1. Introduction

Asymptotic measures of relative efficiencies can be broadly classified into two categories as local and nonlocal efficiency. A measure of performance that requires the alternative to approach the null is known as the local efficiency and a measure that lets the alternative stay fixed as $n \to \infty$ is nonlocal efficiency. Pitman efficiency criterion is the most popular local efficiency criterion. Basawa and Prakasa Rao (1980) have proposed a local efficiency criterion known as Local power criterion. Most popular nonlocal efficiency criterion is Bahadur efficiency criterion proposed by Bahadur (1960). Another nonlocal efficiency criterion is proposed by Hodges and Lehman (1956). They consider the rate of convergence of second kind error going to zero at a fixed alternative, whereas in Bahadur efficiency we consider the rate of the convergence of the level attained by a statistic, to zero when the alternative is fixed. In this chapter we discuss Bahadur efficiency.

Bahadur efficiency of a test statistic is defined in the sense of slope (cf. Bahadur (1960)). The definition of slope will be given in section 2. Bahadur (1965) has obtained the exact slope of the likelihood ratio test statistic for a sequence of independent and identically distributed (i.i.d) random variables (r.v.) and he has shown that it is optimal in the sense of exact slope. Bahadur (1967) has proved the same result for a sequence of independent random variables.
He has defined the approximate slope of a test statistic. Bahadur and Raghavachari (1971) have shown that the likelihood ratio statistic for the general sample space is optimal, in the sense of exact slope. They have discussed the examples of Markov chain also.

In this chapter we obtain exact and approximate slope of the likelihood ratio tests obtained in Chapter II and III. We show that the tests are optimal.

This chapter has six sections. In section 2, Bahadur efficiency of a test statistic is explained for immediate use of the results to be proved in next section. In section 3 exact slopes of the likelihood ratio tests are obtained and the tests are shown to be optimal. In section 4, approximate slopes of the likelihood ratio tests are obtained and the tests are shown to be optimal. In all these sections, we obtain the Bahadur efficiency of the likelihood ratio tests for testing the hypotheses about Markov process observed under scheme 1. In section 5, Bahadur efficiency of these tests for testing the hypotheses about Markov process observed under scheme 2 is obtained. In section 6 some concluding remarks are given.

2. Bahadur efficiency

For immediate use of the results to be proved in next sections we explain Bahadur efficiency in this section.

Let \( S = \{ x_1, x_2, \ldots \} \) be a sequence of random variables (r.v.) with the parameter space \( \Theta \) and probability measure \( \{ P_\theta : \theta \in \Theta \} \).
Let $\mathcal{A}$ denote $\sigma$-field of subsets of $S$ the sample space and $\mathcal{B}_n$ denote the sub $\sigma$-field of subsets $S$. The null hypothesis is that $\theta \in \Theta_0$ obtains against the alternative that $\theta \in \Theta_1$ obtains. Suppose that
\[
P_{\theta} \ll P_{\theta_0} \quad \text{on } \mathcal{B}_n
\]
and let
\[
R_n(s, \theta, \theta_0) = \frac{dP_{\theta}(s)}{dP_{\theta_0}(s)} \quad \text{on } \mathcal{B}_n
\]
where $s$ is the sample point of $S$. Let
\[
K_n(s, \theta, \theta_0) = n^{-1} \log R_n(s, \theta, \theta_0)
\]
where $-\infty \leq K_n \leq \infty$. This is the same as the loglikelihood ratio considered by Basawa and Scott (1983) with the norm $b_n = n$. They have defined generalized Kullback-Leibler number for non-ergodic model. As $n \to \infty$
\[
K_n(s, \theta, \theta_0) + K(\theta, \theta_0) \quad \text{a.s.}
\]
when a non-null $\theta \in \Theta_1$ obtains. $K(\theta, \theta_0)$ is known as the generalized Kullback-Leibler number. Let
\[
J(\theta) = \inf \{ K(\theta, \theta_0) ; \theta_0 \in \Theta_0 \}
\]
where $J(\theta)$ is well defined on $\Theta_1$, $0 \leq J \leq \infty$; $J = 0$ on $\Theta_0$ in typical cases $0 < J(\theta) < \infty$.

Let $T_n = T_n(x_1, x_2, \ldots, x_n)$ be a test statistic. Suppose large values of $T_n$ are significant. Let $T_n$ has exact null distribution. For any $t$ and $\theta$, let
\[
F_n(t, \theta) = P_{\theta}[T_n(s) < t] \quad -\infty < t < \infty
\]
and
\[ G_n(t) = \sup [1 - F_n(t, \theta) : \theta \in \Theta_0] \]  \hspace{1cm} (2.7)

Define
\[ L_n(s) = G_n(T_n(s)). \]  \hspace{1cm} (2.8)

\(L_n(s)\) has been called by Bahadur, the level attained by \(T_n\) and has this interpretation in the framework of tests of hypotheses. Consider the testing problem \(H_0 : \theta \in \Theta_0\) vs. \(H_1 : \theta \in \Theta - \Theta_0\) for which large values of \(T_n\) are significant. Then \(L_n(s)\) is an index of the performance of this test in the sense, it is the maximum probability of obtaining a value of \(T_n\) as large or larger than \(T_n(s)\) when the hypothesis is true. In many cases \(L_n \to 0\) with probability one or in probability as \(n \to \infty\). The rate at which \(L_n \to 0\) is known as slope of a test statistic \(T_n\) (cf. Bahadur (1960)).

**Definition 2.1**: If
\[ -2n^{-1} \log L_n \to C(\theta), \] as \(n \to \infty\) with probability one when a non-null \(\theta\) obtains, then \(C(\theta)\), \(0 < C < \infty\) is known as the exact slope of \(T_n\).

If (2.9) holds in probability when a non-null \(\theta\) obtains, then \(C(\theta)\) is known as weak exact slope.

Bahadur (1967) has defined the approximate slope of a test statistic as follows.

Suppose \(T_n\) has asymptotic null distribution i.e. there exists a probability distribution function \(F\) such that for each \(\theta\) in \(\Theta_0\)
\[ F_n(t, \theta) \to F(t), \] as \(n \to \infty\)  \hspace{1cm} (2.10)
for each t. Then the approximate level is defined as
\[
L_n^{(a)}(s) = 1 - F(T_n(s)) = G(T_n(s)).
\] (2.11)

**DEFINITION 2.2** : If
\[
-2n^{-1}\log L_n^{(a)} \rightarrow C^{(a)}(\theta),
\] (2.12)
as \( n \to \infty \) with probability one when a non-null \( \theta \) obtains, then \( C^{(a)}(\theta) \) is known as the approximate slope.

For a sequence of independent and identically distributed random variables Bahadur (1965) has proved that for each \( \theta \in \Theta_1 \)
\[
\liminf_{n \to \infty} n^{-1} \log L_n \geq -J(\theta),
\] (2.13)
with probability one when \( \theta \in \Theta_1 \) obtains. He has proved this result under certain assumptions. Raghavachari (1970) has proved the same result (2.13) for a sequence of random variables, without any assumptions. Hence we have the following definitions.

**DEFINITION 2.3** : A test statistic \( T_n \) is said to be optimal in the sense of exact slope if (2.9) holds true with probability one when a non-null \( \theta \) obtains with the slope \( C(\theta) = 2J(\theta) \).

**DEFINITION 2.4** : A test statistic \( T_n \) is said to be optimal in the sense of approximate slope if (2.12) holds true with probability one when a non-null \( \theta \) obtains with the slope \( C^{(a)}(\theta) = 2J(\theta) \).
3. Bahadur efficiency of tests in the sense of exact slope

In this section we consider the Markov process observed under Scheme 1 in which case the total number of transitions $n$ is fixed and total time of observation $T$ is a random variable.

We obtain the exact slope of the likelihood ratio tests for testing the hypotheses specified in the theorems of section 2 and 3 of Chapter II and also in sections 2, 3, 4, 5 and 6 of Chapter III, in the following theorem. We assume that the parameter space $(H)$ is finite.

**THEOREM 3.1:** The likelihood ratio tests for testing the hypotheses specified in Theorems (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5) and (2.2.6) are optimal in the sense of exact slopes.

**PROOF:** (i) (a) Consider testing the hypothesis $H_0$ against $H_1$ of Theorem (2.2.1)(a). The likelihood function is given by (2.2.2). The likelihood ratio criterion is given by (2.2.6) and $T_n = -2 \log \Lambda$ is given by (2.2.7).

Now from (2.2) and (2.3) we have

$$R_n(s; \lambda, p^0; \lambda^0, p^0) = R_n(s; \lambda; \lambda^0) = \prod_{i=1}^{m} \frac{\lambda_i}{\lambda_i^0} n e^{-(\lambda_i - \lambda_i^0)Y_i}$$  \hspace{1cm} (3.1)

and

$$K_n(s; \tilde{\lambda}, \lambda^0) = n^{-1} \left\{ \sum_{i=1}^{m} n \log \frac{\lambda_i}{\lambda_i^0} - \sum_{i=1}^{m} (\lambda_i - \lambda_i^0)Y_i \right\}$$  \hspace{1cm} (3.2)

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$. Asymptotically as $n \to \infty$. 

by the ergodicity assumption of the Markov process. \( \pi_i > 0 \) are the stationary probabilities. Thus \( n \to \infty \) implies \( n_i \to \infty \) (\( i = 1, 2, \ldots, m \)).

Also since \( \gamma_1 \), the total time spent in state \( i \), is the sum of \( n_i \) i.i.d. exponential r.v.s., under \( H_i^1 \), by the strong law of large numbers (SLLN) as \( n_i \to \infty \),

\[
\frac{\gamma_1}{n_i} \to \frac{1}{\lambda_1^i} \quad \text{a.s.} \quad (3.4)
\]

From (3.3) and (3.4)

\[
\frac{\gamma_1}{n} \to \frac{\pi_1}{\lambda_1^i} \quad \text{a.s.} \quad (3.5)
\]

Hence from (3.2) and (2.4) as \( n \to \infty \)

\[
K_n(s, \lambda^i, \lambda^0) \to \sum_{i=1}^{m} \pi_i \log \left( \frac{\lambda_i^i}{\lambda_1^i} \right) - \sum_{i=1}^{m} \frac{\lambda_i^i - \lambda_i^0}{\lambda_1^i} \pi_i / \lambda_1^i
\]

\[
= K(\lambda^i, \lambda^0) \quad \text{a.s.,} \quad (3.6)
\]

when a non null \( \lambda^i \) obtains. \( K(\lambda^i, \lambda^0) \) is the generalized Kullback-Leibler number. From (2.5)

\[
J(\lambda^i) = \inf \left\{ K(\lambda^i, \lambda^0) : \lambda^0 \in \Theta_0 \right\}
\]

\[
= K(\lambda^i, \lambda^0), \quad (3.7)
\]

since \( \lambda^0 \) is specified.

Now we obtain the exact slope of \( T_n \) given by (2.2.7). Let

\[
U_n = - n^{-1} \log \Lambda
\]

\[
= \inf_{\lambda^0 \in \Theta_0} \sup_{\lambda^1 \in \Theta_1} \left\{ n^{-1} \log R_n(s, \lambda^i, \lambda^0) \right\},
\]

which is equivalent to \( T_n = -2 \log \Lambda \). \( R_n(s, \lambda^i, \lambda^0) \) is the likelihood
ratio given by (3.1). Let $L_n$ be the level attained by $U_n$. We must show that as $n \to \infty$,

$$n^{-1} \log L_n + J(\lambda^f) \quad \text{a.s.} \quad (3.9)$$

when a non null $\lambda^f$ obtains. We use the same notations as in Bahadur (1967). Let

$$U_n(\lambda^f, \lambda^0) = \max \{ 0, n^{-1} \log R_n(s, \lambda^f, \lambda^0) \}, \quad (3.10)$$

where $U_n(\lambda^f, \lambda^0)$ is the statistic $U_n$ when the entire parameter space is $(\lambda^f, \lambda^0)$. Let

$$V_n(\lambda^0) = \sup_{\lambda \in \Theta_0} U_n(\lambda^f, \lambda^0). \quad (3.11)$$

Then

$$U_n = \inf_{\lambda^f \in \Theta_0} V_n(\lambda^0). \quad (3.12)$$

Now, from (3.8) and (2.2.6) we have

$$U_n = -n^{-1} \left\{ \sum_{i=1}^{m} \pi_i \log \left( \lambda^0 \gamma_i / \pi_i \right) + \sum_{i=1}^{m} \left( \pi_i - \lambda^0 \gamma_i \right) \right\} \quad (3.13)$$

