself with an arbitrary time-dependent velocity when the pressure is uniform and the initial distribution of velocity is an exponential form has been studied by Prakash. Srivastava and Lal extended this
problem in the case of electrically conducting fluid in the presence of transverse magnetic field. Srivastava has studied the same problem when the magnetic lines of force were fixed relative to the plate. Similar problem has also been studied by Rosenou. In the first problem, in Section A, of the present chapter we are concerned with the study of the incompressible laminar, viscous, electrically conducting fluid flow past an infinite flat porous plate moving parallel to itself with an arbitrary time-dependent velocity with uniform suction at the plate, when the pressure is uniform and the initial distribution of velocity is an exponential form, and when the magnetic lines of force are fixed relative to the plate.

SECTION A

5.2 Formulation of the problem and solution

Here we take $x$ and $y$ axes along and transverse to the plate and assume a uniform magnetic field acting along the $y$-axis. Since the magnetic lines of force are fixed relative to the plate, the magnetic field is also moving with the plate, as such, this relative motion must be accounted for. Thus the governing differential equation of motion for the present problem is

$$\frac{\partial u'}{\partial t} - V \frac{\partial u'}{\partial y} + H(u' - g(t)) = \gamma \frac{\partial^2 u'}{\partial y^2} \quad (5.2.1)$$
subject to the boundary and initial conditions:

\[ t = 0 : u^* = a \exp(-by) \text{ for } y \geq 0 \quad (5.2.2) \]
\[ t > 0 : u^* = g(t) \quad \text{ at } y = 0 \quad (5.2.3) \]
\[ t > 0 : u^* = 0 \quad \text{ at } y = \infty \quad (5.2.4) \]

Here \( a, b \) are non-negative constants and \( g(t) \) is bounded continuous or piecewise continuous arbitrary function of \( y \).

To obtain the solution of (5.2.1), subject to the conditions (5.2.2) to (5.2.4), we apply Laplace transform techniques. Then we obtain the solution as:

\[
\begin{align*}
\frac{u^*}{x} &= \exp\left[-\frac{y}{2}Y\right] \int_0^t g(t-t') \frac{e^{-\frac{y^2}{4\pi t'}}}{\sqrt{4\pi t'}} \exp\left\{-\frac{y^2}{4t'} - \frac{\nu^2 t'}{4} - \frac{\nu b t'}{4}ight\} dt' - \frac{a}{2} \exp\left[-\frac{Y}{2}Y\right] \int_0^t g(t-t') \exp\left[-\frac{Y}{2}Y\right] dt' - \\
&- \frac{a}{2} \exp\left[-\frac{Y}{2}Y\right] b^2 t - \nu b t - \nu b t' - \exp\left[-\frac{Y}{2}Y\right] \int_0^t g(t-t') e^{-\nu b t'} dt' - \\
&- \frac{a}{2} \exp\left[-\frac{Y}{2}Y\right] b^2 t - \nu b t - \nu b t' - \exp\left[-\frac{Y}{2}Y\right] \int_0^t g(t-t') e^{-\nu b t'} dt'
\end{align*}
\] (5.2.5)
The solution can also be written in another form as follows.

\[
\begin{align*}
    u' &= \frac{2}{\sqrt{\pi}} \exp \left\{ \frac{-y^2}{2\gamma} \right\} \int_0^\infty g(t - \frac{x^2}{4\gamma^2}) \exp \left\{ \frac{-ny^2}{4\gamma^2} - \frac{v^2y^2}{16\gamma^2} - \eta^2 \right\} \, d\eta - \\
    &\quad - \frac{ny^2}{2\gamma} \exp \left\{ \frac{-y^2}{2\gamma} \right\} \int_0^\infty g(t - \frac{x^2}{4\gamma^2}) \exp \left\{ \frac{-ny^2}{4\gamma^2} \right\} \left\{ \exp \left\{ \frac{-v^2y^2}{4\gamma^2} \right\} \exp \left\{ \eta \right\} \right\} \, d\eta - \\
    &\quad - \frac{vy^2}{2\gamma} \exp \left\{ \frac{-y^2}{2\gamma} \right\} \int_0^\infty g(t - \frac{x^2}{4\gamma^2}) \exp \left\{ \frac{-vy^2}{4\gamma^2} \right\} \exp \left\{ \eta \right\} \frac{1}{\eta^2} \, d\eta - \\
    &\quad + a \exp \left\{ - (\frac{1}{2} - \eta b^2 + \eta b) \gamma t \right\} \left[ \exp \left\{ - \gamma t \right\} - \exp \left\{ - \frac{vy^2}{2\gamma} \right\} + \\
    &\quad + \left( b^2y^2 + \frac{v^2y^2}{4\gamma^2} \right) \gamma t \right\} + \frac{2}{\sqrt{\pi}} \int_0^\infty \exp \left\{ - \eta^2 - \frac{1}{4\gamma^2}(b^2y^2 + \frac{v^2y^2}{4\gamma^2}) + \\
    &\quad + \frac{vy^2}{2\gamma} \right\} \, d\eta \right\} \text{ (if } y > 0) \quad (5.2.6)
\end{align*}
\]

\[
\begin{align*}
    u' &= g(t) \quad (y = 0), \quad (5.2.7)
\end{align*}
\]

where

\[
\eta = \frac{\gamma t}{\sqrt{4\gamma t}}
\]

It is clear from the initial and boundary conditions of the problem that there must be some kind of discontinuity in the
flow at \( y = 0 \) at the time of start of motion unless \( a = g(0) \).

