CHAPTER 1

INTRODUCTION
1.1 BASIC EQUATIONS:

The fluid mechanics theory is based on the continuum hypothesis concept that the fluid mass is distributed throughout the space such that the field theories become applicable mathematical tools in the description of fluid motions. The fundamental laws governing the conservation of mass, momentum and energy are the basic principles of fluid mechanics.

The governing equations for conservation of mass, momentum and energy [11] for the flow region, without sources and sinks etc are,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{V} = 0 , \quad (1.1.1) \]

\[ \rho \frac{\partial \mathbf{V}}{\partial t} = \rho \mathbf{F} + \mathbf{T} , \quad (1.1.2) \]

and

\[ \rho \frac{\partial \mathbf{E}}{\partial t} = \Phi - \nabla \cdot \mathbf{q} , \quad (1.1.3) \]

respectively, where \( \rho \) is the density of the fluid, \( \mathbf{V} \) the components of velocity vector, \( \mathbf{F} \) the components of body force per unit mass, \( \mathbf{T} \) the components of stress vector, \( \frac{\partial}{\partial t} \) is the material time derivative and comma denotes the covariant derivative, \( e \) the internal energy, \( \mathbf{q} \) is the heat flux vector given by

\[ \mathbf{q} = -k \frac{\partial T}{\partial n} , \quad (1.1.4) \]

where \( k \) is the thermal conductivity of the fluid, \( T \) the temperature and \( n \) the normal to the area element across which the heat flux is considered. \( \Phi \) denotes the heat flux in unit volume due to the heating during fluid motion which depends upon the constitutive properties of the fluid material and internal heat sources if present in the flow field.
The description of a fluid motion requires the solution of the basic equations (1.1.1), (1.1.2) and (1.1.3) subject to the initial and boundary conditions imposed by the Physical configurations of the flow problem.

For incompressible fluids, \( \rho \) is constant, and the continuity equation (1.1.1) becomes

\[ \nabla \cdot \mathbf{V} = 0. \] \hspace{1cm} (1.1.5)

In the present study we confine to the isothermal case.

### 1.2 NEWTONIAN AND NON-NEWTONIAN FLUIDS:

Due to the internal friction between the neighbouring fluid elements in relative motion of fluid flow, the stresses developed are related to the state of deformation of shape of fluid elements which can be expressed in terms of velocity gradients and time-rate of change of velocity gradients. The co-efficients (material parameters of the fluid) appearing in the mathematical relation between the stress and strain rate are either constants or, in general, functions of shear stress and temperature of the fluid, which tend to describe the material properties of the fluid.

Newton proposed a simple constitutive equation [12], which states that the shear stress is directly proportional to the shear rate viz,

\[ \tau = \eta \frac{du}{dy} \] \hspace{1cm} (1.2.1)

where \( \eta \) is the viscosity of the fluid and is a constant. The generalization of Newton's equation is [13]

\[ \sqrt{\tau_{ij}} = -p \delta_{ij} + 2 \eta \frac{d_{ij}}{dy} \] \hspace{1cm} (1.2.2)
Fig. 1
where $p$ is the pressure, $\tilde{T}_{ij}$ the stress tensor, $\delta_{ij}$ the Kronecker delta and $d_{ij}$ the deformation rate tensor given by

$$d_{ij} = \frac{1}{2} (V_{i,j} + V_{j,i}),$$

(1.2.3)

where $V_{i,j}$ is the velocity gradient. If $\mu=0$, the equation (1.2.2) reduces to

$$\tilde{T}_{ij} = -p \, \delta_{ij},$$

(1.2.4)

which corresponds to the ideal fluid theory (inviscid). The fluids whose flow behaviour can be adequately described by the liner constitutive equation (1.2.2) are called Newtonian fluids. But however, many real fluids encountered in industrial processes, (such as polymer processing) and biological fluids etc exhibit some bizzare flow characteristics, which cannot be predicted either by the classical Newton's equation (1.2.1) or by its generalization (1.2.2). The fluids whose rheological behaviour deviate from that predicted by Newton's linear constitutive equation (1.2.2) are called as non-Newtonian fluids. For the illustration of strange phenomena of non-Newtonian fluid flows, we quote a few examples of unusual flows [14].