As $n \to \infty$, when $H^f_1$ is true using (3.3), (3.4) and (3.5)

$$U_n \to -\sum_{i=1}^{m} \pi_i \log \left( \lambda^f / \lambda^0 \right) - \sum_{i=1}^{m} \pi_i \left( \lambda^f - \lambda^0 \right) / \lambda^f. \quad (3.14)$$

i.e., from (3.6) and (3.7)

$$U_n \to J(\lambda^f) \quad \text{a.s.}, \quad (3.15)$$

when a non null $\lambda^f$ obtains. Now, let $t > 0$. From (3.11) and (3.12)

$$P_{\lambda^0} [ U_n \leq t ] \leq \sum_{\lambda \in \Theta_0} P_{\lambda} [ V_n(\lambda^0) \leq t ] \quad (3.16)$$
Now
\[ P \left[ \frac{U_n(\lambda^0, X_1)}{X_0} \geq t \right] = P \left[ \frac{n^{-1} \log R_n(s, \lambda^0, X_1)}{X_0} \geq t \right] \]
\[ = P_0 \left[ R_n(s, \lambda, X_0) \geq e^{-nt} \right] \]
\[ = P_0 \left[ f_n(s, \lambda^0) \leq e^{-nt} f_n(s, \lambda^1) \right] \]

where \( f_n(s, \lambda) \) denotes the likelihood function with \( \lambda \) as the parameter and \( s \) is the sample point. Hence

\[ P \left[ \frac{U_n(\lambda^1, X_0)}{X_1} \geq t \right] = \int \frac{dP_0}{X_0} \left[ f_n(s, \lambda^0) \leq e^{-nt} f_n(s, \lambda^1) \right] \]
\[ = \int f_n(s, \lambda^0) ds \left[ f_n(s, \lambda^0) \leq e^{-nt} f_n(s, \lambda^1) \right] \]
\[ \leq e^{-nt} \int f_n(s, \lambda^0) ds \left[ f_n(s, \lambda^0) \leq e^{-nt} f_n(s, \lambda^1) \right] \]
\[ \leq e^{-nt} \] (3.17)

Since \( \lambda^1 \) is fixed but arbitrary we have from (3.16) and (3.17)

\[ 1 - F_n(t) = P \left[ U_n \geq t \right] \leq ke^{-nt} \] (3.18)

From (2.6), (2.8) and (3.18) we have

\[ L_n = \sup [ 1 - F_n(U_n) ] \leq ke^{-nU_n} \] (3.19)

Thus as \( n \to \infty \), using (3.15) and (3.19) we have

\[ n^{-1} \log L_n \leq - J(\lambda^1) \quad a.s. \] (3.20)

when a non null \( \lambda^1 \) obtains.
Raghavachari (1970) has proved that for any statistic $T_n$, as $n \to \infty$

$$n^{-1} \log L_n \geq -J(\theta) \text{ a.s.},$$

when a non null $\theta$ obtains, where $L_n$ and $J(\theta)$ are as defined in (2.5) and (2.8). In this case we have as $n \to \infty$

$$n^{-1} \log L_n \geq -J(\hat{\lambda}) \text{ a.s.}, \tag{3.21}$$

where a non null $\hat{\lambda}$ obtains. From (3.20) and (3.21) it follows that as $n \to \infty$

$$n^{-1} \log L_n \to -J(\hat{\lambda}) \text{ a.s.}, \tag{3.22}$$

when a non null $\hat{\lambda}$ obtains. Thus (2.9) holds true for $U_n$ with the slope $2J(\hat{\lambda})$. Hence by definition (2.3) $U_n$ is optimal in the sense of exact slope. It may be seen that the level attained by $U_n$ is the same as that of $T_n$. Let $L^*_n$ denote the level attained by $T_n = -2\log \lambda$. We have

$$U_n = \frac{1}{2n} T_n$$

From (2.7) we have

$$G_n(t) = P_{\lambda_0} [ T_n > t ]$$

$$= P_{\lambda_0} [ U_n > t/2n ]$$

$$= 1 - F_n(t/2n, \lambda_0).$$

Now

$$L^*_n = G_n(T_n) = 1 - F_n(T_n/2n)$$

$$= 1 - F_n(U_n)$$

$$= G_n(U_n) = L_n.$$
Thus the level attained by $T_n$ is the same as the level attained by $U_n$. Hence $T_n$ is also optimal in the sense of exact slope.

b) Consider testing the hypothesis $H^2_0$ against $H^2_1$ of Theorem (2.2.1)(b). The likelihood ratio criterion is given by (2.2.13) which is the same as (2.2.6). Hence $T_n$ given by (2.2.7) and $U_n$ given by (3.13) are optimal in the sense of exact slope.

(i) (a) Consider testing the hypothesis $H^1_0$ against $H^1_1$ of Theorem (2.2.2)(a). The likelihood ratio criterion is given by (2.2.16) and $T_n$ is given by (2.2.17).

Now from (2.2) and (2.3) we have

$$R_n(s; \lambda^0, p^1; \lambda^0, p^0) = R_n(s, p^1, p^0) = \prod_{i,j=1}^{m} \left( \frac{p_{ij}^1}{p_{ij}^0} \right)^{n_{ij}}$$

(3.23)

and

$$K_n(s, p^1, p^0) = \left\{ \sum_{i,j=1}^{m} n_{ij} \log \left( \frac{p_{ij}^1}{p_{ij}^0} \right) \right\}$$

(3.24)

Since as $n \to \infty$, $n_{ij}/n \to \pi_j$ by (3.3) and

$$n_{ij} / n \to \pi_j P_{ij} \quad a.s.,$$

(3.25)

under $H^1_1$, $n_{ij} / n \to \pi_j P_{ij} \quad a.s.,$

(3.26)

$\pi_j > 0$ are the stationary probabilities. We have by (2.4) and (3.26) as $n \to \infty$

$$K_n(s, P^1, p^0) = \sum_{i,j=1}^{m} \pi_j P_{ij} \left\{ \log \left( \frac{p_{ij}^1}{p_{ij}^0} \right) \right\}$$

(3.27)
when a nonnull $P^1$ obtains. From (2.5), since $P^0$ is specified under $H_0$ we have
\[ J(P^*) = K(P^*, P^0) \]  
(3.28)

Now, we obtain the exact slope of $T_n$ given by (2.2.17). Let
\[ U_n = -n^{-1} \log \Lambda \]
\[ = -n^{-1} \left\{ \sum_{1,j} \pi_j \log(P_{1j}^0 / P_{1j}^1) \right\}. \]
As $n \to \infty$, under $H_1$ using (3.25) and (3.26) we see that
\[ U_n \to \sum_{1,j=1}^m \pi_j^1 P_{1j}^1 \log \left( P_{1j}^1 / P_{1j}^0 \right) \quad a.s. \]
\[ = J(P^1) \]  
(3.29)
when a nonnull $P^1$ obtains. Proceeding as in case (1)(a) of this theorem we get as $n \to \infty$
\[ n^{-1} \log L_n \to -J(P^1) \quad a.s., \]  
(3.30)
when a nonnull $P^1$ obtains. Thus (2.9) holds with probability one and hence $T_n$ and $U_n$ are optimal in the sense of exact slope by definition (2.3).

Similar result has been proved by Bahadur and Raghavachari (1971) for Markov chains. They have proved the optimality of the likelihood ratio statistic for testing Markov chains in the sense of exact slope by a different method. It may be noted that the test statistic $T_n$ is the same as that for testing Markov chains.
b) Now, we consider testing the hypothesis $H^2_0$ against $H^2_1$ of Theorem (2.2.2)(b). The likelihood ratio criterion is given by (2.2.20) which is the same as (2.2.16) and hence $T_n$ is given by (2.2.17). $T_n$ and $U_n$ are optimal in the sense of exact slope as shown in case (1)(a).

Consider testing the hypothesis $H^3_0$ against $H^3_1$ of Theorem (2.2.3). The likelihood ratio criterion is given by (2.2.23) and $T_n$ is given by (2.2.24).

Now from (2.2) and (2.3) we have

$$R_n(s; \lambda^1, p^1; \lambda^0, p^0) = \prod_{i=1}^{m} \left( \frac{\lambda_i^1}{\lambda_i^0} \right)^{n_i^1} e^{-\left(\lambda_i^1 - \lambda_i^0\right)\gamma_1} \prod_{i,j=1}^{m} \left( \frac{p_{ij}^1}{p_{ij}^0} \right)^{n_{ij}^1}$$

and

$$K_n(s; \lambda^1, p^1; \lambda^0, p^0) = n^{-1} \left\{ \sum_{i=1}^{m} n_i \log\left(\frac{\lambda_i^1}{\lambda_i^0}\right) - \sum_{i=1}^{m} \left( \frac{\lambda_i^1}{\lambda_i^0} - \lambda_i^0 \right) \gamma_1 \right. + \sum_{i=1}^{m} \left( \frac{\lambda_i^1}{\lambda_i^0} - \lambda_i^0 \right) \pi_i \lambda_i^1 \left. \right\}$$

As $n \to \infty$ by (2.4) using (3.3), (3.5) and (3.25) we have

$$K_n(s; \lambda^1, p^1; \lambda^0, p^0) \to K(\lambda^1, p^1; \lambda^0, p^0) \text{ a.s.}, \quad (3.33)$$

when a non null $\lambda^1$ and $p^1$ obtains. From (2.5) we have

$$J(\lambda^1, p^1) = K(\lambda^1, p^1; \lambda^0, p^0), \quad (3.34)$$

since $\lambda^0$ and $p^0$ are specified under $H^3_0$. 
Now we obtain the exact slope of $T_n$ given by (2.2.24). Let

$$U_n = -n^{-1} \log \Lambda$$

$$= -n^{-1} \left\{ \sum_{i=1}^{m} n_i \log \frac{\lambda_i^0 y_i}{n_i} - \sum_{i=1}^{m} (\lambda_i^0 y_i - n_i) + \sum_{i,j=1}^{m,n} n_{ij} \log \frac{n_{ij}^0}{n_{ij}} \right\}.$$

Using (3.3), (3.4), (3.5) and (3.6) we see that

$$U_n \rightarrow J(\frac{1}{\Lambda}, P^1) \quad \text{a.s.,} \quad (3.35)$$

when a non null $\frac{1}{\Lambda}$ and $P^1$ obtains. Proceeding as in case (i)(a) we get as $n \rightarrow \infty$

$$n^{-1} \log L_n + J(\frac{1}{\Lambda}, P^1) \quad \text{a.s.,} \quad (3.36)$$

when a non null $\frac{1}{\Lambda}$ and $P^1$ obtains. $L_n$ is the level attained by $U_n$. Thus (2.9) holds with probability one, with the slope equal to $2J(\frac{1}{\Lambda}, P^1)$. Hence $T_n$ and $U_n$ are optimal in the sense of exact slope.

iv)(a) Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (2.2.4)(a). The likelihood ratio criterion is given by (2.2.27)

and $T_n$ is given by (2.2.28).

Now from (2.2) and (2.6) we have

$$R_n(s; \frac{1}{\Lambda}, P^0; \frac{1}{\Lambda}^*, P^0) = R_n(s; \frac{1}{\Lambda}, \frac{1}{\Lambda}^*, P^0) = \prod_{i=1}^{m} \left( \frac{\lambda_i^1}{\lambda_i} \right)^{n_i} e^{-(\lambda_i^1 - \lambda) y_i} \quad (3.37)$$

where $\frac{1}{\Lambda}^*$ is specified in the hypothesis $H_0^1$, and

$$K_n(s, \frac{1}{\Lambda}, \frac{1}{\Lambda}^*) = n^{-1} \left\{ \sum_{i=1}^{m} n_i \log \frac{\lambda_i^1}{\lambda_i} - \sum_{i=1}^{m} (\lambda_i^1 - \lambda) y_i \right\}. \quad (3.38)$$

From (2.4) using (3.3) and (3.5) as $n \rightarrow \infty$

$$K_n(s, \frac{1}{\Lambda}, \frac{1}{\Lambda}^*) \rightarrow \sum_{i=1}^{m} \pi_i \log \frac{\lambda_i^1}{\lambda_i} - \sum_{i=1}^{m} (\lambda_i^1 - \lambda) \pi_i / \lambda_i = K(\frac{1}{\Lambda}, \frac{1}{\Lambda}^*) \quad \text{a.s.,} \quad (3.39)$$
when $\lambda^1$ in $H_1$ obtains. From (2.5) we have
\[ J(\lambda^1) = \inf \{ K(\lambda^1, \lambda^*) : \lambda^* \in \mathcal{H}_0 \}. \]  

(3.40)

Since $\mathcal{H}_0$ is composite and $K(\lambda^1, \lambda^*)$ is minimum at $\lambda = (\Sigma (\pi_i / \lambda_i^1))^{-1}$, we have
\[ J(\lambda^1) = \sum_{i=1}^{m} \pi_i \log \lambda_i^1 \left( \sum_{i=1}^{m} \frac{1}{\lambda_i^1} \right). \]  

(3.41)

Now we obtain the exact slope of $T_n$. Let
\[ U_n = -n^{-1} \log \Lambda \]
\[ = -n^{-1} \left\{ \sum_{i=1}^{m} n_i \log \frac{n_i \gamma_i}{n_1} \right\}. \]

Under $H_1$, we have from (3.5)
\[ \frac{\tau}{n} \to \sum_{i=1}^{m} \left( \pi_i / \lambda_i^1 \right) \text{ a.s.} \]  

(3.42)

Hence using (3.4) and (3.42) we see that
\[ U_n \to J(\lambda^1) \text{ a.s.,} \]  

(3.43)

when a nonnull $\lambda^1$ obtains. $J(\lambda^1)$ is given by (3.41). Proceeding as in case (i)(a), we get as $n \to \infty$
\[ n^{-1} \log L_n \to -J(\lambda^1) \text{ a.s.,} \]  

(3.44)

when a nonnull $\lambda^1$ obtains. Thus (2.9) holds with probability one and hence $U_n$ and $T_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope $2J(\lambda^1)$ is given by (3.41).

b) Consider testing the hypothesis $H_0^2$ against $H_1^2$ of Theorem (2.2.4)(b). The likelihood ratio criterion is given by (2.2.33) which is the same as (2.2.27) and hence $T_n$ is given by (2.2.28). $T_n$ and $U_n$ are optimal in the sense of exact slope as shown in the above case (vi)(a).
Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (2.2.5)(a). The likelihood ratio criterion is given by (2.2.36) and $T_n$ is given by (2.2.37).