This can also be shown by eqs. (5.2.6) and (5.2.7).

We now study the solution at large times. We note that the last two terms in eq. (5.2.6) vanish as \( t \to \infty \) so that solution for large times is given by

\[
\frac{u^*}{\sqrt{\pi}} \exp \left\{- \frac{y v_x}{2y} \right\} \int_{0}^{\infty} \exp \left\{ - \frac{\nu^2}{4\eta^2} - \frac{v^2 y^2}{16 \eta^2} \right\} \frac{1}{\eta^3} \, d\eta - \frac{\nu^2}{2y} \exp \left\{- \frac{y v_x}{2y} \right\} \int_{0}^{\infty} \exp \left\{ - \frac{\nu^2}{4\eta^2} \right\} \frac{1}{\eta^3} \, d\eta - \frac{\nu^2}{2y} \int_{0}^{\infty} \exp \left\{ - \frac{\nu^2}{4\eta^2} \right\} \frac{1}{\eta^3} \, d\eta
\]

From eq. (5.2.8), at \( y = 0 \), we obtain eq. (5.2.7) taking into account that \( \int_{0}^{\infty} \exp \left\{ - \eta^2 \right\} \, d\eta = \frac{\pi}{2} \).

Hence the solution of the problem for large times is given by eq. (5.2.8) for \( y > 0 \). But equation (3.2.6) from which (5.2.8) is derived is valid for \( y > 0 \) only and not for \( y = 0 \). This is because of the fact that at the time of commencement of the motion, there is discontinuity in the flow at \( y = 0 \), as pointed out earlier. This discontinuity reduces gradually with time and finally, totally vanishes after
sufficiently long time has elapsed since the start of the
motion due to viscosity. We also note from eq. (5.2.6) that
velocity depends on both the initial distribution of velocity
and on the motion of the plate in the case of small values of
times whereas, the velocity given by eq. (5.2.8) does not
depend on the initial distribution of velocity and it depends
on the velocity of the plate for large values of times. The
equation (5.2.8) is said to be steady state solution of the
problem.

5.3 some special cases

a) solution for ordinary hydrodynamic flow (n = 0):
If \( M = \frac{G}{P_c} \quad \mu_0^2 = 0 \) eqs. (5.2.6), (5.2.7) and (5.2.8)
respectively reduce to

\[
\begin{align*}
u' &= \frac{2}{\sqrt{\pi}} \exp \left\{ -\frac{y^2}{2} \right\} \int_0^\infty g(t-\frac{y^2}{4\eta^2}) \exp \left\{ -\frac{y^2}{16 \eta^2} - \eta^2 \right\} \, d\eta + \\
&+ a \exp \left\{ -t(y_b - y_b^2) \right\} \left[ \exp \left\{ -b\eta \right\} - \exp \left\{ -\frac{y}{2}\right\} + \frac{y}{y_b} + \frac{y^2 y_b^2}{4} \right] \\
&+ \frac{2}{\sqrt{\pi}} \int_0^\infty \exp \left\{ -\eta^2 - \frac{1}{4 \eta^2} \left( b^2 y^2 + \frac{v_b^2}{4 \gamma^2} + \frac{v_b y^2}{y} \right) \right\} \, d\eta \quad (y > 0)
\end{align*}
\]
These correspond to the solutions for hydrodynamic flow given by Srivastava.\(^7\)

\[ u^* = g(t) \quad (y = 0) \]

\[
u^* = \frac{2}{\sqrt{\pi}} \exp \left( -\frac{y^2}{4\gamma t} \right) \int_0^\infty g(\tau - \frac{y^2}{4\gamma \eta^2}) \exp \left[ -\frac{\gamma^2}{16\eta^2} - \frac{\gamma^2}{4\eta^2} \right] d\eta
\]

\[ \ldots \quad (5.3.2) \]

b) An infinite porous flat plate sets impulsively into uniform motion with constant velocity \( u_1 \) with uniform suction \( V \) on the plate, in the fluid at rest in the presence of a transverse uniform magnetic field \( H_0 \):

The solution for this problem may be obtained by putting \( g(t) = u_1 \) and \( a = 0 \), in eq. (5.2.6) and (5.2.7).

Thus we have,

\[
u^* = u_1 \left( 1 - e^{-\gamma t} \right) - \frac{u_1 e^{-\gamma t}}{2} \exp \left\{ -\frac{y^2}{2\gamma} \right\} \left\{ \exp \left\{ -\frac{y^2}{2\gamma} \right\} \times \right.
\]

\[
\times \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} - \frac{u_1 t}{\sqrt{4\gamma}} \right] + \exp \left\{ \frac{y^2}{2\gamma} \right\} \times \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} + \frac{u_1 t}{\sqrt{4\gamma}} \right]
\]

\[ \begin{cases} \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} - \frac{u_1 t}{\sqrt{4\gamma}} \right] + \exp \left\{ \frac{y^2}{2\gamma} \right\} \times \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} + \frac{u_1 t}{\sqrt{4\gamma}} \right] \\ \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} - \frac{u_1 t}{\sqrt{4\gamma}} \right] + \exp \left\{ \frac{y^2}{2\gamma} \right\} \times \text{erfc} \left[ \frac{y}{\sqrt{4\gamma t}} + \frac{u_1 t}{\sqrt{4\gamma}} \right] \end{cases} \quad (y > 0)
\]

\[ u^* = u_1 \quad (y = 0) \quad (5.3.4) \]

c) An infinite porous flat plate moving in non-conducting fluid with time-dependent velocity \( U(t) \) with
uniform suction $v$ on the plate in fluid at rest:

The solution for this problem is obtained by setting $g(t) = U(t)$, $a = 0$, and $u = 0$ in equation (5.2.6) and (5.2.7).

\[
u' = \frac{2}{\sqrt{\pi}} \exp \left\{ -\frac{v^2}{2\gamma} \right\} \int_0^\infty U(t-\frac{y^2}{4\gamma^2}) \exp \left\{ -\frac{v^2y^2}{16\gamma^2} - \eta^2 \right\} d\eta,
\]

\[(y > 0) \quad (5.3.5)\]

\[
u' = U(t) \quad (y = 0) \quad (5.3.6)
\]

These correspond to the expressions given by Hasimoto\(^7\) namely

\[
u = \sqrt{\gamma} \int_0^\infty u(t-\eta) \frac{v}{\sqrt{4\pi}\gamma \eta^3} \exp \left\{ -\frac{v^2\eta^2}{2\gamma^2} - k^2\gamma \eta \right\} d\eta
\]

\[d) \text{ An infinite porous flat plate oscillating (linear-harmonically) parallel to itself with velocity, } U \cdot \text{coant with uniform suction } v \text{ in the fluid at rest:}

The solution for this problem for large values of times is obtained by putting $g(t) = U \cdot \text{coant}$, $a = 0$, $b = 0$ in (5.2.8) which gives

\[
u' = \frac{2i}{\sqrt{\pi}} \exp \left\{ -\frac{v^2}{2\gamma} \right\} \int_0^\infty \cos \left\{ \alpha(t-\frac{y^2}{4\gamma^2}) \right\} \exp \left\{ -\frac{v^2y^2}{16\gamma^2} - \eta^2 \right\} d\eta
\]

\[(y > 0) \quad (5.3.6)\]
This solution can be compared to the solution obtained by Srivastava and Lai, namely,

\[ u = u_0 \exp \left\{ \frac{V_0}{2 \gamma_0} - (x^2 - \sin \frac{y}{2}) \right\} \cos \left\{ \left( x^2 - \cos \frac{y}{2} \right) y + nt \right\} \]

where

\[ \rho = \left( \frac{b}{\gamma_0} \right)^2 + \left( \frac{V_0}{4 \gamma_0} \right)^2 \]

\[ \sinh \left( \frac{4 \gamma_0}{4 \gamma_0} \right) = \tan^{-1} \left( \frac{4 \gamma_0 V_0}{V_0} \right) \]

**e) Classical Stokes first problem**

By putting \( g(t) = U; a=0, \) =0, \( V = 0 \) in (5.2.6) we find

\[ u' = U \left[ 1 - \frac{2}{\sqrt{\pi}} \int_0^y \exp \left\{ -\eta^2 \right\} \eta \, d\eta \right] = U \operatorname{erf} \eta \quad (y > 0) \tag{5.3.7} \]

which is Schlichting solution\textsuperscript{32} (page 72).

**f) Classical Stokes second problem**

For large values of times, Schlichting solution\textsuperscript{32} (page 75) is obtained with \( g(t) = U \) count, \( a = 0, \) \( b = 0 \) and \( V = 0 \) in (5.2.6) as

\[ u' = \frac{2U}{\sqrt{\pi}} \int_0^y \cos \left\{ \eta \left( t - \frac{y^2}{4} \right) \right\} \exp \left\{ -\eta^2 \right\} \, d\eta \quad (y > 0) \]

\[ \ldots \tag{5.3.8} \]

which is equivalent to
g) Stokes' first problem in magnetohydrodynamics:

For this problem the solution is obtained by putting

\[ g(t) = U; \quad a = 0; \quad v = 0, \text{ in (5.2,5)} \text{ which yield,} \]

\[ u' = u (1 - e^{-\alpha t}) - U e^{-\alpha t} \text{ erfc} \left( \frac{y}{\sqrt{4\nu t}} \right); \quad (y > 0) \quad (5.3.9) \]

and for \( y = 0 \) \( u = u \)

h) Stokes' second problem in mhd:

In this case for large values of time \( t \), solution is obtained by putting \( g(t) = U \text{ coant}, \quad a = 0, \quad \text{ and } v = 0 \text{ in (5.2,9)} \text{ which gives}

\[
\begin{align*}
u' &= \frac{2\nu}{\sqrt{\pi}} \int_0^\infty \cos \left( n(t - \frac{y^2}{4
u\eta^2}) \right) \exp \left\{ -\frac{ny^2}{4\nu\eta^2} - n^2 \right\} \, d\eta - \\
-\nu' &= \frac{\nu y^2}{2\nu} \int_0^\infty \cos \left( n(t - \frac{y^2}{4\nu\eta^2}) \right) \exp \left\{ -\frac{ny^2}{4\nu\eta^2} \right\} \text{ erfc} \eta \cdot \frac{1}{\eta^3} \, d\eta \\
-\nu' &= \frac{\nu y^2}{2\nu} \int_0^\infty \cos \left( n(t - \frac{y^2}{4\nu\eta^2}) \right) \exp \left\{ -\frac{ny^2}{4\nu\eta^2} \right\} \frac{1}{\eta^3} \, d\eta \quad (y > 0) \\
\end{align*}
\] 

\[ \ldots \quad (5.3.10) \]
SECTION B

Often the velocity of the main stream fluid is not uniform as in a rocket nozzle, it increases with distance downstream, whilst in a ram-jet intake it decreases, the main stream pressure is accordingly non-uniform. Prakash has studied the unsteady incompressible viscous flow under a time-varying pressure gradient in a straight channel with two parallel porous walls with uniform suction and injection at the walls. The same problem has been extended by Srivastava for the case. In this section the case of fluid flow past a flat porous plate moving parallel to itself with uniform velocity under the influence of time-varying pressure gradient is studied.

5.4 Formulation of the problem and solution

Consider an unsteady, electrically conducting, two dimensional incompressible, viscous fluid flow past an infinite porous flat plate insulated plate which sets impulsively into uniform motion with constant velocity \( v_1 \) with uniform suction \( v(v > 0) \) when the initial distribution of the velocity is an exponential form in the region occupied by the fluid on one side of the plate, under the influence of time varying pressure gradient. We take x and y axes along and normal to the plate.
At sufficiently large distances from the origin (taken arbitrarily on the plate), the flow is fully developed and the physical quantities depend on \( y \) and \( t \) only. Then the governing equations of the problem are:

\[
\begin{align*}
\frac{\partial u'}{\partial t} - v \frac{\partial u'}{\partial y} + \gamma u' &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u'}{\partial y^2} \\
0 &= \frac{\partial p}{\partial y}
\end{align*}
\]  
\tag{5.4.1, 5.4.2}

The initial and boundary conditions for this problem are

\[
\begin{align*}
t = 0: \quad u &= a \exp(-by) \text{ for } y \geq 0 \\
t > 0: \quad u &= V_1 \text{ for } y = 0 \\
t > 0: \quad u &\to 0 \text{ as } y \to \infty
\end{align*}
\]  
\tag{5.4.3, 5.4.4}

Now, if we assume \( \frac{\partial p}{\partial x} = -f(t) \), eq. (5.4.1) reduces to

\[
\begin{align*}
\frac{\partial u'}{\partial t} - v \frac{\partial u'}{\partial y} + \gamma u' &= \frac{1}{\rho} f(t) + \gamma \frac{\partial^2 u'}{\partial y^2}
\end{align*}
\]  
\tag{5.4.5}

We obtain the solution of (5.4.5) subject to initial and boundary conditions (5.4.3) and (5.4.4) with Laplace transform techniques and is given by...
\[ u' = \frac{V}{2} \exp\left\{ -\frac{vy}{2}\right\} \left\{ \exp\left\{ -\frac{y}{\sqrt{4y}} \left( \frac{V^2}{4y} + m t\right) \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} - \sqrt{\left(\frac{V^2}{4y} + m\right)t} \right] - \right. \\
- \left. \frac{v}{2} \exp\left\{ -\frac{vy}{2}\right\} \left\{ \exp\left\{ \frac{y}{\sqrt{4y}} \left( \frac{V^2}{4y} + m\right) t\right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} + \sqrt{\left(\frac{V^2}{4y} + m\right)t} \right] \right\} - \frac{a}{2} \exp\left\{ -\frac{vy}{2y} + \gamma b^2 t - \nu t \right\} \left\{ \exp\left\{ yr \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} + k \sqrt{y} t \right] + \right. \\
+ \left. \exp\left\{ -y k \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} - k \sqrt{y} t \right] \right\} - \right. \\
- \frac{1}{2y} \exp\left\{ -\frac{vy}{2y}\right\} \left\{ \int_{0}^{t} f(t-t') e^{-\gamma t'} \left\{ \exp\left\{ -\frac{vy}{2y} \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt'}} + \frac{V^2}{\sqrt{4yt'}} \right] + \right. \\
+ \left. \exp\left\{ \frac{vy}{2y} \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt'}} - \frac{V^2}{\sqrt{4yt'}} \right] \right\} dt' + \right. \\
+ \left. \frac{1}{2} \int_{0}^{t} f(t-t') e^{-\gamma t'} dt' + a \exp\left\{ -by - \nu t - \gamma b^2 t - \nu t \right\} \right\}; \right. \\
\right) \quad \quad (5.4.6) \]

where

\[ k = (\gamma b - \frac{V}{2y}) \]