The climbing of the fluid along the rod, which is called "Weissenberg effect" is shown in fig. 1(a). The surface of the Newtonian fluid is depressed near the rod, whereas in the case of the non-Newtonian fluid it tries to climb the rod. Fig. 1(b) shows how a non-Newtonian fluid swells when it emerges from a tube or slit, called "Merrington effect". Fig 1(c) shows the tendency of the non-Newtonian fluid to recoil when the applied shearing force is stopped - called the "Memory effect", whereas the Newtonian fluid comes to rest. The inadequacy of equation (1.2.2) to explain these flow behaviour of certain real
fluids leads to the necessity of considering non-linear relationships between the stress and the strain-rate tensors. Several non-linear constitutive equations have been proposed for mathematical modelling of non-Newtonian fluids in order to explain the anomalous behaviour of real fluids. However, no single model accounts for all the characteristic properties of non-Newtonian fluids. Hence, to understand the different properties of non-Newtonian fluids, on the basis of their rheological behaviour the real fluids are customarily classified into four categories [15-18]. The four groups are as follows

I. Time independent or purely viscous fluids:

This class of fluids is characterized by the fact that, the shear rate depends only on the shear stress and is a single valued function of it.

(a) Newtonian fluids: The viscosity is independent of shear rate.
(b) Non-Newtonian fluids: The viscosity is a single valued function of shear rate. The broad classification of the fluids appearing in the above categories are shown in Fig. 1. (d,e).

II. Time dependent fluids: The shear rate and viscosity depend on both the shear stress and the duration of the stress.

III. Viscoelastic fluids: The shear rate and the viscosity depend on the shear stress and the extent of deformation of the fluid.

IV. Complex rheological fluids: These fluids exhibit a combination of properties from at least any two of the above.

Within Time-independent fluids, there are three main types:
Shear-thinning (pseudoplastic) and shear-thickening (dilatant) fluids depending on whether the apparent viscosity of the fluids is a decreasing or increasing function of the shear rate respectively. Examples of shear-thinning fluids include dilute polymer solutions, rubber, paints, detergent slurries and biological fluids. Examples of shear-thickening materials include: sugar solutions, gum arabic in water and wet beach sand. Sometimes the fluids show yield stresses and such fluids are referred to as "Bingham Plastic" fluids. Many concentrated solutions such as detergent slurries, oil well drilling muds, tooth paste, soap and fermentation broths show this property.

Time-dependent non-Newtonian fluids are usually classified into two groups, namely 'thixotropic' fluids and 'rheopetic' fluids, depending upon whether the shear stress decreases or increases with time at a given shear rate and constant temperature. The typical examples of thixotropic fluids can be found among paints and printing inks. The property of rheopetic fluids is observed primarily in suspension system such as gypsum in water.

Viscoelastic fluids exhibit both creep and stress relaxation phenomena. The rheological behaviour of these fluids at any instant will be a function of the recent history of the material and cannot be described by relationship between shear stress and shear rate alone but also the time derivative of these quantities. The attempts for inclusion of elastic response ('fluid memory') in the rheological models have resulted in several theories of viscoelasticity developed by Maxwell, Oldroyd, Walters, Rivlin and Ericksen, Noll, Coleman and Noll, to mention a few. The examples of viscoelastic fluids include polymer
and polymer melts such as nylon and many polymer solutions and napalm.

The fluids with microstructures such as emulsions, suspensions, biofluids, clay and slurries exhibit complex rheological properties. These fluids sustain couple stresses and body couples apart from the Cauchy linear stresses giving rise to size-dependent effects. The attempts to describe the rheology of fluids with microstructures in the realm of continuum mechanics has led to the development of several micro-continuum theories of fluids such as polar fluids, micropolar fluids, dipolar fluids and couple stress fluids. The names of Eringen, Cosserat, Truesdell and Toupin; and Stokes can be mentioned here for their work in connection with these theories. An excellent review of these micro-continuum theories of fluids can be found in Ariman et al [19-20].