From (2.2) and (2.3) we have

$$R_n(s; A^0, P^1; A^0, P) = R_n(s, P^1, P) = \prod_{i,j=1}^m (p_{ij}/p_j)^{n_{ij}}$$

(3.45)

and

$$K_n(s, P^1, P) = n^{-1} \sum_{i,j=1}^m n_{ij} \log (p_{ij}/p_j).$$

(3.46)

As $n \to \infty$, under $H_1^1$, using (3.25) by (2.4) we have

$$K_n(s, P^1, P) \xrightarrow{a.s.} K(P^1, P).$$

(3.47)

when a nonnull $P'$ obtains. Now from (2.5)

$$J(P') = \inf \{ K(P^1, P) : P \in H_0 \}.$$

(3.48)

Since $H_0$ is composite and $K(P^1, P)$ is minimum at $\hat{P}_0 = \sum_{i=1}^m \pi_i \frac{1}{P_{ij}}$, we have

$$J(P') = \sum_{i,j=1}^m \pi_i \frac{1}{P_{ij}} \log \left( \frac{p_{ij}}{\sum_{i=1}^m \pi_i \frac{1}{P_{ij}}} \right).$$

(3.49)

Now we obtain the exact slope of $T_n$. Let $U_n$ be defined as in (3.8) i.e.

$$U_n = -n^{-1} \log \Lambda$$

$$= -n^{-1} \left\{ \sum n_{ij} \log (n_{ij}/n_{1j}) \right\}.$$

Under $H_1^1$ we have

$$\frac{n_{ij}}{n} \xrightarrow{a.s.} \sum_{i=1}^m \pi_i \frac{1}{P_{ij}}$$

(3.50)

Using (3.50) and (3.25) we see that
Un \rightarrow \frac{1}{m} \sum_{i=1}^{m} \pi_i \frac{1}{p_{ij}} \log \left( \frac{1}{\sum_{i=1}^{m} \pi_i p_{ij}} \right)
= J(P^i) \quad \text{a.s.,} \quad (3.51)

when a non null \( P^i \) obtains. \( J(P^i) \) is given by (3.49). By the same argument as in case (i)(a) we get as \( n \rightarrow \infty \)

\[ n^{-1} \log L_n \rightarrow J(P^i) \quad \text{a.s.,} \quad (3.52) \]

when a non null \( P^i \) obtains. Thus (2.9) holds with probability one and hence \( T_n \) and \( U_n \) are optimal in the sense of exact slope by definition (2.3). The exact slope \( 2J(P^i) \) is given by (3.49).

b) Consider testing the hypothesis \( H_0^2 \) against \( H_1^2 \) of Theorem (2.2.5)(b).
The likelihood ratio criterion is given by (2.2.40) which is the same as (2.2.36). Hence \( T_n \) given by (2.2.37) and also \( U_n \) are optimal in the sense of exact slope as shown in case (v)(a) above. The exact slope \( 2J(P^i) \) is given by (3.49).

vi) Consider testing the hypothesis \( H_0^1 \) against \( H_1^1 \) of Theorem (2.2.6).
The likelihood ratio criterion is given by (2.2.43) and \( T_n \) is given by (2.2.44).

From (2.2) and (2.3) we have

\[ R_n(s; \lambda^*, p^i; \lambda^i, P) = \prod_{i=1}^{m} \left( \frac{\lambda_i}{\lambda} \right)^{n_i} e^{-(\lambda_i - \lambda)\gamma_i} \prod_{i,j=1}^{m} \left( \frac{p_{ij}}{p_{ij}^o} \right)^{n_{ij}} \]

... (3.53)

and

\[ K_n(s; \lambda^*, p^i; \lambda^i, P) = n^{-1} \left\{ \sum_{i=1}^{m} n_i \log(\lambda_i/\lambda) - \sum_{i=1}^{m} (\lambda_i - \lambda)\gamma_i \right. \\
+ \sum_{i,j=1}^{m} n_{ij} \log(p_{ij}^i/p_{ij}^o) \right\}. \quad (3.54) \]
Now as \( n \to \infty \), under \( H_1^\perp \) from (3.56') and (2.4) using (3.3), (3.5) and (3.25) we have

\[
K_n(s; \lambda, \pi, \lambda^*, P) = m \sum_{i=1}^{m} \pi_i \log(\lambda_i / \lambda) - \sum_{i=1}^{m} (\lambda_i - \lambda) \pi_i / \lambda_i
\]

\[
+ \sum_{i,j=1}^{m} \pi_i p_{1j} \log(p_{1j} / p_j)
\]

\[
= K(\lambda_L^L, P^L; \lambda^*, P)
\]

with probability one, when a nonnull \( P^L \) and \( \lambda^L \) obtains. From (2.5)

\[
J(\lambda^L, P^L) = \inf \{ K(\lambda^L, P^L; \lambda^*, P), (\lambda^*, P) \in \mathcal{H}_0 \}
\]

Since \( \mathcal{H}_0 \) is composite and \( K(\lambda^L, P^L; \lambda^*, P) \) is minimum at

\[
\hat{\lambda} = \left[ \sum_{i=1}^{m} \left( \frac{\pi_i}{\lambda_i^*} \right) \right]^{-1}
\]

and \( \hat{p}_{ij} = \frac{\sum_{i=1}^{m} \pi_i p_{1j}}{\sum_{i=1}^{m} \pi_i} \), we have

\[
J(\lambda^L, P^L) = \sum_{i=1}^{m} \pi_i \log \lambda_i \left( \sum_{i=1}^{m} \left( \frac{\pi_i}{\lambda_i} \right) \right) + \sum_{i,j=1}^{m} \pi_i p_{1j} \log(p_{1j} / \Sigma_{i=1}^{m} \pi_i p_{1j})
\]

... (3.56)

Now we obtain the exact slope of \( T_n \). Let \( U_n \) be as defined in (3.8) i.e.

\[
U_n = -n^{-1} \log \lambda
\]

\[
= -n^{-1} \left\{ \sum_{i=1}^{m} n_{1i} \log \frac{n_{1j}}{n_{1i}} + \sum_{i,j=1}^{m} n_{1j} \log \frac{n_{1j}}{n_{1i}} \right\}
\]

Using (3.5), (3.25), (3.42) and (3.50) we see that

\[
U_n = \sum_{i=1}^{m} \pi_i \log \lambda_i \left( \sum_{i=1}^{m} \left( \frac{\pi_i}{\lambda_i} \right) \right) + \sum_{i,j=1}^{m} \pi_i p_{1j} \log(p_{1j} / \Sigma_{i=1}^{m} \pi_i p_{1j})
\]

\[
= J(\lambda^L, P^L) \quad \text{a.s.,} \quad \text{(3.57)}
\]

when a nonnull \( \lambda^L \) and \( P^L \) obtains. \( J(\lambda^L, P^L) \) is given by (3.56). By the same argument as in case (1)(a) we get as \( n \to \infty \)

\[
n^{-1} \log \lambda_n \to -J(\lambda^L, P^L) \quad \text{a.s.,} \quad \text{(3.58)}
\]
hen a non null $\lambda'$ and $P'$ obtains. Thus (2.9) holds with probability ne and hence $T_n$ and $U_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope $2J(\lambda', P')$ is given by (3.56).

Now we obtain the exact slope of the likelihood ratio tests for testing the hypotheses specified in Theorem (2.3.1).

**Theorem 3.2**: The likelihood ratio test for testing the hypothesis

(a) $H_0^1$ against $H_1^1$, (b) $H_0^2$ against $H_1^2$ and (c) $H_0^3$ against $H_1^3$, specified in Theorem (2.3.1), is optimal in the sense of exact slope.

**Proof** (a) Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (2.3.1)(a). The likelihood ratio criterion is given by (2.3.6) and $\gamma_1$ is given by (2.3.7).

Now from (2.2) and (2.3) we have

$$N(h; \lambda(h), \rho^0; \lambda', \rho^0) = R_N(s, \lambda(h), \lambda) = \prod_{h=1}^{k} \prod_{i=1}^{m} (\lambda_i(h)/\lambda_1)^{n_1(h)} e^{-(\lambda_1(h) - \lambda_1)\gamma_1(h)}$$

and

$$N(s, \lambda(h), \lambda) = \frac{1}{n_1(h) \log(\lambda_1(h)/\lambda_1)} \sum_{h=1}^{k} \sum_{i=1}^{m} n_i(h) \log(\lambda_i(h)/\lambda_1)$$

$$+ \frac{k}{n_1(h) \log(\lambda_1(h)/\lambda_1)} \sum_{h=1}^{k} \sum_{i=1}^{m} (\lambda_i(h) - \lambda_1)\gamma_1(h)$$

$s N \rightarrow \infty$, under $H_1^1$ we have

$$\frac{n_1(h)}{N_h} \rightarrow \pi_1(h) \quad a.s.,$$

by the ergodicity assumption of $h^{th}$ Markov process. Also by assumption given in section (2.3)

$$\frac{N_h}{N} + \varepsilon_h,$$

where $0 < \varepsilon_h < 1$ and $\sum_{h=1}^{k} \varepsilon_h = 1$. From (3.61) and (3.62) we have,
\[
\frac{n_1(h)}{N} \to \pi_1(h) \xi_h \text{ a.s.} \quad (3.63)
\]
From (3.63) and (2.3.8)
\[
\frac{\gamma_1(h)}{N} \to \frac{\pi_1(h) \xi_h / \lambda_1(h)}{a.s.}, \quad (3.64)
\]
and hence
\[
\gamma_1 = \frac{k}{N} \sum_{h=1}^{k} \gamma_1(h) \to \frac{k}{N} \left( \sum_{h=1}^{k} \pi_1(h) \xi_h / \lambda_1(h) \right) \text{ a.s.} \quad (3.65)
\]
Thus from (2.4) and (3.62) using (3.63) and (3.65) we have
\[
\begin{align*}
K_N(s, \lambda(h), \lambda) &= \sum_{h=1}^{k} m \sum_{i=1}^{m} \pi_1(h) \xi_h \log(\lambda_1(h) / \lambda_1) \\
&\quad - \frac{k}{N} \sum_{h=1}^{k} \left( \lambda_1(h) - \lambda_1 \right) \left( \sum_{i=1}^{m} \pi_1(h) \xi_h / \lambda_1(h) \right) \\
&= K(\lambda(h), \lambda) \text{ a.s.}, \quad (3.66)
\end{align*}
\]
when a non null \( \lambda(h) \) obtains. From (2.5) we have
\[
J(\lambda(h)) = \inf \{ K(\lambda(h), \lambda) : \lambda \in \Theta \}
\]
Since \( \Theta \) is composite and \( K(\lambda(h), \lambda) \) is minimum at
\[
\lambda_1(h) = \frac{k}{N} \sum_{h=1}^{k} \pi_1(h) \xi_h / \sum_{h=1}^{k} \left( \pi_1(h) \xi_h / \lambda_1(h) \right) = c_1(h) \text{ say, we have}
\]
\[
J(\lambda(h)) = K(\lambda(h), \lambda_1) = \frac{k}{N} \sum_{h=1}^{k} \sum_{i=1}^{m} \pi_1(h) \xi_h \log(\lambda_1(h) / c_1(h)). \quad (3.67)
\]
We obtain the exact slope of \( T_N \). Let \( U_N \) be a statistic as defined in (3.8) i.e.
Under $H^2_1$, as $N \to \infty$, we have

$$\frac{\gamma_1}{n_1} = \frac{\sum_{h=1}^{k} \gamma_1(h)}{\sum_{h=1}^{k} n_1(h)} \to \frac{\sum_{h=1}^{k} (\pi_1(h) \xi_h / \lambda_1(h))}{\sum_{h=1}^{k} \pi_1(h) \xi_h} = \frac{1}{C_1(h)} \quad \text{a.s.} \quad (3.68)$$

Thus as $N \to \infty$, using (2.3.8) and (3.68) we see that

$$U_N \to \sum_{h=1}^{k} \sum_{i=1}^{m} \pi_1(h) \xi_i \log(\lambda_1(h) / C_1(h)) = J(\lambda(h)) \quad \text{a.s.}, \quad (3.69)$$

when a non null $\lambda(h)$ obtains. By the same argument as in case (i)(a) of Theorem (3.1) we get as $N \to \infty$

$$N^{-1} \log L_N \to -J(\lambda(h)) \quad \text{a.s.}, \quad (3.70)$$

when a non null $\lambda(h)$ obtains. Here we have taken $N$ as norming constant because we have $N = \sum_{h=1}^{k} n_h$ observations of $k$ samples of Markov processes. $L_N$ is the level obtained by $U_N$ or equivalently it is the level attained by $T_N = -2 \log A$ given by (2.3.7). Hence $T_N$ and $U_N$ are optimal in the sense of exact slope, by definition (2.3). The exact slope $2J(\lambda(h))$ is given by (3.67).

b) Consider testing the hypothesis $H^2_0$ against $H^2_1$ of Theorem (2.3.1)(b). The likelihood ratio criterion is given by (2.3.17) and $T_N$ is given by (2.3.18).

