1. Introduction

In this Chapter we obtain a non parametric test for testing the hypothesis that the process is Markov against that it is semi-Markov. In the context of reliability Proschan and Pyke (1967) have obtained a nonparametric test for testing the hypothesis that a system has constant failure rate against the alternative that there is monotone increasing failure rate. They consider a sample of $n$ independent observations and obtain a test criterion for testing $H_0$ against $H_1$ and obtain its asymptotic distribution under $H_0$.

Consider a stochastic process $\{X(t), \ t > 0\}$ in which the time parameter is continuous and the state space is finite $\{1, 2, \ldots, m\}$ (say) containing no instantaneous state. Suppose the process is completely of jump type. Then the system represented by this process stays in a state for random duration of time having the same distribution at every visit.

The process can be observed as per two schemes scheme 1 and scheme 2 which are explained in Chapter I. Under either of the two schemes, let the transitions occur at the epochs $0 < t_0 < t_1 < \ldots < t_n \leq T$. The interval $(t_{j-1} - t_j)$ $(j = 1, 2, \ldots, n)$ are independent random variables. Let $\lambda_1(t)$ denote the rate of transition from state 1 at $t = 0$ to some other state at time $t$. We assume that $\lambda_1(t)$ is monotone function of $t \in (0, T)$.

The process described above will be a Markov process if the intervals are independent exponential random variables. Under this model $\lambda_1(t) = \lambda_1$
is a constant ($i = 1,2,\ldots,m$). The Markov process is assumed to be ergodic and regular possessing a stationary distribution, $\{\pi_i\}$ ($\pi_i > 0$, $\sum_{i=1}^{m} \pi_i = 1$).

The process $\{X(t), t > 0\}$ will be a semi-Markov process if given $X_1$, ... the intertransition times are independent r.v.'s and have the same arbitrary distribution $F_{ij}(t)$ belonging to a certain class $\mathcal{F}_i$ (say) whenever the state $i$ is followed by the state $j$, $(i,j) \in I$. (cf. Levy (1954) and Smith (1955)). For $\mathcal{F}_i$, we take those distributions for which $F_{ij}(t) = F_j(t)$ not depending on $j$ and whose transition rates $\lambda_i(t) = f_i(t)/1 - F_i(t)$ ($i = 1,2,\ldots,m$) are monotone increasing function of $t \in (0,T)$.

We consider testing the hypothesis $H_0$ that the observed realization $\{X(t)\}$ is from a Markov process against the hypothesis $H_1$ that the realization is from the above semi-Markov process. We note that for the process $\{X(t)\}$ mentioned above, successive durations of stay in state $i$ constitute a sample of independent and identically distributed (i.i.d.) random variables (r.v), having the distribution function $F_i(t) \in \mathcal{F}$ which is exponential under $H_0$ and arbitrary with rate $\lambda_i(t)$ monotone increasing under $H_1$ ($i = 1,2,\ldots,m$). As in Proschan and Pyke (1967) we propose a test criterion for testing $H_0$ against $H_1$ and obtain its asymptotic distribution under $H_0$. We show that the test is consistent and unbiased.

This chapter has five sections. In section 2 the model considered is described and the test criterion is proposed. In section 3 asymptotic distribution of the criterion under $H_0$ is obtained. In section 4 the test is shown to be consistent and unbiased. In all these sections
2, 3 and 4 we consider the process observed continuously up to the
nth transition (scheme 1). In section 5 all the results discussed
in the earlier section are obtained for the scheme 2.

2. Testing $H_0$ vs. $H_1$

Suppose the observation on $X(t)$ is continued until the nth
transition. Let $n_i$ denote the number of transitions from state
1 to other states. Let $x_{i1}, x_{i2}, \ldots, x_{in_i}$ be the lengths of the
interval during which the process is in state $i$. These intervals
are independent and identically distributed (i.i.d) r.v.'s and
constitute a sample of $n_i$ ($i = 1, 2, \ldots, m$) i.i.d. r.v.'s having
the distribution function $F_1 \in \mathcal{F}$. Evidently
$$n = \sum_{i=1}^{m} n_i.$$

Let $x_1(1) < x_1(2) < \ldots < x_1(n_1)$ be the ordered values of the
lengths of these intervals for state $i$. Now obtain the differences
$$D_{i1} = x_1(1), \quad D_{i2} = x_1(2) - x_1(1), \quad \ldots, \quad D_{in_i} = x_1(n_1) - x_1(n_1-1)$$
and the normalized differences
$$D_{i1} = n_i, \quad D_{i2} = (n_i - 1)D_{i1}, \quad \ldots, \quad D_{in_i} = D_{in_i}.$$

Next, define the r.v.'s
$$Y_{p,q}^i = \begin{cases} 1, & \text{if } D_{ip} > D_{iq} \\ 0, & \text{otherwise} \end{cases} \quad (2.1)$$
($p, q = 1, 2, \ldots, n_i \quad i = 1, 2, \ldots, m$).