We assume the pressure gradient to be constant i.e.,

\[ \frac{\partial P}{\partial x} = -f(t) = C, \]

where C is positive constant. We obtain the exact solution of the problem for uniform pressure gradient as:

\[ u' = \frac{1}{2} \left( V_1 - \frac{C}{\rho y} \right) \exp\left\{ -\frac{vy}{2y}\right\} \left\{ \exp\left\{ -\frac{yl}{\sqrt{4yt}} \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} - \sqrt{yt} \right] + \right. \\
+ \left. \exp\left\{ \frac{yl}{\sqrt{4yt}} \right\} \text{erfc}\left[ \frac{y}{\sqrt{4yt}} + \sqrt{yt} \right] \right\}; \]

\[ (y > 0) \]
The steady state solution of this problem is obtained by taking the limit of eq. (5.4.7) as \( t \to \infty \). Thus by doing so the steady state solution is obtained as:

\[
\begin{align*}
\frac{d u}{d t} &= \left( V_1 - \frac{C}{\rho M} \right) \exp \left( -\frac{V}{2y} \right) \exp \left( -\frac{x}{\sqrt{4y}} \right) (V^2/4y + h)^{\frac{1}{2}} + \frac{C}{\rho M} \quad (5.4.9)
\end{align*}
\]

Here also, as in section A, we find that there exists a discontinuity in the flow at \( y=0 \). We also note from (5.4.7) that the velocity depends on the initial distribution of
velocity, on the motion of the plate, and on the pressure gradient for small values of time whereas the steady state velocity given by (5.4.9) does not depend on the initial distribution of the velocity but depends on the velocity of the plate as well as the pressure gradient for \( y > 0 \). And for \( y = 0 \) it is just the impulsive velocity \( v_1 \) of the plate.

**SECTION C**

In this section we generalise the problem considered in section B with the motion of the plate taken arbitrarily time-dependent. Thus the present problem to be considered is concerned with the study of problem of incompressible laminar, viscous, electrically conducting fluid flow past an infinite flat porous plate moving parallel to itself with an arbitrary time dependent velocity with uniform suction at the plate, under constant pressure gradient, when the initial distribution of velocity is an exponential form.

### 5.5 solution of the problem

The equation of motion for this problem is given by (5.4.1). The initial and boundary conditions are given by eqs. (5.2.2) to (5.2.4).

After assuming the pressure gradient constant i.e.,

\[
\frac{\partial p}{\partial x} = -f(t) = c, \quad \text{where} \ c \ \text{is constant, the solution to the}
\]
equation (5.4.1) subject to the boundary conditions and
initial condition (5.2.2) to (5.2.4) of the present problem,
is obtained using Laplace transform techniques and is given by,

\[ u' = \exp(-\frac{v^2}{2\gamma}) \int_0^t g(t-t') \frac{v^2}{4\gamma t'} \exp \left\{ \left(-\frac{v^2}{4\gamma} t - \frac{v^2}{2\gamma} \right) \right\} dt' - \frac{a}{2} \exp \left\{ \left(-\frac{v^2}{2\gamma} \right) + b^2 t - vbt - mt \right\} \left\{ \exp \left\{ y(b-\frac{v^2}{2\gamma}) \right\} \right\} \times \]

\[ \times \text{erfc} \left[ \frac{v}{4\gamma t} + \left(b-\frac{v^2}{2\gamma} \right) \sqrt{\gamma t} \right] \times \exp \left\{ \left(-\frac{v^2}{2\gamma} \right) \right\} \times \]

\[ \times \text{erfc} \left[ \frac{v}{4\gamma t} - \left(b-\frac{v^2}{2\gamma} \right) \sqrt{\gamma t} \right] \right\} + \]

\[ + \frac{C}{2\pi} \exp(-mt) x \left\{ \exp(-\frac{V}{2\gamma}) \text{erfc} \left[ \frac{v}{4\gamma t} - \frac{V}{4\gamma} \right] \right\} + \text{erfc} \left[ \right. \]

\[ \frac{v}{4\gamma t} + \frac{V}{4\gamma} \right\} - \frac{C}{2\pi} \exp(-\frac{V}{2\gamma}) \left\{ \exp \left\{ \frac{v}{\sqrt{\gamma}} \right\} \text{erfc} \left[ \frac{v}{\sqrt{\gamma}} - \sqrt{\gamma t} \right] \right\} + \]

\[ + \exp \left\{ \frac{v}{\sqrt{\gamma}} \right\} \times \text{erfc} \left[ \frac{v}{\sqrt{\gamma}} + \sqrt{\gamma t} \right] \right\} + \frac{C}{2\pi} (1 - \exp(-\gamma t)) + \]

\[ + a \exp \left\{ -by + b^2 t + vbt - mt \right\} ; \]

\[ (y > 0) \]

\[ u' = g(t) ; (y = 0) \]

where

\[ l = \left( \frac{v^2}{4\gamma} + m \right) \]
The steady state solution is obtained by taking the limit of eq. (5.5.1)
as \( t \to \infty \):

\[
u' = \exp\left(-\frac{\nu y}{2}\right) \int_0^\infty g(t-t') \frac{v}{(4\pi t')^{\frac{3}{2}}} \exp\left\{-\frac{\nu^2}{4\pi t'} - \frac{v^2}{4y} - \nu t'\right\} dt' =
\]

\[
- \frac{c}{\sqrt{n}} \exp\left(-\frac{\nu y}{2}\right) \exp\left\{-\frac{v}{\sqrt{v^2}} \sqrt{\frac{1}{l}} + \frac{c}{\sqrt{n}}\right\}
\]

(5.5.2)

5.6 Discussion

Now we find that solution (5.5.2) is valid for both \( y > 0 \) and \( y = 0 \). However, this solution is derived from solution (5.5.1) which is only valid for \( y > 0 \). This is due to discontinuity in the flow at \( y = 0 \) since the start of motion.