From (2.2) and (2.3) we have
\[ R_N(s; \lambda^0, P(h); \lambda^0, P) = R_N(s, P(h), P) = \prod_{h=1}^{k} \prod_{i,j=1}^{m} \left( \frac{p_{ij}(h)}{p_{ij}} \right)^{n_{ij}(h)} (3.71) \]

and

\[ K_N(s, P(h), P) = N^{-1} \left\{ \sum_{h=1}^{k} \sum_{i,j=1}^{m} n_{ij}(h) \log \left( \frac{p_{ij}(h)}{p_{ij}} \right) \right\} (3.72) \]

Under \( H_1^2 \) as \( n \to \infty \), we have

\[ \frac{n_{ij}(h)}{n_1(h)} \to p_{ij}(h) \quad \text{a.s.} \] (3.73)

and hence by (3.63) and (3.73)

\[ \frac{n_{ij}(h)}{N} \to \pi_1(h) \xi_h p_{ij}(h) \] (3.74)

Thus by (2.4) and (3.72) using (3.74) we have as \( N \to \infty \)

\[ K_N(s, P(h), P) \to \sum_{h=1}^{k} \sum_{i,j=1}^{m} \pi_1(h) \xi_h p_{ij}(h) \log \left( \frac{p_{ij}(h)}{p_{ij}} \right) \]

\[ = K(P(h), P) \quad \text{a.s.,} \] (3.75)

when a non null \( P(h) \) is true. From (2.5) we have

\[ J(P(h)) = \inf \{ K(P(h), P) : P \in \Theta_0 \} \]

Since \( \Theta_0 \) is composite and \( K(P(h), P) \) is minimum at

\[ \hat{p}_{ij} = \sum_{h=1}^{k} \pi_1(h) \xi_h p_{ij}(h) / \sum_{h=1}^{k} \pi_1(h) \xi_h = C_{ij}(h) \quad \text{(say)}, \]

we have

\[ J(P(h)) = \sum_{h=1}^{k} \sum_{j=1}^{m} \pi_1(h) \xi_h p_{ij}(h) \log(p_{ij}(h)/C_{ij}(h)). \] (3.76)

Now we obtain the exact slope of \( T_N \) given by (2.3.10). Let \( U_N \)

be a statistic defined as
Under $H_1^h$, from (3.63) and (3.74) we have

$$\frac{k \sum_{h=1}^{m} \Pi(1) \cdot \Pi(h) \cdot \Xi(h)}{k \sum_{h=1}^{m} \Pi(1) \cdot \Pi(h) \cdot \Xi(h) \cdot \Pi(h)} = \frac{1}{\Pi(h)}.$$  \hfill (3.77)

Thus using (3.73) and (3.77) we see that

$$U_N + J(P(h)) \quad a.s., \quad (3.78)$$

when a non null $P(h)$ is true. By the same argument as in case (1)(a) of Theorem (3.1) we get as $N \to \infty$

$$-N^{-1} \log L_N + J(P(h)) \quad a.s., \quad (3.79)$$

when a non null $P(h)$ is true. Thus (2.9) holds with probability one and $J(P(h))$, hence $U_N$ and $T_N$ are optimal in the sense of exact slope. The exact slope $2J(P^a)$ is given by (3.76).

c) Consider testing the hypothesis $H_0^h$ against $H_1^h$ of Theorem (2.3.1)(c). The likelihood ratio criterion is given by (2.3.21) and $T_N$ is given by (2.3.22).

From (2.2) and (2.3) we have

$$R_N(s; \lambda(h), P(h); \lambda, P) = \prod_{h=1}^{m} \left( \frac{\lambda_1(h)}{\lambda_1} \right)^{n_1(h)} \cdot e^{-\lambda_1(h) \cdot \gamma_1(h) \cdot x} \cdot \prod_{i,j=1}^{m} \left( \frac{\Pi(1)}{\Pi(h)} \right)^{n_1(h)} \cdot (3.80)$$

and
\[ K_N(s; \lambda(h) ; P(h) ; \lambda, P) = N^{-1} \left\{ \sum_{h=1}^{k} \sum_{i=1}^{m} n_1(h) \log \frac{\lambda_i(h)}{\lambda_1} - \sum_{h=1}^{k} \sum_{i=1}^{m} \left( \lambda_i(h) - \lambda_1 \right) \gamma_1(h) + \sum_{h=1}^{k} \sum_{i,j=1}^{m} n_{ij}(h) \log \frac{P_{ij}(h)}{P_{ij}} \right\}. \] (3.81)

As \( N \to \infty \), by (2.4) and (3.81) using (3.63), (3.65) and (3.74) we get
\[ K_N(s; \lambda(h) ; P(h) ; \lambda, P) = K(\lambda(h), \lambda) + K(P(h), P) \]
\[ = K(\lambda(h), P_{h}; \lambda, P) \quad \text{a.s.,} \] (3.82)
when a non null \( \lambda(h) \) and \( P_h \) is true. From (2.5) we have
\[ J(\lambda(h), P(h)) = \inf \{ K(\lambda(h), P(h); \lambda, P) : \lambda, P \in \mathcal{H}_0 \}. \]

Since \( \mathcal{H}_0 \) is composite and \( K(\lambda(h), P(h); \lambda, P) \) is minimum at
\[ \hat{\lambda}_1 = C_1(h) \quad \text{and} \quad \hat{P}_{ij} = C_{ij}(h), \]
where \( C_1(h) \) and \( C_{ij}(h) \) are given by (3.68) and (3.77), we get
\[ J(\lambda(h), P(h)) = J(\hat{\lambda}(h)) + J(P(h)) \] (3.83)
where \( J(\hat{\lambda}(h)) \) and \( J(P(h)) \) are given by (3.67) and (3.76) respectively.

Now we obtain the exact slope of \( T_N \) given by (2.3.24). Let \( U_N \)
be a statistic as defined in (3.8) i.e.,
\[ U_N = -N^{-1} \log \Lambda \]
\[ = -N^{-1} \left\{ \sum_{h=1}^{k} \sum_{i=1}^{m} n_1(h) \log \left( \frac{\sum_{h=1}^{k} n_1(h)}{n_1(h)} \right) \gamma_1(h) \right\} \]
\[ + \sum_{h=1}^{k} \sum_{i,j=1}^{m} n_{ij}(h) \log \left( \frac{\sum_{h=1}^{k} n_{ij}(h)}{n_{ij}(h)} \right) \]
\[ \text{Using (2.3.24), (3.68) and (3.74), we see that} \]
\[ U_N + J(\lambda(h), P(h)) \quad \text{a.s.,} \] (3.84)
when a non null $P(h)$ and $\lambda(h)$ are true. By the same argument as in case (i)(a) of Theorem (3.1) we get as $N \to \infty$

$$N^{-1} \log L_N \to J(\lambda(h), P(h)) \quad \text{a.s.,} \quad (3.85)$$

when a non null $\lambda(h)$ and $P(h)$ is true. Thus (2.9) holds with probability one with the slope $2J(\lambda(h), P(h))$ and hence $U_N$ and $T_N$ are optimal in the sense of exact slope by definition (2.3). The exact slope is $2J(\lambda(h), P(h))$. $J(\lambda(h), P(h))$ is given by $(3.83)$.

Now we obtain the exact slope of the likelihood ratio tests for testing the hypotheses considered in Chapter III. Let $\lambda_r$ denote a vector of rates of transitions and $P_r$ denote the transition probability matrix for $r$th order Markov process.

**THEOREM 3.3.** The likelihood ratio tests for testing the hypothesis

(a) $H_0^1$ against $H_1^1$, (b) $H_0^2$ against $H_1^2$ and (c) $H_0^3$ against $H_1^3$ specified in Theorem (3.2.1), is optimal in the sense of exact slope.

**PROOF:** (a) Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (3.2.1)(a). The likelihood ratio criterion is given by $(2.2.27)$, since these hypotheses $H_0^1$ and $H_1^1$ are the same as those of Theorem (2.2.4)(a). Hence $T_n$ is given by $(2.2.28)$ and is optimal in the sense of exact slope as shown in case (iv)(a) of Theorem (3.1). The exact slope is $2J(\lambda^1)$ and $J(\lambda^1)$ is given by $(3.41)$.

b) Consider testing the hypothesis $H_0^2$ against $H_1^2$ of Theorem (3.2.1)(b). These hypotheses are the same as those of Theorem (2.2.5)(a). The likelihood ratio criterion is given by $(2.2.36)$ and $T_n$ is given by $(2.2.37)$. 
Hence $T_n$ is optimal in the sense of exact slope as shown in case (v)(a) of Theorem (3.1). The exact slope is $2J(P^*)$ and $J(P^*)$ is given by (3.49).

c) Consider testing the hypothesis $H_0^3$ against $H_1^3$ of Theorem (3.2.1)(c). These hypotheses are the same as those of Theorem (2.2.6). The likelihood ratio criterion is given by (2.2.43) and $T_n$ is given by (2.2.44). $T_n$ is optimal in the sense of exact slope as shown in case (vi) of Theorem (3.1). The exact slope is $2J(P^*, \lambda^*)$. $J(P^*, \lambda^*)$ is given by (3.56).

**THEOREM 3.4.** The likelihood ratio tests for testing the hypothesis
(a) $H_0^1$ against $H_1^1$, (b) $H_0^2$ against $H_1^2$ and (c) $H_0^3$ against $H_1^3$, specified in Theorem (3.3.1) is optimal in the sense of exact slope.

**PROOF:** (a) Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (3.3.1)(a). The likelihood ratio criterion is given by (3.3.5) and $T_n$ is given by (3.3.6).