The literature of non-Newtonian fluids is presented in the text books by Skelland [21], [12, 13], Leigh [22], Coleman et al [23], Moore [24], Harris [25] and Schowalter [26]. Because of the occurrence of many non-Newtonian fluids in nature and industrial processes, it is therefore evident that the study of non-Newtonian flows enables substantial economic development in a wide diversity of fluid engineering techniques. In the present study we confine to three non-Newtonian fluids viz., the fluids described by Second-order viscoelastic fluid model proposed by Coleman and Noll [27], the fluids described by the couple stress fluid model proposed by Stokes [28] and the micropolar fluids, proposed by Eringen [29]. These fluid models fall in the categories III and IV, respectively, in the above classification.
1.3 SECOND-ORDER FLUIDS:

Noll [30] proposed 'Simple fluid' model which accounts for fluid 'memory' by assuming that at time $t$ the stress state of the fluid depends on its kinematic state at all times up to and including $t$. The simple fluid exhibits the phenomena of gradual stress relaxation as well as shear dependent viscosity and normal stress effects. With the assumptions that Noll's simple fluid has a short-lived memory and is moving slowly, Coleman and Noll [27] evolved a second approximation scheme to a simple fluid in the sense of retardation giving the same stress equations as those obtained for Rivlin-Ericksen fluids. The constitutive equation of an incompressible second-order fluid proposed by Coleman and Noll [27] based on the postulate of gradually fading memory, is given by

$$
\tilde{T}_{ij} = -p \delta_{ij} + \phi_0 A(1)_{ij} + \phi_1 A(2)_{ij} + \phi_2 A(1)_{i1}^k A(1)_{kj}.
$$

(1.3.1)

where $\tilde{T}_{ij}$ is the stress tensor, $\phi_0$, $\phi_1$, $\phi_2$ are the co-efficients of viscosity, viscoelasticity and cross viscosity respectively; usually the material constants $\phi_1$ and $\phi_2$ are referred to as the normal stress moduli with $\phi_1 < 0$. Further, Rivlin-Ericksen tensors $A(1)_{ij}$ and $A(2)_{ij}$ are defined as

$$
A(1)_{ij} = V_{i,j} + V_{j,i},
$$

(1.3.2)

$$
A(2)_{ij} = a_{1,i} + a_{j,i} + 2V_{m,i} V_{m,j}.
$$

(1.3.3)
where $V_i$ and $a_i$ are the velocity and acceleration components respectively. The acceleration $a_i$ is given by

$$a_i = \frac{\partial V_i}{\partial t} + V_j V_{i,j}. \quad (1.3.4)$$

While $A_{ij}$ is the rate of strain tensor, $A_{(1)ij}$ represents the memory effect of the fluid.

The governing equations in cartesian co-ordinates for two-dimensional incompressible flow of second-order fluid, represented by Coleman and Noll's constitutive equation [27], neglecting body forces, are

**Continuity equation:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (1.3.5)$$

**Momentum equation:**

$$\left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

$$+ 3(\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2})$$

$$+ \frac{\partial^2}{\partial x \partial y} \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \right)$$

$$+ \frac{\partial^2}{\partial y^2} \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} \right)$$
\[ + 2 \frac{\partial}{\partial y} \left( \frac{2 u}{\partial x} \frac{2 u}{\partial y} + \frac{2 v}{\partial x} \frac{2 v}{\partial y} + \frac{2 u}{\partial x^2} \frac{2 u}{\partial y} \right) \]

\[ + \gamma \left[ 8 \frac{2 u}{\partial x} \frac{2 u}{\partial y} + 2 \left( \frac{2 u}{\partial y} + \frac{2 v}{\partial x} \right) \left( \frac{2 u}{\partial x} \frac{2 v}{\partial y} + \frac{2 v}{\partial x} \frac{2 v}{\partial y} \right) \right] \quad (1.3.6) \]

and

\[ \frac{\partial v}{\partial t} \frac{\partial v}{\partial x} + v \frac{\partial^2 v}{\partial y^2} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + v \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta \left[ 2 \frac{\partial^2 (u \frac{2 v}{\partial x} + v \frac{2 v}{\partial y})}{\partial y^2} \right] \]

\[ + \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right) + \frac{\partial^2}{\partial x \partial y} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial^2}{\partial x \partial y} \left( u \frac{\partial v}{\partial y} + v \frac{\partial v}{\partial x} \right) \]

\[ + \frac{\partial^3 v}{\partial y^3} \right) + \gamma \left[ 8 \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} + 2 \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right) \right] \quad (1.3.7) \]

If \( h \) and \( L \) are the length scales across the film and in the plane of the film, respectively, then for conventional lubricant films, we have \( h \ll L \).