In order to test the hypothesis $H_{10} : \{X(t), \ t > 0\},$ is
Markov process with $\lambda_1(t) = \lambda_1$ a constant, vs. $H_{11} : \{X(t), \ t > 0\}$
is semi-Markov process with $\lambda_1(t)$ a monotone increasing function of
For a certain state \( i \), we define the test criterion

\[
V_{n_1} = \sum_{p < q = 1}^{n_1} V_{p,q}^2
\]

and the critical region

\[
\left[ V_{n_1} > C_1 \right] = W_1, \quad \text{(say).} \tag{2.3}
\]

Since \( H_0 \) is the intersection of \( H_{10} \) (\( i = 1, 2, \ldots, m \)) by Roy's union-intersection principle (cf. Kshirsagar (1972), pp.131), the critical region for testing \( H_0 \) vs. \( H_1 \) is given by

\[
W = \bigcup_{i=1}^{m} W_i = \bigcup_{i=1}^{m} \left[ V_{n_1} > C_1 \right] \tag{2.4}
\]

The sizes of \( W_i \)'s are chosen such that the size of \( w \) is \( \alpha \).

Suppose the critical value \( C_1 \) of (2.3) is determined such that

\[
P\left[ V_{n_1} > C_1 / H_0 \right] = \alpha^* \tag{2.5}
\]

which is the same for all \( i = 1, 2, \ldots, m \). By the mutual independence (at least asymptotically) of \( V_{n_1} \) (\( i = 1, 2, \ldots, m \)), it follows that

\[
\alpha^* = 1 - (1 - \alpha)^{1/m} \tag{2.6}
\]

i.e. if \( \alpha \) is the size of the overall critical region (2.4), the sizes of (2.3) are given by (2.6).

The test (2.3) may be justified as follows. Under \( H_0 \), for each state \( i \), the r.v.'s \( X_i(1), X_i(2), \ldots, X_i(n_i) \) are i.i.d. exponential r.v.'s with parameters \( \lambda_i \). Then the normalized differences \( D_{11}, D_{12}, \ldots, D_{n_1} \) will also be i.i.d. exponential r.v.'s with parameter \( \lambda_1 \) (cf. Epstein and Sobel (1953)). Therefore under \( H_0 \)
\[ P[ D_{1p} > D_{1q} ] = 1/2 , \quad i = 1,2,\ldots, m , \]
\[ p, q = 1,2,\ldots, n_1 \]
or
\[ P[ V_{p,q}^1 = 1 ] = 1/2 \quad (2.7) \]

Under as is well known
\[ P[ V_{p,q}^1 = 1 ] > 1/2 , \quad p, q = 1,2,\ldots, n_1 \quad (2.8) \]
\((p < q)\).

Thus under \( H_1 \), \( V_{n_1} \) defined by (2.2) tends to be large. Hence we reject \( H_0 \) for large values of \( V_{n_1} \) for each \( i \).

Now, we obtain the asymptotic distribution of \( V_{n_1} \) under \( H_0 \) in the following section.

3. Distribution of \( V_{n_1} \) under \( H_0 \)

In this section we show that \( V_{n_1} \) is asymptotically normally distributed as \( n \to \infty \) under \( H_0 \), on the same lines as in section 2 of Proschan and Pyke (1967). We obtain the result in the following theorem.

**THEOREM 3.1.** The test statistic \( V_{n_1} \) defined in (2.2), is asymptotically normally distributed as \( n \to \infty \) under the null hypothesis \( H_0 \).

**PROOF.** We prove the theorem with the same notations of Proschan and Pyke (1967). For given \( n_1 \), \( V_{n_1} \) is a sum of \( n_1 \) independent r.v.'s such that
\[ E(V_{n_1}) = \frac{n_1(n_1 - 1)}{4} = \nu_{n_1} \]
and
\[ \text{Var}(V_{n_1}) = \frac{n_1(n_1 - 1)(2n_1 + 5)}{72} = \sigma_{n_1}^2 , \quad (i = 1,2,\ldots,m) \]
By the ergodicity assumption of the Markov process, \( n_{i_1}/n \to \pi_i > 0 \) for \( i = 1, 2, \ldots, m \). Thus \( n \to \infty \) implies \( n_{i_1} \to \infty \) for \( i = 1, 2, \ldots, m \). By the central limit theorem for a random sum of i.i.d. r.v.'s (see Chung (1965)) as \( n_{i_1} \to \infty \)

\[
V_{n_{i_1}}^* = \frac{V_{n_{i_1}} - \mu_{n_{i_1}}}{\sigma_{n_{i_1}}} \to N(0,1). \tag{3.1}
\]

Thus \( V_{n_{i_1}}^* \) is asymptotically normally distributed with mean \( \mu_{n_{i_1}} = n_{i_1}^2 \pi_i^2/4 \) and variance \( \sigma_{n_{i_1}}^2 = n_{i_1}^3 \pi_i^3/36 \). The critical values \( C_i \)'s of (2.5) can be obtained using normal tables.