Now from (2.2) and (2.3) we have

$$R_n(s, \lambda_2, \lambda_1) = \prod_{i,j=1}^{m} \left( \frac{\lambda_{i,j}}{\lambda_j} \right)^{n_{i,j}} e^{-(\lambda_{i,j} - \lambda_j) \gamma_{i,j}}$$  \hspace{1cm} (3.86)

and

$$K_n(s, \lambda_2, \lambda_1) = \sum_{i,j=1}^{m} n_{i,j} \log(\lambda_{i,j}/\lambda_j) - \sum_{i,j=1}^{m} (\lambda_{i,j} - \lambda_j) \gamma_{i,j}$$  \hspace{1cm} (3.87)

As $n \to \infty$ under $H_1^1$,

$$\frac{\gamma_{i,j}}{n_{i,j}} \to 1/\lambda_{i,j} \quad \text{a.s.},$$  \hspace{1cm} (3.88)

since $\gamma_{i,j}$, the time spent in state $(i,j)$ is the sum of $n_{i,j}$ i.i.d.

exponential random variables. Also by the ergodicity assumption

$$\frac{n_{i,j}}{n} \to \pi_{i,j} \quad \text{a.s.},$$  \hspace{1cm} (3.89)
\( \pi_{1j} > 0 \) are the stationary probabilities of 2nd order Markov chain. From (3.88) and (3.89) we have

\[
\frac{Y_{1j}}{n} + \frac{\pi_{1j}}{\lambda_{1j}} \quad \text{a.s.} \quad (3.90)
\]

Thus from (3.87) and (2.4) using (3.89) and (3.90) we have

\[
K_n(s, \lambda_2, \lambda_1) = \sum_{i,j=1}^{m} \pi_{1j} \log(\lambda_{1j}/\lambda_{ij}) - \sum_{i,j=1}^{m} (\lambda_{1j} - \lambda_{ij}) \pi_{1j}/\lambda_{1j}
\]

\[
= K(\lambda_2, \lambda_1) \quad \text{a.s.,} \quad (3.91)
\]

when a non null \( \lambda_2 \) obtains. From (2.5) we have

\[
J(\lambda_2) = \inf \{ K(\lambda_2, \lambda_1), \lambda_1 \in \Theta_0 \}
\]

Since \( \Theta_0 \) is composite and \( K(\lambda_2, \lambda_1) \) is minimum at

\[
\hat{\lambda}_j = \left\{ \sum_{i=1}^{m} \pi_{1j} / \sum_{i=1}^{m} (\pi_{1j}/\lambda_{1j}) \right\} = C_{1j}^* \quad \text{(say)}, \quad \text{we have}
\]

\[
J(\hat{\lambda}_2) = \sum_{i,j=1}^{m} \pi_{1j} \log(\lambda_{1j}/C_{1j}^*) \quad (3.92)
\]

Now we obtain the exact slope of \( T_n \) given by (3.3.6). Let \( U_n \) be a statistic as defined in (3.8) i.e.,

\[
U_n = -n^{-1} \log \Lambda
\]

\[
= -n^{-1} \left\{ \sum_{i,j=1}^{m} n_{1j} \log \frac{\pi_{1j}Y_{1j}}{n_{1j}Y_{1j}} \right\}
\]

Since as \( n \to \infty \), under \( H_1^1 \)

\[
\frac{Y_{1j}}{n_{1j}} \to \frac{\sum_{i=1}^{m} (\pi_{1j}/\lambda_{1j})}{\sum_{i=1}^{m} \pi_{1j}} = \frac{1}{C_{1j}^*} \quad \text{a.s.,} \quad (3.93)
\]

we see that using (3.93) and (3.88)
\[ U_n = \sum_{i,j=1}^{m} \pi_{ij} \log(\lambda_{ij}/C_{ij}^*) \]
\[ = J(\lambda_2) \quad \text{a.s.}, \quad (3.94) \]
when a non null \( \lambda_2 \) obtains. Proceeding as in case (1)(a) of Theorem (3.1) we get as \( n \to \infty \)
\[ n^{-1} \log L_n = -J(\lambda_2) \quad \text{a.s.}, \quad (3.95) \]
when a non null \( \lambda_2 \) obtains. Thus (2.9) holds with probability one and hence \( U_n \) and \( T_n \) are optimal in the sense of exact slope by definition (2.3). The exact slope is \( 2J(\lambda_2) \). \( J(\lambda_2) \) given by (3.92).

b) Consider testing the hypothesis \( H_0^2 \) against \( H_1^2 \) of Theorem (3.3.1)(b). The likelihood ratio criterion is given by (3.3.13) and \( T_n \) is given by (3.3.14).

Now from (2.2) and (2.3) we have
\[ R_n(s, p_2, p_1) = \prod_{i,j,k=1}^{m} \left( \frac{p_{ijk}}{p_{jk}} \right)^{n_{ijk}} \quad (3.96) \]
and
\[ K_n(s, p_2, p_1) = n^{-1} \sum_{i,j,k=1}^{m} n_{ijk} \log \left( \frac{p_{ijk}}{p_{jk}} \right) \quad (3.97) \]
As \( n \to \infty \) under \( H_1^1 \), we have
\[ \frac{n_{ijk}}{n_{ij}} \to p_{ijk} \quad \text{a.s.} \quad (3.98) \]
and hence from (3.89) and (3.98) we get
\[ \frac{n_{ijk}}{n} \to \pi_{ij} p_{ijk} \quad \text{a.s.}, \quad (3.99) \]
where \( \pi_{ij} \)'s are the stationary probabilities. From (3.97) and (2.4) using (3.99), we get
\[ K_n(\mathbf{s}, P_2, P_1) = \sum_{i,j,k=1}^{m} \pi_{i,j,k} P_{i,j,k} \log \left( \frac{P_{i,j,k}}{c_{i,j,k}} \right) = K(P_2, P_1) \quad \text{a.s.,} \quad (3.100) \]

when a non null \( P_2 \) obtains. From (2.5),

\[ J(P_2) = \inf \{ K(P_2, P_1) : P_1 \in \mathcal{B}_0 \} . \]

Since \( \mathcal{H}_0 \) is composite and \( K(P_2, P_1) \) is minimum at

\[ \hat{P}_{i,j,k} = \frac{\sum_{i=1}^{m} \pi_{i,j,k}}{\sum_{i,j,k=1}^{m}} P_{i,j,k} = c_{i,j,k} \quad \text{(say),} \]

\[ J(P_2) = \sum_{i,j,k=1}^{m} \pi_{i,j,k} P_{i,j,k} \log \left( \frac{P_{i,j,k}}{c_{i,j,k}} \right) \quad (3.101) \]

Now we obtain the exact slope of \( T_n \). Let \( U_n \) be a statistic as defined in (3.8) i.e.

\[ U_n = -n^{-1} \log \Lambda \]

\[ = -n^{-1} \left\{ \sum_{i,j,k=1}^{m} \pi_{i,j,k} \log \frac{n_{i,j,k}}{n_{i,j} n_{i,k}} \right\} \]

Since as \( n \to \infty \), under \( H_1 \),

\[ \frac{n_{i,j,k}}{n_{i,j}} \to \frac{\sum_{i=1}^{m} \pi_{i,j,k}}{\sum_{i,j,k=1}^{m} \pi_{i,j,k}} \quad \text{a.s.,} \quad (3.102) \]

we see that using (3.98) and (3.102)

\[ U_n \to J(P_2) \quad \text{a.s.,} \quad (3.103) \]

when a non null \( P_2 \) obtains. Proceeding as in case (1)(a) of Theorem (3.1) we get as \( n \to \infty \),

\[ n^{-1} \log L_n \to -J(P_2) \quad \text{a.s.,} \quad (3.104) \]
when a non null $P_2$ obtains. Thus (2.9) holds with probability one and hence $U_n$ and $T_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope is $2J(P_2)$ and $J(P_2)$ is given by (3.101).

c) Consider testing the hypothesis $H_0^3$ against $H_1^3$ of Theorem (3.3.1)(c). The likelihood ratio criterion is given by (3.3.17) and $T_n$ is given by (3.3.18).

From (2.2) and (2.3) we have

$$R_n(s; A_2, P_2; A_1, P_1) = \prod_{i,j=1}^{m} \left( \frac{\lambda_{ij}}{\lambda_j} \right)^{n_{ij}} e^{-(\lambda_{ij} - \lambda_j)Y_{ij}}.$$  

and

$$K_n(s; A_2, P_2; A_1, P_1) = n^{-1} \left\{ \sum_{i,j=1}^{m} n_{ij} \log \left( \frac{\lambda_{ij}}{\lambda_j} \right) 
- \sum_{i,j=1}^{m} (\lambda_{ij} - \lambda_j)Y_{ij} + \sum_{i,j,k=1}^{m} n_{ijk} \log \left( \frac{p_{ijk}}{p_{jk}} \right) \right\}.$$  

As $n \to \infty$ by (2.4) and (3.108) using (3.89), (3.90) and (3.99) we get

$$K_n(s; A_2, P_2; A_1, P_1) \to \sum_{i,j=1}^{m} \pi_{ij} \log \left( \frac{\lambda_{ij}}{\lambda_j} \right)$$

$$- \sum_{i,j=1}^{m} (\lambda_{ij} - \lambda_j)\pi_{ij} / \lambda_{ij} + \sum_{i,j,k=1}^{m} \pi_{ijk} p_{ijk} \log \left( \frac{p_{ijk}}{p_{jk}} \right)$$

$$= K(A_2, P_2; A_1, P_1) \text{ a.s.},$$  

when a non null $A_2$ and $P_2$ obtains. From (2.5)
\[ J(\lambda_2, P_2) = \inf \{ K(\lambda_2, P_2 ; \lambda_1, P_1) : (\lambda_1, P_1) \in \Omega_0 \} \]

Since \( \Omega_0 \) is composite and \( K(\lambda_2, P_2 ; \lambda_1, P_1) \) is minimum at

\[ \hat{\lambda}_j = \frac{\sum_1^m \pi_{1j}}{\sum_1^m (\pi_{1j} / \lambda_{1j})} = C_{1j}^* \]

and

\[ \hat{p}_{jk} = \frac{\sum_1^m \pi_{1j} p_{1jk}}{\sum_1^m \pi_{1j}} = C_{1jk}^* \]

we have

\[ J(P_2, \lambda_2) = J(P_2) + J(\lambda_2) \quad \ldots \ldots \quad (3.138) \]

where \( J(P_2) \) and \( J(\lambda_2) \) are given by (3.101) and (3.92) respectively.

Now we obtain the exact slope of \( T_n \). Let \( U_n \) be a statistic as defined in (3.8), i.e.,

\[ U_n = -n^{-1} \log \Lambda \]

\[ = -n^{-1} \left\{ \sum_{i,j=1}^m n_{ij} \log \left( \frac{\gamma_{ij} n_{ij}}{Y_j n_{ij}} \right) + \sum_{i,j,k=1}^m n_{ijk} \log \left( \frac{n_{ijk} n_{ij}}{n_{ijk} n_j} \right) \right\} \]

using (3.93), (3.88), (3.98) and (3.102) we see that

\[ U_n + J(\lambda_2, P_2) \text{ a.s.,} \quad (3.109) \]

when a non null \( \lambda_2 \) and \( P_2 \) obtains. Proceeding as in case (i)(a) cf Theorem (3.1) we get as \( n \to \infty \)

\[ n^{-1} \log L_n \to J(\lambda_2, P_2) \text{ a.s.,} \quad (3.100) \]

when a non null \( P_2 \) and \( \lambda_2 \) obtains. Thus (2.9) holds with probability one and hence \( U_n \) and \( T_n \) are optimal in the sense of exact slope by definition (2.3). The exact slope \( 2J(\lambda_2, P_2) \) is given by (3.108).
THEOREM 3.4. The likelihood ratio test for testing the hypothesis
(a) \( H_0^1 \) against \( H_1^1 \), (b) \( H_0^2 \) against \( H_1^2 \) and (c) \( H_0^3 \) against \( H_1^3 \),
specified in Theorem (3.4.1), is optimal in the sense of exact slope.

PROOF: (a) Consider testing the hypothesis \( H_0^1 \) against \( H_1^1 \) of
Theorem (3.4.1)(a). The likelihood ratio criterion is given by (3.4.5).
Taking logarithm on both sides of (3.4.5) and expanding by Taylor's
series we get

\[
T_n = -2 \log A \approx \sum_{i_1, i_2, \ldots, i_r = 1}^{m} n_{i_1 i_2 \ldots i_r} \left( \frac{n_{123 \ldots r} y_{112 \ldots r}^2}{n_{112 \ldots r} y_{112 \ldots r}^2} \right) + O(1/n).
\]

Now, from (2.2) and (2.3) we have

\[
R_n(s, \lambda_r, \lambda_{r-1}) = \prod_{i_1, i_2, \ldots, i_r = 1}^{m} \left( \frac{\lambda_{112 \ldots r}}{\lambda_{123 \ldots r}} \right)^{n_{112 \ldots i_r}}
\]

and

\[
K_n(s, \lambda_r, \lambda_{r-1}) = n^{-1} \left( \sum_{i_1, i_2, \ldots, i_r = 1}^{m} n_{112 \ldots i_r} \log \left( \frac{\lambda_{112 \ldots i_r}}{\lambda_{123 \ldots i_r}} \right) - \sum_{i_1, i_2, \ldots, i_r = 1}^{m} (\lambda_{112 \ldots i_r} - \lambda_{123 \ldots i_r}) y_{112 \ldots i_r} \right) \]
\[
\frac{n}{n_1 n_2 \ldots n_r} \rightarrow \frac{\pi_1 \pi_2 \ldots \pi_r}{n_1 n_2 \ldots n_r} \text{ a.s.,} \quad (3.114)
\]
\[
\frac{\gamma_1 \gamma_2 \ldots \gamma_r}{n_1 n_2 \ldots n_r} \rightarrow \frac{1}{\lambda_1 \lambda_2 \ldots \lambda_r} \text{ a.s.} \quad (3.115)
\]

Hence from (3.114) and (3.115)
\[
\frac{\gamma_1 \gamma_2 \ldots \gamma_r}{n} \rightarrow \frac{\pi_1 \pi_2 \ldots \pi_r}{\lambda_1 \lambda_2 \ldots \lambda_r} \text{ a.s.} \quad (3.116)
\]