From (5.5.1) we note that the velocity field depends on the initial distribution of velocity, motion of the plate and the pressure gradient, whereas the steady state solution does not depend on the initial distribution of velocity but on plate motion and the pressure's gradient. To study the effect of suction and magnetic field on the velocity profiles we plot equation (5.5.1) by giving suitable value to \( g(t) \). We consider the plate to be uniformly accelerated i.e., \( g(t) = a_0 t \). By taking the values of the constants, \( a_1, b_1, c \) and \( a_0 \) each as unity and also taking \( \frac{v}{l} = 1 \) (as for water) the solution for velocity profiles, eq. (5.5.1) and (5.5.2) becomes:
\[ u' = \frac{\eta}{2} \left[ t + \frac{\eta}{2\sqrt{t}} - \frac{1}{4} \right] \exp \left( -\frac{\eta}{2} \right) \times \]

\[ \times \left\{ \exp \left\{ -\eta \sqrt{1} \right\} \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} - \sqrt{1t} \right] + \exp \left\{ \eta \right\} \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} + \sqrt{1t} \right] \right\} - \]

\[ - \frac{1}{2} \exp \left\{ -\frac{\eta}{2} + t - V_1 t - \eta t \right\} \left\{ \exp (\eta k) \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} + k\sqrt{t} \right] + \right\] \]

\[ + \exp (-\eta k) \times \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} - k\sqrt{t} \right] \left[ \frac{1}{\eta} (1 - e^{-\eta t}) + \frac{1}{2\eta} \exp (-\eta t) \times \right\] \]

\[ \times \left\{ \exp (-\eta) \times \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} - \frac{V_1 \sqrt{t}}{2} \right] + \text{erfc} \left[ \frac{\eta}{2\sqrt{t}} + \frac{V_1 \sqrt{t}}{2} \right] \right\} + \]

\[ + \exp (-\eta + t - V_1 t - \eta t) ; \quad (\eta > 0) \] \quad (5.6.1)

\[ u' = t \quad (\eta = 0) \] \quad (5.6.2)

where

\[ \eta = \frac{V}{\sqrt{t}} , \quad V_1 = \frac{V}{\sqrt{t}} , \quad b_1 = V\sqrt{t} \cdot b = 1 \]

and

\[ l = \left( \frac{V_1^2}{4} + h \right) , \quad k = (1 - \frac{V_1}{2}) \]

The velocity profiles are shown in Fig. (5.1)(a) and (b), for time \( t = 0.5 \) and \( t = 1 \) respectively. We find from the figure that \( u' \) is just \( t \) as given by equation (5.6.2) for \( \eta = 0 \). For increasing value of \( \eta \), the velocity decreases and for large value of \( \eta \), \( u' \) attains a steady value determined by the magnetic field parameter \( h \). For higher value of \( t \), the
steady value is attained quickly compared with lower value of $t$. It has already been shown that for large values of time the solution attains a steady state value. With increasing value of suction, the value of $u'$ decreases before it attains the steady state value. Thus the tendency of separation increases with increase in suction. The effect of magnetic field is more prominent, the increase in magnetic field decreases the velocity field and the decrease in the velocity at a point is more for higher magnetic field for the same difference in the value of magnetic field strength. This is also true for higher value of time $t$.

SECTION D

In the present section the extension of the problem considered in section C, is studied. The problem considered here is the study of incompressible, laminar viscous flow past an infinite flat insulated porous plate moving parallel to itself with an arbitrary time dependent velocity with uniform suction at the plate in the presence of transverse magnetic field under the influence of time-varying pressure gradient and the initial distribution of velocity is an exponential form and the magnetic lines of force are taken fixed relative to the plate. The solution of the problem has been obtained when the pressure gradient is (i) varying linearly with time (ii) varying exponentially with time (iii) varying harmonically.
with time (iv) varying impulsively with time.

5.7 Formulation of the problem and solution

Consider unsteady laminar viscous fluid flow past an infinite insulated porous plate with uniform suction \((v > 0)\) moving parallel to itself with an arbitrary time-dependent velocity in the presence of transverse magnetic field, under the influence of a time-varying pressure gradient, when the magnetic lines of force are fixed relative to the plate, the initial distribution of velocity is taken as an exponential form. The \(x\)-axis is taken along and \(y\)-axis normal to the plate, and the magnetic field is acting along \(y\)-axis.

Since the magnetic field is moving, the relative motion must be accounted for. Thus the governing equation of motion of the problem is the following.

\[
\frac{\partial u'}{\partial t} - v \frac{\partial u'}{\partial y} + (u' - g(t)) \nu = \frac{1}{\rho} \frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u'}{\partial y^2} \tag{5.7.1}
\]

The initial and the boundary conditions are as given by eqs. (5.2.3) to (5.2.4).