The appropriate dimensionless scheme for the analysis of fluid film lubrication will be

\[ \epsilon = \frac{h}{L}, \quad \bar{u} = \frac{u}{V}, \quad \bar{v} = \frac{v}{V}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \]

\[ \bar{p} = \frac{\epsilon^2 L}{\mu V} p \quad \text{and} \quad \bar{w} = \frac{\epsilon^2 w}{\mu V}. \quad (1.3.8) \]
Which yields the simplified forms of governing equations by considering only the dominant terms i.e. the terms of $O(e^{-2})$, where $L$ is the characteristic length of the bearing and $V$ is the characteristic velocity.

The governing equations (1.3.5) to (1.3.7), for steady incompressible homogeneous fluid, after employing the transformations (1.3.8), are

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.3.9)$$

Momentum equation:

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = v \left( \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right)$$

$$+ v \frac{\partial^3 u}{\partial y^3} + 2 \gamma \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y}$$

$$+ \frac{1}{\rho} \frac{\partial p}{\partial y} = 2(2\beta + \gamma) \frac{\partial^2 u}{\partial y \partial x} \frac{\partial^2 u}{\partial y \partial x} \quad (1.3.10)$$

$$+ 2\gamma \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \quad (1.3.11)$$

The governing equations in cylindrical polar co-ordinates $(x, \theta, z)$ for two dimensional incompressible flow of second-order fluid represented by Coleman and Noll's constitutive equation (27), neglecting body forces, are

Continuity equation:

$$\frac{\partial (xu)}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (1.3.12)$$
Momentum equation:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{x^2} \right)
\]

\[
+ \beta \left[ 2 \frac{\partial^2 u}{\partial x^2} + \frac{2}{x} \frac{\partial u}{\partial x} - \frac{2A}{x^2} + \frac{2}{x} \frac{\partial^2 u}{\partial x \partial z} - \frac{2w}{x^3} \right] + \frac{\partial \mathbf{b}}{\partial x} \cdot \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} \right)
\]

\[
+ \frac{2}{x} \left( \frac{\partial u^2}{\partial x} + \frac{\partial w^2}{\partial x} \right) + \gamma \left( \frac{\partial^2 u}{\partial x^2} - \frac{2u}{x} \frac{\partial^2 w}{\partial x^2} - \frac{2w}{x} \frac{\partial^2 u}{\partial x \partial z} \right)
\]

\[
+ \frac{1}{x} (\frac{\partial u}{\partial z})^2 + \frac{1}{x} (\frac{\partial w}{\partial z})^2 + 4 (\frac{\partial u}{\partial x})^2 + 2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} - \frac{4u^2}{x^3}
\]

\[
+ 2 \left( \frac{\partial u}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x \partial z} \right) \right) \right) \right). \tag{1.3.13}
\]

and

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
+ \beta \left[ 2 \frac{\partial^2 u}{\partial x^2} + \frac{2}{x} \frac{\partial u}{\partial x} - \frac{2A}{x^2} + \frac{2}{x} \frac{\partial^2 u}{\partial x \partial z} - \frac{2w}{x^3} \right] + \frac{\partial \mathbf{b}}{\partial x} \cdot \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z} \right)
\]