Thus by (2.4) and (3.113), using (3.114) and (3.116) as \( n \rightarrow \infty \)
\[
K_n(s, \lambda_r, \lambda_{r-1}) = \sum_{\gamma_1 \gamma_2 \ldots \gamma_r = 1}^{m} \frac{\lambda_1 \lambda_2 \ldots \lambda_r}{\pi_1 \pi_2 \ldots \pi_r / \lambda_1 \lambda_2 \ldots \lambda_r} \log\left(\frac{\lambda_1 \lambda_2 \ldots \lambda_r}{\pi_1 \pi_2 \ldots \pi_r / \lambda_1 \lambda_2 \ldots \lambda_r}\right)
\]

\[
= K(\lambda_r, \lambda_{r-1}) \text{ a.s.,} \quad (3.117)
\]

when a non null \( \lambda_r \) obtains. Since \( H_0 \) is composite and \( K(\lambda_r, \lambda_{r-1}) \)
is minimum at \( \lambda_r = \frac{\sum_{1=1}^{m} \pi_1 \pi_2 \ldots \pi_r}{\sum_{1=1}^{m} (\pi_1 \pi_2 \ldots \pi_r / \lambda_1 \lambda_2 \ldots \lambda_r)} = \gamma^*_{1, \ldots, r} \)
(say) we have by (2.5)
\[
J(\lambda_r) = \sum_{\gamma_1 \gamma_2 \ldots \gamma_r = 1}^{m} \pi_1 \pi_2 \ldots \pi_r \log\left(\frac{\lambda_1 \lambda_2 \ldots \lambda_r}{\gamma^*_{1, \ldots, r}}\right) \quad (3.118)
\]
Now, let $U_n$ be a statistic as defined in (3.8) i.e.,
\[ U_n = -n^{-1} \log \Lambda \]
\[ = -n^{-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq m} n_{i_1 i_2 \ldots i_r} \log \frac{n_{i_1 i_2 \ldots i_r} \gamma_{i_1 i_2 \ldots i_r}}{n_{i_1 i_2 \ldots i_r} \gamma_{i_1 i_2 \ldots i_r}} \]

As $n \to \infty$, we have
\[ \frac{n_{i_1 i_2 \ldots i_r}}{n_{i_1 i_2 \ldots i_r}} \to \gamma_{i_1 i_2 \ldots i_r} \]

and using (3.119) and (3.115) we see that
\[ U_n \to J(\lambda_r) \quad a.s., \quad (3.120) \]
when a non null $\lambda_r$ obtains. Proceeding as in case (1)(a), we get as
\[ n \to \infty \]
\[ n^{-1} \log L_n \to -J(\lambda_r) \quad a.s., \]
when a non null $\lambda_r$ obtains. Thus (2.9) holds with probability one and hence $U_n$ and $T_n$ are optimal in the sense of exact slope. The exact slope $2J(\lambda_r)$ is given by (3.118).

b) Consider testing the hypothesis $H_0^2$ against $H_1^2$ of Theorem (3.4.1)(b).

The likelihood ratio criterion is given by (3.4.9) and $T_n$ is given by (3.4.10).

From (2.2) and (2.3) we have
\[ R_n(s, p_r, p_{r-1}) = \prod_{l=1}^{m} \left( \frac{p_{i_1 i_2 \ldots i_{r+1}}}{p_{i_1 i_2 \ldots i_{r}+1}} \right) \]

where $i_1, i_2, \ldots , i_{r+1} = 1$.
and

\[ K_n(s, P_r, P_{r-1}) = n^{-1} \sum_{i=1}^{m} n_{i1, i2, \ldots, i_{r+1}} \log \left( \frac{p_{i1, i2, \ldots, i_{r+1}}}{p_{i1, i2, \ldots, i_{r+1}}} \right) \]

\[ = \frac{n_{i1, i2, \ldots, i_{r+1}}}{n_{i1, i2, \ldots, i_r}} \rightarrow p_{i1, i2, \ldots, i_{r+1}} \text{ a.s.} \quad (3.122) \]

As \( n \to \infty \) under \( H_1^2 \),

\[ \frac{n_{i1, i2, \ldots, i_{r+1}}}{n_{i1, i2, \ldots, i_r}} \rightarrow p_{i1, i2, \ldots, i_{r+1}} \text{ a.s.} \quad (3.123) \]

and from (3.114) and (3.123)

\[ \frac{n_{i1, i2, \ldots, i_{r+1}}}{n} + \frac{p_{i1, i2, \ldots, i_{r+1}}}{p_{i1, i2, \ldots, i_{r+1}}} \text{ a.s.} \quad (3.124) \]

Thus by (2.4), using (3.124), we get

\[ K_n(s, P_r, P_{r-1}) \rightarrow \sum_{i=1}^{m} \frac{p_{i1, i2, \ldots, i_{r+1}}}{p_{i1, i2, \ldots, i_{r+1}}} \log \left( \frac{p_{i1, i2, \ldots, i_{r+1}}}{p_{i1, i2, \ldots, i_{r+1}}} \right) \]

\[ = K(P_r, P_{r+1}) \text{ a.s.} \quad (3.126) \]

when a non null \( P_r \) obtains. Since \( H_0 \) is composite and \( K(P_r, P_{r-1}) \) is minimum at

\[ \hat{p}_{i1, i2, \ldots, i_{r+1}} = \frac{m}{\sum_{i=1}^{m} \frac{p_{i1, i2, \ldots, i_{r+1}}}{p_{i1, i2, \ldots, i_{r+1}}}} \]

\[ = C^*_{i1, i2, \ldots, i_{r+1}} \text{ (say), we have by (2.5)} \]
\[ J(\pi) = \sum^m \frac{n_{1_2}\cdots r}{n_{12}\cdots r+1} \log \left( \frac{p_{1_2}\cdots r+1}{C^*_{12}\cdots r+1} \right) \quad \ldots \quad (3.127) \]

Now, let \( U_n \) be a statistic as defined in (3.8) i.e.
\[ U_n = -n^{-1} \log \Lambda \]
and using (3.123), (3.124) and (3.125) we see that as \( n \to \infty \)
\[ U_n \to J(\pi) \quad \text{a.s.,} \quad (3.128) \]
when a non null \( \pi \) obtains. Proceeding as in case (i)(a), we get as \( n \to \infty \)
\[ n^{-1} \log L_n \to -J(\pi) \quad \text{a.s.,} \quad (3.129) \]
when a non null \( \pi \) obtains. Thus (2.9) holds with probability one and hence \( U_n \) and \( T_n \) are optimal in the sense of exact slope. The exact slope \( 2J(\pi) \) is given by (3.127).

c) Consider testing the hypothesis \( H_0^3 \) against \( H_1^3 \) of Theorem (3.4.1)(c).
The likelihood ratio criterion is given by (3.4.13), which is the product of (3.4.5) and (3.49). Hence \( T_n \) is given by the sum of (3.111) and (3.4.10).

From (2.2) and (2.3) we have
\[ R_n(s; \lambda_r, \pi_r; \lambda_{r-1}, \pi_{r-1}) = \prod \left( \frac{\lambda_{1_2}\cdots r}{\lambda_{12}\cdots r+1} \right)^{n_{1_2}\cdots r+1} \]
\[ \ldots \quad (3.130) \]
and
\[ K_n(s; \lambda_r, P_r ; \lambda_{r-1}, P_{r-1}) = n^{-1} \sum_{i_1, i_2, \ldots, i_r=1}^{m} \frac{\lambda^1_{i_1} \lambda^2_{i_2} \cdots \lambda^r_{i_r}}{\lambda^1_{1} \lambda^2_{1} \cdots \lambda^r_{1}} \log\left(\frac{\lambda^1_{i_1} \lambda^2_{i_2} \cdots \lambda^r_{i_r}}{\lambda^1_{1} \lambda^2_{1} \cdots \lambda^r_{1}}\right) \]

\[ = K(\lambda_r, P_r ; \lambda_{r-1}, P_{r-1}), \text{ a.s.,} \]

when a non null \( \lambda_r \) and \( P_r \) obtains. From (2.5) we have

\[ J(\lambda_r, P_r) = \inf \{ K(\lambda_r, P_r ; \lambda_{r-1}, P_{r-1}) : (\lambda_{r-1}, P_{r-1}) \in \mathcal{H}_o \} \]
Since $\mathcal{H}_0$ is composite and $K(X^*, P; X_{r-1}^*, P_{r-1})$ is minimum at $\hat{\lambda}_{12\ldots 1r} = C^*_112\ldots 1r$ and $\hat{P}_{12\ldots 1r} = C^*_112\ldots 1r+1$, where $C^*_112\ldots 1r$ is given by (3.119) and $C^*_112\ldots 1r+1$ is given by (3.125), we have

$$J(\hat{\lambda}_{r}, P_r) = J(\lambda_r) + J(P_r)$$

(3.133)

where $J(\lambda_r)$ is given by (3.118) and $J(P_r)$ is given by (3.127).

Now, let $U_n$ be a statistic as defined in (3.8) i.e.,

$$U_n = -n^{-1} \log A = -n^{-1} \left\{ \sum n_112\ldots 1r \log \left( \frac{Y_{112\ldots 1r} n_{112\ldots 1r}}{n_112\ldots 1r} \right) \right\} \log \left( \frac{n_{12\ldots 1r} n_{112\ldots 1r+1}}{n_112\ldots 1r+1} \right)$$

As $n \to \infty$, using (3.119), (3.115), (3.123), (3.124) and (3.125) we see that

$$U_n \to J(P_r, \lambda_r) \quad \text{a.s.},$$

(3.134)

when a non null $P_r$ and $\lambda_r$ obtains. Proceeding as in case (i)(a) of Theorem 3.1 we get as $n \to \infty$

$$n^{-1} \log L_n \to -J(P_r, \lambda_r) \quad \text{a.s.},$$

(3.135)

when a non null $P_r$ obtains. Thus (2.9) holds with probability one and $T_n$ and $U_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope $2J(P_r, \lambda_r)$ is given by (3.133).
THEOREM 3.5. The likelihood ratio test for testing the hypothesis
(a) $H_0^1$ against $H_1^1$, (b) $H_0^2$ against $H_1^2$ and (c) $H_0^3$ against $H_1^3$, specified
in Theorem (3.5.1), is optimal in the sense of exact slope.

PROOF: (a). Consider testing the hypothesis $H_0^1$ against $H_1^1$ of
Theorem (3.5.1). The likelihood ratio criterion is given by (3.5.5)
and $T_n$ is given by (3.5.6).

From (2.2) and (2.3) we have

$$R_n(s, \lambda_3, \lambda_1) = \prod_{i,j,k=1}^m \left( \frac{\lambda_{1jk}}{\lambda_k} \right)^{n_{1jk}} e^{-(\lambda_{1jk} - \lambda_k)\gamma_{1jk}} \quad (3.136)$$

and

$$K_n(s, \lambda_3, \lambda_1) = n^{-1} \left\{ \sum_{i,j,k=1}^m n_{1jk} \log\left(\frac{\lambda_{1jk}}{\lambda_k}\right) - \sum_{i,j,k=1}^m (\lambda_{1jk} - \lambda_k)\gamma_{1jk} \right\} \quad (3.137)$$

As $n \to \infty$, under $H_1^1$ we have

$$\frac{n_{1jk}}{n} \to p_{1jk} \quad \text{a.s.}, \quad (3.138)$$

$$\frac{\gamma_{1jk}}{n_{1jk}} \to 1/\lambda_{1jk} \quad \text{a.s.} \quad (3.139)$$

and hence

$$\frac{\gamma_{1jk}}{n} \to p_{1jk}/\lambda_{1jk} \quad \text{a.s.}, \quad (3.140)$$

also

$$\frac{n_k}{n_k} \to \frac{\sum_{j=1}^m \left( \frac{p_{1jk}}{\lambda_{1jk}} \right)}{\sum_{j=1}^m \frac{n_{1jk}}{\lambda_{1jk}}} = \frac{1}{C_{1jk}} \quad \text{(say).} \quad (3.141)$$

Thus by (2.4) and (3.138), using (3.138) and (3.140) we have as $n \to \infty$
\begin{align*}
K_n(s, \lambda_3, \lambda_1) &= \frac{m}{1, J, k=1} \pi_1 j k \log \frac{\lambda_{1 j k}}{\lambda_k} - \frac{m}{1, J, k=1} (\lambda_{1 j k} - \lambda_k) \frac{\pi_1 j k}{\lambda_{1 j k}} \\
&= K(\lambda_3, \lambda_1) \quad \text{a.s.,} \quad (3.142)
\end{align*}

when a non null \( \lambda_3 \) obtains. From (2.5) we have

\[ J(\lambda_3) = \inf \{ K(\lambda_3, \lambda_1) : \lambda_1 \in \Theta_0 \} \]