Now, if we assume \(-\frac{\partial u}{\partial x} = f(t)\), (5.7.1) reduces to

\[
\frac{\partial u'}{\partial t} - v \frac{\partial u'}{\partial y} + (u' - g(t)) \nu = \frac{1}{\rho} f(t) + \gamma \frac{\partial^2 u'}{\partial y^2} \tag{5.7.2}
\]
The solution of (5.7.2) with conditions (5.2.2) to (5.2.4) is obtained with Laplace transform techniques, and is given by

\[ u' = c - e^{-\frac{t}{2\gamma}} \int_0^t f(t-t')e^{-\lambda t'} \left\{ \exp\left(-\frac{\lambda^2}{4\gamma^2}t' \right) \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} - \frac{\lambda \sqrt{\gamma^2}}{\sqrt{4\gamma^2}} \right] + \right. \]

\[ + \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} + \frac{\lambda \sqrt{\gamma^2}}{\sqrt{4\gamma^2}} \right] \right\} dt' + e + \frac{1}{\gamma} \int_0^t f(t-t')e^{-\lambda t'} dt' g(y > 0) \]

\[ \left. \right\} \]...

(5.7.3)

\[ u' = g(t) \quad (y = 0) \]

where

\[ c = \exp\left(-\frac{\lambda^2}{4\gamma^2}\right) \int_0^t g(t-t') \frac{\lambda^2}{\sqrt{4\pi \gamma^2}} \exp \left\{ -\frac{\lambda^2}{4\gamma^2}t' - \frac{\lambda^2 t'}{4\gamma} - \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} \right] \right\} dt' \]

\[ \left. \right\} \]...

(5.7.4)

\[ d = \frac{\lambda}{2} \int_0^t g(t-t')e^{-\lambda t'} \left\{ \exp\left(-\frac{\lambda^2}{4\gamma^2}t' \right) \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} - \frac{\lambda \sqrt{\gamma^2}}{\sqrt{4\gamma^2}} \right] + \right. \]

\[ + \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} + \frac{\lambda \sqrt{\gamma^2}}{\sqrt{4\gamma^2}} \right] \right\} dt' \]

\[ \left. \right\} \]...

(5.7.5)

\[ e = -\frac{3}{2} \exp \left\{ -\frac{\lambda^2}{2\gamma} + \sqrt{b^2 - y^2} \right\} \exp \left\{ y(b-\frac{\lambda}{2\gamma}) \right\} \times \]

\[ \times \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} + \frac{\lambda \sqrt{b^2 - y^2}}{\sqrt{4\gamma^2}} \right] + \exp \left\{ -y(b-\frac{\lambda}{2\gamma}) \right\} \text{erfc} \left[ \frac{\lambda}{\sqrt{4\gamma^2}} \right] \]

\[ - (b-\frac{\lambda}{2\gamma}) \sqrt{b^2 - y^2} \right\} \]

\[ \left. \right\} \]...

(5.7.6)

\[ f = \frac{\lambda}{2} \int_0^t g(t-t')e^{-\lambda t'} dt' \]

(5.7.7)
and
\[ g = a \exp \left( -by - vbt - \frac{1}{2}b^2 t - kt \right) \tag{5.7.8} \]

Solution for large values of time from (5.7.3) is obtained by taking its limit as \( t \to \infty \). Thus,
\[ u' = \exp \left\{ -\frac{y^2}{27} \int_0^\infty g(t-t') \frac{d}{\sqrt{4\pi t'}} \exp \left\{ -\frac{y^2 t'}{47 t'} - \frac{y^2 t'}{4 t} - k t' \right\} dt' \right\} \]
\[ = \left[ \frac{1}{2} \int_0^\infty \left[ N g(t-t') + \frac{f(t-t')}{y} \right] e^{vt'} \exp \left\{ -\frac{y^2 t'}{47} \right\} \exp \left\{ -\sqrt{\frac{y^2 t'}{47}} \right\} \right] \]
\[ \times \text{erfc} \left[ \frac{y}{\sqrt{47 t'}} \right] + \text{erfc} \left[ \frac{y}{\sqrt{47 t'}} + \frac{\sqrt{47 t'}}{\sqrt{47}} \right] dt' + \left[ \int g(t-t') \right] + \frac{f(t-t')}{y} \right] \] \tag{5.7.9}

Here also we find the discontinuity in the flow at \( y = 0 \) at the time of start of motion, which reduces gradually with time and vanishes with large values of time. Eq. (5.7.9), thus represents the solution at steady state.

5.8 Solution for velocity with different variations of pressure gradients

Case 1: Pressure gradient varying linearly with time
we assume that
\[ \frac{\partial p}{\partial x} = f(t) = a_0 + a_1 t \tag{5.8.1} \]
Substituting this in equation (5.7.2) we solve it with the conditions (5.2.2) to (5.2.4). By doing so we obtain the solution when the pressure gradient varies linearly with time given by

\[
   u' = c + d + e + \left\{ \left( \frac{a_1}{f^{\mu}} - \frac{a_0}{f^{\mu}} \right) \hat{u} \exp \left( -\frac{\nu x}{2\gamma} \right) - \frac{a_1 t}{2 f^{\mu}} \right\} \exp \left( \frac{y x}{\gamma} \right) \text{erfc} \left( \frac{\nu x}{4\gamma t} + \sqrt{k} t \right) + 
\]

\[
   + \left\{ \left( \frac{a_1}{f^{\mu^2}} - \frac{a_0}{f^{\mu}} \right) x \hat{u} \exp \left( -\frac{\nu x}{2\gamma} \right) + \frac{a_0}{f^{(4\gamma/k)}} - \frac{a_1 t}{2 f^{\mu}} \right\} x \exp \left( -\frac{y x}{\gamma} \right) \text{erfc} \left( \frac{\nu x}{\sqrt{4\gamma t}} \right) + \frac{a_0}{f^{\mu}} - \frac{a_1 t}{2 f^{\mu}} \right\} \hat{u} e^{-nt} \gamma 
\]