\[
+ \frac{\partial w}{\partial x} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} \right) \right) \right) \right) \right) \right). \tag{1.3.14}
\]
\[ \begin{align*}
+ \gamma & \left[ \frac{2}{z} \frac{\partial u}{\partial z} - \frac{2}{x} \frac{\partial w}{\partial x} \frac{\partial^2 y}{\partial x^2} - \frac{2}{x} \frac{\partial w}{\partial x} \frac{\partial^2 z}{\partial x^2} - \frac{2}{x} \frac{\partial^2 u}{\partial x^2} - \frac{2}{x} \frac{\partial u}{\partial x} \frac{\partial^2 z}{\partial x^2} - \frac{2}{x} \frac{\partial u}{\partial x} \frac{\partial^2 z}{\partial x^2} - \frac{2}{x} \frac{\partial u}{\partial x} \frac{\partial^2 z}{\partial x^2} \right] \\
+ 2 \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \left( \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z} \right),
\end{align*} \tag{1.3.14} \]

where \( u \) and \( w \) are the velocity components in radial and axial directions respectively,

\[ A = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \]

\[ B = \frac{\partial u}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \]

and

\[ \nu = \frac{\phi u}{\rho} = \frac{u}{\rho}, \quad \beta = \frac{\phi_1}{\rho}, \quad \gamma = \frac{\phi_2}{\rho} \]

1.4 COUPLE STRESS FLUIDS:

By assuming the existence of couple stress in addition to the classical Cauchy stress, several theories [31-36] of fluid micro-continua have developed which are basically intended to explain the flow behaviour of such rheologically complex fluids as liquid crystals, polymeric suspensions, physiological fluids and fluids with large molecular structures. In 1966, Stokes [28] proposed one such couple stress theory of fluids which represents the simplest generalization of the classical theory and allows for polar effects such as the presence of couple stresses and body couples. Eringen's micropolar fluid theory [29] and the asymmetric hydrodynamics of Aero et al. [32] define the rotational field in terms of a kinematically independent rotation vector, called as micro-rotation vector, for setting up the stress constitutive equations. On the contrary,
Stokes' theory of couple stress fluid defines the rotational field in terms of the velocity field. Second-order gradient of the velocity vector, instead of a kinematically independent rotation vector, is introduced in the constitutive relationship between stress and strain rate, and hence the theory yields only one vector equation to describe the flow. Stokes' theory has been shown to predict size-dependent effects, observed in the flow of fluids with microstructures, which do not exist in the nonpolar case (couple stress absent). The constitutive equations proposed by Stokes\cite{28} are,

\[ \tilde{T}_{(1j)} = -p \delta_{jj} - \lambda c_k \delta_{jj} + 2 \mu d_{jj}, \quad (1.4.1) \]

\[ \tilde{T}_{[1j]} = -2 \eta W_{jj} + \frac{D}{2} \varepsilon_{jj} \varepsilon_s, \quad (1.4.2) \]

and

\[ \tilde{M}_{jj} = 4 \eta \omega_{jj} + 4 \eta \varepsilon_{jj} \omega_{jj}, \quad (1.4.3) \]

where

\[ d_{jj} = \frac{1}{2} \left( \nu_{1j} + \nu_{jj} \right), \quad (1.4.4) \]

\[ W_{jj} = -\frac{1}{2} \left( \nu_{1j} - \nu_{jj} \right), \quad (1.4.5) \]

\[ \omega_{1j} = \frac{1}{2} \varepsilon_{jj} \nu_{jj}, \quad (1.4.6) \]

\( \tilde{T}_{(1j)} \) and \( \tilde{T}_{[1j]} \) are the symmetric and anti-symmetric parts of stress tensor \( \tilde{T}_{jj} \), \( M_{jj} \) the couple stress tensor, \( d_{jj} \) the deformation rate tensor, \( \omega_{1j} \) the anti-symmetric part of the deformation rate tensor, and \( W_{jj} \) the symmetric part of the deformation rate tensor.
\( \mathbf{\omega}_{ij} \) the vorticity tensor, \( \epsilon_{ijs} \) the alternating unit tensor, \( V_i \) the components of velocity vector, \( G \) the body couple, \( \delta_{ij} \) the Kronekar delta, \( \Omega_1 \) the vorticity vector, \( \rho \) the density, \( P \) the pressure, \( \lambda_c \) and \( \mu \) are the material constants of the dimension of viscosity and \( \eta \) and \( \eta' \) are the material constants having the dimension of momentum. The ratio \( \eta/\mu \) has the dimension of length squared.