Since \( \Theta_0 \) is composite and \( K(\lambda_3, \lambda_1) \) is minimum at

\[ \lambda_k = \frac{m}{1, J, k=1} \pi_1 j k / \frac{m}{1, J, k=1} (\pi_1 j k / \lambda_{1 j k}) = \lambda^{*}_{1 j k} , \]

we have

\[ J(\lambda_3) = \frac{m}{1, J, k=1} \pi_1 j k \log(\lambda_{1 j k} / \lambda^{*}_{1 j k}) \quad (3.143) \]

Now, let \( U_n \) be a statistic as defined in (3.8) i.e.

\[ U_n = -n^{-1} \log \Lambda = -n^{-1} \left\{ \sum_{1, J, k=1} \pi_1 j k \log \left( \frac{n_k Y_{1 j k}}{\pi_1 j k Y_k} \right) \right\} \]

Using (3.138), (3.139) and (3.141) as \( n \to \infty \) we see that

\[ U_n \to J(\lambda_3) \quad \text{a.s.,} \quad (3.144) \]

when a non null \( \lambda_3 \) obtains. Proceeding as in case (1)(a) of Theorem (3.1) we get as \( n \to \infty \)

\[ n^{-1} \log L_n \to -J(\lambda_3) \quad \text{a.s.,} \]

when a non null \( \lambda_3 \) obtains. Thus (2.9) holds with probability one and hence \( U_n \) and \( T_n \) are optimal in the sense of exact slope by definition (2.3). The exact slope is \( 2J(\lambda_3) \) and \( J(\lambda_3) \) is given by (3.143).
b) Consider testing the hypothesis $H_0^2$ against $H_1^2$ of Theorem (3.5.1)(b). The likelihood ratio criterion is given by (3.5.19) and $T_n$ is given by

$$T_n = -2 \log \Lambda \approx \sum_{i,j,k,l=1}^{m} \frac{(n_{ijkl} n_k - n_{k1} n_{1jk})^2}{n_{ijkl}^2 n_k^2}$$

From (2.2) and (2.3) we have

$$R_n(s, p_3, p_1) = \prod_{i,j,k,l=1}^{m} \left( \frac{p_{ijkl}}{p_{k1}} \right)$$

and

$$K_n(s, p_3, p_1) = n^{-1} \left\{ \sum_{i,j,k,l=1}^{m} n_{ijkl} \log \left( \frac{p_{ijkl}}{p_{k1}} \right) \right\} \quad (3.145)$$

As $n \to \infty$, under $H_1^2$ we have

$$\frac{n_{ijkl}}{n_{i1jk}} + p_{ijkl} \quad (3.146)$$

From (3.138) and (3.146)

$$\frac{n_{ijkl}}{n} = \pi_{i1jk} p_{ijkl} \quad (3.147)$$

and also

$$\frac{n_{kl}}{n_k} + \sum_{j=1}^{m} \frac{\pi_{1jk} p_{ijkl}}{\pi_{i1jk} p_{ijkl}} = c_{i1jk}^* \quad (3.148)$$

By (2.4) and (3.145), using (3.147) as $n \to \infty$

$$K_n(s, p_3, p_1) \to \sum_{i,j,k,l} n_{ijkl} \pi_{i1jk} p_{ijkl} \log \left( \frac{p_{ijkl}}{p_{k1}} \right)$$
when a non null $P_3$ obtains. From (2.5)

$$\mathcal{J}(P_3) = \inf \{K(P_3, P_1) : P_1 \in \mathcal{H}_0\}$$

Since $\mathcal{H}_0$ is composite and $K(P_3, P_1)$ is minimum at

$$P_{k1} = \frac{\sum_{i,j=1}^{m} \pi_{ijk} P_{ijkl}}{\sum_{i,j,k,l=1}^{m} \pi_{ijk} P_{ijkl}} = c_{ijk}^* \text{ (say), we have}$$

$$\mathcal{J}(P_3) = \mathcal{J}(P_0) = -\frac{1}{m} \sum_{i,j,k,l=1}^{m} \pi_{ijk} P_{ijkl} \log\left(\frac{P_{ijkl}}{c_{ijk}^*}\right) \quad (3.150)$$

Now, let $U_n$ be a statistic as defined in (3.8), i.e.

$$U_n = -\frac{1}{n-1} \log A$$

$$= -\frac{1}{n-1} \left\{ \sum_{i,j,k,l=1}^{m} \pi_{ijkl} \log\left(\frac{n_{ijkl}}{n_{ikl}} \cdot \frac{n_{ijk}}{n_{k}}\right) \right\}$$

Using (3.147), (3.146) and (3.148), as $n \to \infty$ we see that

$$U_n \to \mathcal{J}(P_3) \quad \text{a.s.,} \quad (3.151)$$

when a non null $P_3$ obtains. Proceeding as in case (1)(a) of Theorem (3.1) we get as $n \to \infty$

$$n^{-1} \log L_n \to -\mathcal{J}(P_3) \quad \text{a.s.,} \quad (3.152)$$

when a non null $P_3$ obtains. Thus (2.9) holds with probability one and $T_n$ is optimal in the sense of exact slope. The exact slope is $2\mathcal{J}(P_3)$ and $\mathcal{J}(P_3)$ is given by (3.150).
c) Consider testing the hypothesis $H_0^3$ against $H_1^3$ of Theorem (3.5.1)(c). The likelihood ratio criterion is given by (3.5.22) and $T_n$ is given by (3.5.23).

From (2.2) and (2.3) we have

$$R_n(s; \lambda_3, \lambda_1, P_1) = \prod_{i,j,k=1}^m \left( \frac{\lambda_{ijk}}{\lambda_k} \right)^{n_{ijk}} e^{-(\lambda_{ijk} - \lambda_k)Y_{ijk}}$$

$$R_n(s; \lambda_3, \lambda_1, P_1) = \prod_{i,j,k=1}^m \left( \frac{\lambda_{ijk}}{\lambda_k} \right)^{n_{ijk}} e^{-(\lambda_{ijk} - \lambda_k)Y_{ijk}}$$

and

$$K_n(s; \lambda_3, \lambda_1, P_1) = n^{-1} \left\{ \sum_{i,j,k=1}^m n_{ijk} \log \frac{\lambda_{ijk}}{\lambda_k} - \sum_{i,j,k=1}^m (\lambda_{ijk} - \lambda_k)Y_{ijk} + \sum_{i,j,k=1}^m n_{ijk} \log \left( \frac{\lambda_{ijk}}{\lambda_k} \right) \right\}$$

$$K_n(s; \lambda_3, \lambda_1, P_1) = n^{-1} \left\{ \sum_{i,j,k=1}^m n_{ijk} \log \frac{\lambda_{ijk}}{\lambda_k} - \sum_{i,j,k=1}^m (\lambda_{ijk} - \lambda_k)Y_{ijk} + \sum_{i,j,k=1}^m n_{ijk} \log \left( \frac{\lambda_{ijk}}{\lambda_k} \right) \right\}$$

... (3.154)

As $n \to \infty$, by (2.4) and (3.154) using (3.138), (3.140) and (3.147), we have

$$K_n(s; \lambda_3, \lambda_1, P_1) = K(\lambda_3, \lambda_1) + K(P_3, P_1)$$

$$K_n(s; \lambda_3, \lambda_1, P_1) = K(\lambda_3, \lambda_1) + K(P_3, P_1)$$

when a non null $\lambda_3$ and $P_3$ obtains. From (2.5)

$$J(\lambda_3, P_3) = \inf \{ K(\lambda_3, \lambda_1, P_1) : (\lambda_1, P_1) \in \Theta_0 \}$$

Since $\Theta_0$ is composite and $K(\lambda_3, P_3; \lambda_1, P_1)$ is minimum at

$\hat{\lambda}_k = c_{1jk}^*$ and $\hat{p}_{kl} = c_{1jk1}^*$ where $c_{1jk}^*$ and $c_{1jk1}^*$ are given by (3.141) and (3.148) respectively, we have
where $J(\lambda_3)$ is given by (3.143) and $J(P_3)$ is given by (3.150).

Now, let $U_n$ be a statistic as defined in (3.8), i.e.

$$U_n = -n^{-1} \log L_n$$

and as $n \to \infty$ using (3.139), (3.141), (3.138), (3.147), (3.146) and (3.148) we see that

$$U_n \to J(\lambda_3, P_3) \quad \text{a.s.,}$$

when a non null $\lambda_3$ and $P_3$ obtains. Proceeding as in case (i)(a) we see that as $n \to \infty$

$$n^{-1} \log L_n = -J(\lambda_3, P_3)$$

with probability one when a non null $\lambda_3$ and $P_3$ obtains. Thus (2.9) holds with probability one and hence $T_n$ and $U_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope is $2J(\lambda_3, P_3)$ and $J(\lambda_3, P_3)$ is given by (3.156).

**THEOREM 3.6.** The likelihood ratio test for testing the hypothesis (a) $H_0^1$ against $H_1^1$, (b) $H_0^2$ against $H_1^2$ and (c) $H_0^3$ against $H_1^3$, specified in Theorem (3.6.1) is optimal in the sense of exact slope.

**PROOF:** (a) Consider testing the hypothesis $H_0^1$ against $H_1^1$ of Theorem (3.6.1)(a). The likelihood ratio criterion is given by (3.6.3) and $T_n$ is given by (3.6.4).
From (2.2) and (2.3) we have

\[
R_n(s; \lambda_r, \lambda_u) = \prod_{i=1}^{m} \left( \frac{\lambda_{1,1}^{1,2} \cdots 1_r}{\lambda_{1,1}^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r} \right) \prod_{i=1}^{m} \left( \frac{1_r^{1,2} \cdots 1_r}{1_r^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r} \lambda_{1,1}^{1,2} \cdots 1_r \right)
\]

and

\[
K_n(s; \lambda_r, \lambda_u) = n^{-1} \left\{ \prod_{i=1}^{m} \left( \frac{\lambda_{1,1}^{1,2} \cdots 1_r}{\lambda_{1,1}^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r} \right) \prod_{i=1}^{m} \left( \frac{1_r^{1,2} \cdots 1_r}{1_r^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r} \lambda_{1,1}^{1,2} \cdots 1_r \right) \right\}
\]

As \( n \to \infty \) by (2.4) and (3.160) using (3.114), (3.116), we have

\[
K_n(s; \lambda_r, \lambda_u) = \sum_{i=1}^{m} \pi_{1,1}^{1,2} \cdots 1_r \log \left( \frac{\lambda_{1,1}^{1,2} \cdots 1_r}{\lambda_{1,1}^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r} \right) \]

\[
= \sum_{i=1}^{m} \pi_{1,1}^{1,2} \cdots 1_r \left( \lambda_{1,1}^{1,2} \cdots 1_r - \lambda_{1,1}^{1,2} \cdots 1_r \right) \frac{\pi_{1,1}^{1,2} \cdots 1_r}{\lambda_{1,1}^{1,2} \cdots 1_r}
\]

when a non null \( \lambda_r \) obtains. From (2.5)

\[
J(\lambda_r) = \inf \{ K(\lambda_r, \lambda_u) : \lambda_u \in \Theta_0 \}
\]

Since \( \Theta_0 \) is composite and \( K(\lambda_r, \lambda_u) \) is minimum at

\[
\lambda_{1,1}^{1,2} \cdots 1_r = \left\{ \sum_{i=1}^{m} \pi_{1,1}^{1,2} \cdots 1_r \lambda_{1,1}^{1,2} \cdots 1_r \right\} / \sum_{i=1}^{m} \pi_{1,1}^{1,2} \cdots 1_r
\]

\[
= C_{1,1}^{1,2} \cdots 1_r \quad \text{(say), we have}
\]
Now, let $U_n$ be a statistic as defined in (3.8), i.e.

$$U_n = -n^{-1} \log \Lambda$$

$$= -n^{-1} \sum_{1 \leq i_1, i_2, \ldots, i_r \leq n} n_{i_1 \ldots i_r} \log \frac{n_{i_1 \ldots i_r - u i_4 \ldots i_r}}{n_{i_1 \ldots i_r - u i_4 \ldots i_r}}$$

As $n \to \infty$, under $H^1$, we have

$$E \left( \frac{\gamma_{i_1 \ldots i_r}}{\lambda_{i_1 \ldots i_r}} \right) = 1$$

and hence using (3.163), (3.114) and (3.115) we see that

$$U_n \to J(\lambda_r) \text{ a.s.,} \quad (3.164)$$

when a non null $\lambda_r$ obtains. Proceeding as in case (1)(a), we get

$$n^{-1} \log L_n \to -J(\lambda_r) \text{ a.s.,} \quad (3.165)$$

when a non null $P_r$ obtains. Thus (2.9) holds with probability one and hence $U_n$ and $T_n$ are optimal in the sense of exact slope by definition (2.3). The exact slope is $2J(\lambda_r)$ and $J(\lambda_r)$ is given by (3.161).
b) Consider testing the hypothesis $H_0^2$ against $H_1^2$ of Theorem (3.6.1)(b).