\[
   \times \left\{ \exp(-\frac{\nu x}{\gamma}) \text{erfc} \left( \frac{\nu x}{\sqrt{4\gamma t}} \right) + \text{erfc} \left( \frac{\nu x}{\sqrt{4\gamma t}} + \frac{y x}{\gamma} \right) \right\} + 
\]

\[
   + f + \left( \frac{a_0}{f^{\mu}} - \frac{a_1}{f^{\mu^2}} \right) (1-e^{-nt}) + \frac{a_1 t}{2 f^{\mu}} + g 
\]

(5.3.2)

where \( c, d, e, f \) and \( g \) are given by (5.7.4) to (5.7.8) respectively and \( k = \left( \frac{\nu^2}{4\gamma} \right) + h \).

Case II: Pressure gradient varying exponentially with time:

we assume that.
Thus the solution in this case is given by

\[ u^* = c - d - c = \frac{1}{2} \sum_{l=0}^{\infty} \frac{a_l}{(n-1)^2} \exp \left( - \frac{v^2}{2l^2} \right) + \]

\[ + \exp \left( \frac{v^2}{2l^2} \right) \frac{\sqrt{\pi}}{\sqrt{v}} \left[ \text{erfc} \left( \frac{v}{\sqrt{4l^2}} \right) \right] + \frac{1}{2} \sum_{l=0}^{\infty} \frac{a_l}{(n-1)^2} e^{-lt} \times \]

\[ \exp \left( - \frac{v^2}{2l^2} \right) \frac{\sqrt{\pi}}{\sqrt{v}} \left[ \text{erfc} \left( \frac{v}{\sqrt{4l^2}} \right) + \text{erfc} \left( \frac{v}{\sqrt{4l^2}} + \sqrt{v}l \right) \right] + \]

\[ + \sum_{l=0}^{\infty} \frac{a_l}{f(n-1)} \left[ e^{-lt} - e^{-lt} \right] \times f + g \quad (5.8.4) \]

where

\[ q = \left( \frac{v^2}{2l^2} \right) - 1 \]

**Case III : Pressure gradient varying harmonically**

with time,

we assume that

\[ \frac{d\theta}{dx} = f(t) = B_1 \sin wt. \quad (5.8.5) \]

Then the exact solution for the problem when the pressure gradient is varying harmonically with time, is given by
\[ u' = c-d-e - \frac{B_1}{w f (\xi^2 + \omega^2)} \exp \left\{ -\frac{vy}{2 \gamma} - \frac{vy}{2 \gamma \sqrt{\gamma}} \right\} \]

\[ x \left\{ \sin (\omega t - \frac{vy}{\sqrt{\gamma}}) - \omega \cos (\omega t - \frac{vy}{\sqrt{\gamma}}) \right\} + \\
+ e^{-\omega t} \left[ \Omega \sin \omega y + \omega \cos \omega y \right] \}

\[ + \frac{B_1}{w f (\xi^2 + \omega^2)} \]

\[ x \left[ - \Omega \sin \omega t - \omega \cos \omega t + e^{-\omega t} \right] + f + g \] (5.6.6)

Where

\[ r = \frac{1}{\sqrt{2}} \left\{ \left[ \left( \frac{v^2}{4\gamma} + \omega^2 \right) + \omega^2 \right] - \left( \frac{v^2}{4\gamma} + \omega^2 \right) \right\}^{1/2} \]

Case IV : Pressure gradient varies impulsively with time:

Here we assume that

\[ \frac{\partial p}{\partial x} = f(t) = \Pi \delta(t) \] (5.6.7)

Where \( \delta(t) \) is Dirac delta function.

The solution in this case of pressure gradient varying impulsively with time is obtained as follows:

\[ u' = c-d-e - \frac{H}{2f} e^{-\omega t} \left\{ \exp \left( -\frac{vy}{\gamma} \right) \erfc \left[ \frac{y}{4\gamma t} \right] - \frac{y \sqrt{t}}{\sqrt{\gamma}} \right\} + \\
+ \erfc \left[ \frac{y}{4\gamma t} + \frac{y \sqrt{t}}{\sqrt{\gamma}} \right] + \frac{H}{S} e^{-\omega t} + f + g \] (5.6.8)
5.9 Conclusions

1. The solutions obtained are the generalisations of several earlier works such as of Stokes problem 1 and 2.

2. There is a discontinuity in the flow at \( y = 0 \) since the start of motion, and this discontinuity disappears for large time.

3. The solutions for velocity depend on the initial distribution of velocity, the motion of the plate and in the presence of pressure gradient, it also depends on the pressure gradient for the case of small values of time, whereas it does not depend on the initial distribution of velocity for large values of time, where it attains steady state.

4. For large distance from the plate the velocity reaches steady state value determined by magnetic field parameter \( M \). The impact of the magnetic field on the flow is to retard the flow.

5. With increased suction there is decrease in the velocity of the fluid.
Figure 5: Velocity Profiles

(a) $t = 0.5$

(b) $t = 1$