The governing equations in cartesian co-ordinates for two dimensional steady, incompressible flow of Stokes' couple stress fluid \([28]\), in the absence of body couples and body forces are,

Continuity equation:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.4.7)
\]

Momentum equations:

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
- K_c \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u, \quad (1.4.8)
\]

and

\[
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
- K_c \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v, \quad (1.4.9)
\]
where \( u, v \) are the velocity components in the \( x \) and \( y \) directions respectively,

\[ v = \frac{u}{\rho}, \quad \text{and} \quad K_c = \frac{\Pi}{\rho}. \]

1.5 MICROPOLAR FLUIDS:

Eringen [37, 38] put forth a general theory which is counted as the earliest formulation of fluid microcontinua in which he considered the mechanics of fluids with deformable microelements. He called these fluids as "Simple microfluids". In the theory of simple microfluids, Eringen has developed a Physical model in which each material volume element contains microvolume elements which can translate, rotate and deform independently of the motion of the macrovolume element. However, the theory is too complicated and the underlying mathematical problem is not easily amenable to solution of the non-trivial flow problems. This forced Eringen and other researchers to consider subclasses of these fluids to be obtained by simplifying and restricting the general theories. Eringen [29] himself obtained a subclass of microfluids, called the 'micropolar' fluids, by imposing the condition of microisotropy and skew-symmetric property of the gyration tensor on the general theory of simple microfluids. In the theory of micropolar fluids, Eringen introduced two basic and kinematically independent vector fields.

1. The 'velocity vector' field representing the translational velocity of the fluid particles;

2. The 'microrotation vector' field representing the angular (or spin) velocity of the particles.
Thus, in micropolar fluids, rigid particles contained in a small volume element can rotate about the centroid of the volume element in an average sense described by the microrotation vector in addition to the usual rigid body motion of the entire volume element.

The constitutive equations for the micropolar fluids are

\[
\tau_{ij} = \left( -p + \lambda \delta_{ij} \right) \delta_{ij} + \left( 2\mu + k \right) d_{ij} + k \varepsilon_{ijk} \left( \omega_k - \nu_k \right) \tag{1.5.1}
\]

and

\[
m_{ij} = \alpha \nu_{i,j} \delta_{ij} + \beta \nu_{i,j} + \gamma \nu_{i,j} \tag{1.5.2}
\]

where \(\tau_{ij}\) is the stress tensor, \(d_{ij}\) deformation rate tensor, \(m_{ij}\) couple stress tensor, \(\nu_i\) the components of microrotation vector, \(\omega_i\) the components of vorticity vector, \(\delta_{ij}\) the Kronecker delta, \(p\) the thermodynamic pressure, \(\lambda, \mu, k, \sigma, \beta, \gamma\) are the material constants (coefficients of viscosity) governed by the following thermodynamic restrictions, namely,

\[
3\lambda + 2\mu + k \geq 0, \quad 2\mu + k \geq 0, \quad k \geq 0, \tag{1.5.3}
\]

\[
3\sigma + \beta + \gamma \geq 0, \quad -\gamma \leq \beta \leq \gamma, \quad \gamma \geq 0,
\]

the comma denotes covariant differentiation, \(\varepsilon_{ijk}\) is an alternating tensor.

**BASIC EQUATIONS:**

The field equations of the micropolar fluid dynamics in the vectorial
Continuity equation:
\[ \frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0 \]  \hspace{1cm} (1.5.4)

Balance of linear momentum:
\[ \rho \frac{D\mathbf{v}}{Dt} = \left( \lambda + 2 \mu + k \right) \text{grad} \text{div} \mathbf{v} - (\mu + k) \cdot \text{curl} \text{curl} \mathbf{v} + k \text{curl} \mathbf{v} - \text{grad} p + \rho \mathbf{l} \]  \hspace{1cm} (1.5.5)

Balance of moment of momentum:
\[ \rho j \frac{D\mathbf{w}}{Dt} = (\alpha + \beta + \gamma) \text{grad} \text{div} \mathbf{w} - \gamma \text{curl} \text{curl} \mathbf{w} + k \text{curl} \mathbf{v} - 2k \mathbf{v} + \rho \mathbf{l} \]  \hspace{1cm} (1.5.6)

Where \( \mathbf{v}, \mathbf{w}, \mathbf{f}, \mathbf{l} \) are respectively the velocity, microrotation, body force, and body couple vector, \( \rho, j \) are the density and microinertia respectively. The remaining symbols have their usual meaning.