We may note that \( V_{n_{i_1}}^* \)'s are correlated because \( n_{i_1} \)'s are. But since the asymptotic distributions of \( V_{n_{i_1}}^* \)'s do not depend on \( n_{i_1} \), \( V_{n_{i_1}}^* \)'s are asymptotically independent.

4. Consistency and Unbiasedness

In this section we show that the test criterion \( V_{n_{i_1}} \) given by (2.2) is consistent and unbiased.

**THEOREM 4.1.** The over all test given by the critical region (2.4) for testing the hypothesis \( H_0 \) against \( H_1 \) is consistent.

**PROOF.** We prove the theorem on lines similar to that of Mann (1945). The test statistic \( V_{n_{i_1}} \) is the same as \( T \) defined in section 4 of the paper by Mann (1945). Hence we see that \( V_{n_{i_1}} \) is consistent for each state \( i \) \( (i = 1, 2, \ldots, m) \). By the mutual independence of \( V_{n_{i_1}} \)'s (asymptotically) and by the property of Roy's union-intersection principle (see Nandi (1965)) consistency of the over all test given by the critical region (2.4) follows.
THEOREM 4.2. The overall test given by the critical region (2.4) for testing the hypothesis $H_0$ against $H_1$ is unbiased.

PROOF. We prove the theorem on lines similar to that of Proschan and Pyke (1967). We now proceed to show the test statistic $V_{n_1}$ is unbiased for each state $i$. ($i = 1,2,...,m$).

We make the transformation

$$X_{1p} = -\log F(x_{1p})$$

where $\bar{F}(x_{1p}) = 1 - F(x_{1p})$, where $F$ is defined in section 2.

$x_{1p}$'s ($p = 1,2,...,n_1$) are independent and are distributed according to exponential distribution with unit mean. Proceeding on the same lines of section 2, the normalized differences of $x_{1p}$'s are given by

$$\overline{D}_{1p} (p = 1,2,...,n_1) \text{.}$$

From (4.1), it follows that for $p < q$

$$\overline{D}_{1p} \geq \overline{D}_{1q} \text{ implies } D_{1p} \geq D_{1q} \ldots$$

From (2.1), this implies that

$$V_{p,q}^i \geq \overline{V}_{p,q}^i$$

where $\overline{V}_{p,q}^i$ is defined as in (2.1), for the normalized differences $\overline{D}_{1p}$'s. Summing over $p,q = 1,2,...,n_1$ ($p < q$), we see that under $H_1$ for all $n_1 = 2,3,...$

$$V_{n_1} \geq V_{n_1}^i \text{, (a.s.)}$$

Therefore the inequality holds even when $n_1$ is random and by (2.5)

$$P \left[ V_{n_1} > C_1 \mid H_1 \right] \geq \alpha^*$$

Thus for each state $i$, the test given by the critical region $w_1$ of
(2.3) is unbiased. Since $V_n$'s are mutually independent (asymptotically) by the property of Roy's union-intersection principle it follows that the overall critical region given by (2.4) is unbiased. (see Nandi (1965)).

5. Testing $H_0$ vs. $H_1$ when the process is observed under scheme 2

In this section we consider the case, when the process is observed under scheme 2, in which case $n$ is a r.v. and $T$ is fixed. As in section 2, let $x_{11}, x_{12}, \ldots, x_{1n}$ denote the lengths of the intervals during which the process stays in state 1 ($i = 1, 2, \ldots, m$).

We consider the information given by the Markov process up to $t_n$ only, where $t_n$ is the last transition epoch before $T$. We ignore that of the interval $(T - t_n)$ because its contribution becomes negligible as $T \to \infty$. Asymptotically as $T \to \infty$, $n_{1j} \to \infty$ (cf. Adke and Manjunath (1984)), where $n_{1j}$ is the number of transitions from state 1 to state $j$. Hence

$$n_1 = \sum_{j=1}^{m} n_{1j} \quad \text{and} \quad n \to \infty.$$

Proceeding as in section 2, for each state $i$, the critical region for testing $H_{10}$ vs. $H_{11}$ is given by (2.3). The critical region for testing $H_0$ vs. $H_1$ is given by (2.4). Sizes of the critical regions given by (2.3) for each state $i$ are such that (2.6) holds and the size of (2.4) is $\alpha$. As shown in section 2, 3 and 4 the test criterion $V_{n_1}$ has asymptotic normal distribution and $w$ given by (2.4) is consistent and unbiased.
Similarly we can test the hypothesis that the process \{X(t)\} is semi-Markov process with transition rates \(\lambda_i(t)\) as monotone decreasing functions of \(t \in (0,T)\) for all \(i\), \((i = 1, 2, ..., m)\) by reversing the inequality in the definition (2.1) and in the result (2.3).