1.6 POROUS REGION:

Porous metal bearings are often used where plain metal bearings are impractical because of inaccessibility for lubrication. Self lubricating porous bearings have advantages in overcoming the need for oilpipes, pumps etc and simplifies the problems concerned with machine design, also,
it can be easily fitted and maintenance can be minimised.

For a homogeneous and isotropic porous medium the differential form of Darcy's law can be expressed as [39]

\[ \frac{d_1}{dt} = -\frac{k_1}{\mu} \text{grad} p_1, \]  

(1.6.1)

where \( k_1 \) is the permeability of the porous medium and \( \mu \) is the viscosity of the fluid. In equation (1.6.1), \( \frac{d_1}{dt} \) is local filter velocity or seepage velocity and \( p_1 \) is the pressure in the porous region. The equation of continuity in porous medium is

\[ \text{div} \frac{d_1}{dt} = 0. \]  

(1.6.2)

It is well known that the above form of Darcy's law is generally valid for moderate flow rates and hence it serves the basic requirements of our analysis of the theory of the process of filtration.

The two dimensional flow of a viscous fluid in a porous matrix is governed by the equations,

\[ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0. \]  

(1.6.3)

where \( u_1 \) and \( v_1 \) are the velocity components which are given by

\[ u_1 = -\frac{k_1}{\mu} \frac{\partial p_1}{\partial x}, \]  

(1.6.4)

\[ v_1 = -\frac{k_1}{\mu} \frac{\partial p_1}{\partial y}. \]  

(1.6.5)

1.7 SYNOVIAL FLUID AND ARTICULAR CARTILAGE:

Synovial fluid is a clear, yellowish, and tacky substance found in the cavities of freely movable joints and interacting with the cartilage.
to provide lubricating action. Its volume changes from joint to joint. The largest synovial joint, the knee, contains about 1-2 ml of fluid when aspirated. Chemical composition of synovial fluid indicates that it is a dialysate of blood plasma with the addition of hyaluronic acid protein complex and a small cellular component. The most significant property of synovial fluid is its viscosity, and this appears to be due largely to the hyaluronic acid component. Synovial fluid is classified as a non-Newtonian liquid having the property of shear thinning. It also possesses normal stress effects which can be verified from the fact that if one shears synovial fluid between plates, it can exert a force on the plates that tries to push them apart [40, 41]. The review by Lai et al [42] gives the comprehensive report of series of rational constitutive equations linear integral, non-linear integral, linear rate, non-linear rate differential and also some adhoc constitutive equations to describe the existing experimental rheological data.

Articular cartilage is the smooth gristle which lines the articulating surfaces of the bones in synovial joints. Its functions are to absorb the wear which results from joint movement over a period of many years, to minimize joint friction by contributing to a lubricating mechanism, and to transmit some of the highest loads developed in the body. In the mechanical process of the transport of the fluid through cartilage matrix, the interstitial fluid will be transported through the porous permeable cartilage matrix via the application of direct fluid pressure gradient across the tissue. The transport of fluid is directly proportional to the pressure gradient. The permeability parameter can be measured in a classical permeability experiment for porous media. The flow of
fluid within the cartilage is assumed to be governed by Darcy's law [39]. Ultra structural investigation of cartilage [43] shows that cartilage matrix is mainly consisting of three layers viz., superficial tangential zone, middle zone and deep zone depending on permeability of respective regions. Experiments by Maroudas et al [44] confirm the variation of permeability $k$, with depth. It varies from $7.65 \times 10^{-13}$ cm$^2$ near the surface to $4.2 \times 10^{-13}$ cm$^2$ in the middle zone to $1.2 \times 10^{-13}$ cm$^2$ in the deep zone. The decrease in surface permeability is attributed to the dense network of collagen fibres in the superficial layers, whereas the decrease in permeability of the deep zone is due to the increase of charge density.