The likelihood ratio criterion is given by (3.6.7) and $T_n$ is given by

$$T_n = -2 \log \Lambda \approx \sum_{l_1, l_2, \ldots, l_r+1} n_{l_1} l_2 \ldots l_r+1 \left( \frac{\hat{p}_{l_1 \ldots l_r+1} - \hat{p}_{l_1 \ldots l_r+1}}{\hat{p}_{l_1 \ldots l_r+1}} \right)^2$$

... (3.166)

From (2.2) and (2.3) we have

$$R_n(s, P_r, P_u) = \prod_{l_1, l_2, \ldots, l_r+1} \left( \frac{p_{l_1 \ldots l_r+1}}{p_{l_1 \ldots l_r+1}} \right)^{n_{l_1} l_2 \ldots l_r+1}$$

... (3.167)

and

$$K_n(s, P_r, P_u) = n^{-1} \left( \sum_{l_1, l_2, \ldots, l_r+1} n_{l_1} l_2 \ldots l_r+1 \log \left( \frac{p_{l_1 \ldots l_r+1}}{p_{l_1 \ldots l_r+1}} \right) \right)$$

... (3.168)

As $n \to \infty$, by (2.4) and (3.168), using (3.124) we have

$$K_n(s, P_r, P_u) = \sum_{l_1, l_2, \ldots, l_r+1} p_{l_1 \ldots l_r+1} \log \left( \frac{p_{l_1 \ldots l_r+1}}{p_{l_1 \ldots l_r+1}} \right)$$

... (3.169)

when a non null $P_r$ obtains. From (2.5),

$$J(P_r) = \inf \{ K(P_r, P_u) \ , \ P_u \in \mathcal{H}_0 \}$$

Since $\mathcal{H}_0$ is composite and $K(P_r, P_u)$ is minimum at

$$C^{**}_{l_1 l_2 \ldots l_r+1} = \frac{\sum_{l_1, l_2, \ldots, l_r+1=1}^{m} p_{l_1 l_2 \ldots l_r+1}}{\sum_{l_1, l_2, \ldots, l_r+1=1}^{m} p_{l_1 l_2 \ldots l_r+1}}$$

... (3.169)

we have

$$C^{**}_{l_1 l_2 \ldots l_r+1}$$
\[ J(P_r) = \sum_{i_1, i_2, \ldots, i_{r+1}} \log \left( \frac{p_{i_1 i_2 \cdots i_{r+1}}}{c_{i_1 i_2 \cdots i_{r+1}}} \right) \]

Let \( U_n \) be a statistic as defined in (3.8) i.e.

\[ U_n = -n^{-1} \log \Lambda \]

\[ = -n^{-1} \left( \sum_{i_1, i_2, \ldots, i_{r+1}} \log \left( \frac{n_{i_1 i_2 \cdots i_{r+1}}^{n_{i_{r+1}}} \cdots n_{r+1}^{n_{r+1}}}{n_{1}^{n_{1}} \cdots n_{r+1}^{n_{r+1}}} \right) \right) \]

As \( n \to \infty \), under \( H_1^2 \) we have

\[ \frac{n_{r+1}^{n_{r+1}} \cdots n_{r+1}^{n_{r+1}}}{n_{r}^{n_{r}} \cdots n_{r}^{n_{r}}} \to \left( \sum_{i_1, i_2, \ldots, i_{r+1}} \log \left( \frac{p_{i_1 i_2 \cdots i_{r+1}}}{c_{i_1 i_2 \cdots i_{r+1}}} \right) \right) = C^* \]

\[ \frac{n_{r+1}^{n_{r+1}} \cdots n_{r+1}^{n_{r+1}}}{n_{r}^{n_{r}} \cdots n_{r}^{n_{r}}} \to \left( \sum_{i_1, i_2, \ldots, i_{r+1}} \log \left( \frac{p_{i_1 i_2 \cdots i_{r+1}}}{c_{i_1 i_2 \cdots i_{r+1}}} \right) \right) = C^* \]

Using (3.124), (3.123) and (3.171) we see that

\[ U_n \to J(P_r) \quad \text{a.s.} \quad (3.172) \]

with probability one, when a non null \( P_r \) obtains. Proceeding as in case (i)(a), we get as \( n \to \infty \)

\[ n^{-1} \log L_n = -J(P_r) \quad \text{a.s.,} \quad (3.173) \]

when a non null \( P_r \) obtains. Thus (2.9) holds with probability one and hence \( T_n \) and \( U_n \) are optimal in the sense of exact slope by definition (2.3). The exact slope is \( 2J(P_r) \) and \( J(P_r) \) is given by (3.170).
C) Consider testing the hypothesis \( H_0^3 \) against \( H_1^3 \) of Theorem (3.6.1)(a). The likelihood ratio criterion is given by (3.6.10) which is the product of (3.6.3) and (3.6.7) and hence \( T_n \) is given by the sum of (3.6.4) and (3.166).

From (2.2) and (2.3) we have:

\[
R_n(s; \lambda_r, P_r; \lambda_u, P_u) = \prod_{1,1'2,\ldots, r+1} \frac{\lambda^{1_i2_i\ldots r_i+1}}{1^{u-r_i+1}} \left( \frac{\lambda^{1_i2_i\ldots r_i-1}}{1^{u-r_i+1}} \right) \gamma^{1_i2_i\ldots r_i+1}_{1,1'}
\]

and

\[
K_n(s; \lambda_r, P_r; \lambda_u, P_u) = n^{-1} \left\{ \sum_{1,1'2,\ldots, r+1} \log\left( \frac{\lambda^{1_i2_i\ldots r_i+1}}{1^{u-r_i+1}} \right) \right\} - \sum_{1,1'2,\ldots, r+1} \left( \frac{\lambda^{1_i2_i\ldots r_i-1}}{1^{u-r_i+1}} \right) \gamma^{1_i2_i\ldots r_i+1}_{1,1'}
\]

As \( n \to \infty \), by (2.4) and (3.175), using (3.114), (3.116) and (3.124) we get
\[ K_n(s; \lambda_r, P_r; \lambda_u, P_u) = K(\lambda_r, \lambda_u) + K(P_r, P_u) \]
\[ = K(\lambda_r, \lambda_r; \lambda_u, \lambda_u) \quad \text{a.s.,} \quad (3.176) \]
when a non null \( \lambda_r \) and \( P_r \) obtains. From (2.5)
\[ J(\lambda_r, P_r) = \inf \{ K(\lambda_r, P_r; \lambda_u, \lambda_u) : (\lambda_u, \lambda_u) \in \Theta \} \]
Since \( \Theta \) is composite and \( K(\lambda_r, P_r; \lambda_u, \lambda_u) \) is minimum at
\[ \lambda_r = \hat{\lambda}_{r-1} \ldots \lambda_r = \hat{\lambda}_{r-1} \ldots \lambda_r \]
and \( \hat{p}_{1-r} \) is given by (3.163) and \( \hat{c}_{1-r} \) is given
(3.171), we get
\[ J(\lambda_r, P_r) = J(\lambda_r) + J(P_r) \quad (3.177) \]
Let \( U_n \) be a statistic as defined in (3.8), i.e.
\[ U_n = -n^{-1} \log \Lambda \]
\[ = -n^{-1} \left( \sum_{i=1}^{r} \frac{n_{1}^{1} \ldots i_{r}}{n_{1}^{1} \ldots i_{r}} \log \left( \frac{\gamma_{1}^{1} \ldots i_{r}}{\gamma_{1}^{1} \ldots i_{r}} \right) \right) \]
\[ + \sum_{i=1}^{r+1} \frac{n_{1}^{1} \ldots i_{r+1}}{n_{1}^{1} \ldots i_{r+1}} \log \left( \frac{\gamma_{1}^{1} \ldots i_{r+1}}{\gamma_{1}^{1} \ldots i_{r+1}} \right) \]
Using (3.114), (3.115), (3.16), (3.124), (3.123) and (3.171) we
see that
\[ U_n + J(\lambda_r, P_r) \quad \text{a.s.,} \quad (3.178) \]
with probability one when a non null \( \lambda_r \) and \( P_r \) obtains. Proceeding
as in case (i)(a), we get as \( n \to \infty \)
\[ n^{-1} \log L_n - J(\lambda_r, P_r) \quad \text{a.s.,} \quad (3.179) \]
when a non null \( \lambda_r \) and \( P_r \) obtains. Thus (2.9) holds with probability
one and hence \( T_n \) is optimal in the sense of exact slope by definition
(2.3). The exact slope is \( 2J(P_r, \lambda_r) \) and is given by (3.177).
We now proceed to obtain the approximate slopes of all these likelihood ratio tests for testing the hypotheses considered in chapter II and III.

4. Bahadur efficiency of tests in the sense of approximate slope

In this section we consider the Markov process observed continuously over a period \((0,T)\) under scheme 1. In this case total number of transitions \(n\) is fixed and total time of observation \(T\) is a r.v. We obtain the approximate slope of the likelihood ratio tests for testing the hypotheses specified in the theorems of Chapter II and III. We obtain the results as particular cases of the following general theorem, which we prove on the lines similar to that of Bahadur (1967).

**THEOREM 4.1.** Let \(T_n\) be a test statistic. If

1. \(T_n\) has asymptotic chi-square distribution with \(p\) degrees of freedom \(\chi^2(p)\), under the null hypothesis \(H_0\),

2. \(U_n = T_n / 2n \to J(\theta)\) a.s., when a non null \(\theta\) obtains, where \(J(\theta)\) is as defined in (2.5), then \(T_n\) and \(U_n\) are optimal in the sense of approximate slope. The approximate slope is given by \(2J(\theta)\).

**PROOF.** Let \(L_n^{(a)}\) denote the approximate level attained by \(T_n\) as defined in (2.11). To prove the optimality we must show that, as \(n \to \infty\)

\[
    n^{-1} \log L_n^{(a)} \to - J(\theta) \quad \text{a.s.,} \quad (4.1)
\]

when a non null \(\theta\) obtains.
Now we proceed to obtain the approximate level attained by $U_n$. We have
\[
P_{\theta_0} [ U_n \geq t ] = P_{\theta_0} [ T_n \geq 2nt ]. \tag{4.2}
\]
Since, by condition (ii), $T_n$ has asymptotic $\chi^2(p)$ distribution under the null hypothesis $H_0$, we have as $n \to \infty$
\[
P_{\theta_0} [ U_n \geq t ] = P_{\theta_0} [ X \geq 2nt ] \tag{4.3}
\]
where $X \sim \chi^2(p)$ distribution. Hence
\[
P_{\theta_0} [ U_n \geq t ] = \frac{1}{\Gamma p/2} \int_{2nt}^{\infty} e^{-x/2} x^{p/2-1} dx + o(1/n) \tag{4.4}
\]
Thus as $n \to \infty$, $U_n$ has also asymptotic $\chi^2(p)$ distribution and hence the approximate level attained by $U_n$ is the same as the level attained by $T_n$. To simplify the integral in (4.4) we use the following result.
\[
\int_{u}^{\infty} x^{\alpha-1} e^{-\beta x} dx = \beta^{-1} u^{\alpha-1} e^{-\beta u} [ 1 + o(1/u) ]
\]
for large $u$. Using this result with $\beta = 1/2$ and $\alpha = p/2$ we get for large $n$
\[
P_{\theta_0} [ U_n \geq t ] = \frac{1}{\Gamma p/2} e^{-nt} (nt)^{p/2-1} + o(1/n) \tag{4.5}
\]
From (2.10) and (2.11), (4.5) is written as
\[
I_n^{(a)} (t) = G(t) = \frac{1}{\Gamma p/2} e^{-nt} (nt)^{p/2-1}
\]
or
\[
I_n^{(a)} ( U_n ) = \frac{1}{\Gamma p/2} e^{-nU_n} (nU_n)^{p/2-1} \tag{4.6}
\]
Thus as $n \to \infty$, we see that $I_n^{(a)}$ tends to zero at an exponential rate.
Since $U_n \to J(\theta)$ a.s. by condition (ii), we get as $n \to \infty$
\begin{align*}
n^{-1} \log L(a) & \to -J(\theta) \quad \text{a.s.,}
\end{align*}
when a non null $\theta$ obtains. Hence by definition (2.2), $T_n$ and $U_n$ are optimal in the sense of approximate slope. The approximate slope is given by $2J(\theta)$.

It may be seen that the likelihood ratio tests obtained in Chapter II and Chapter III satisfy both the conditions given in the Theorem 4.1. Hence they are optimal in the sense of approximate slope. We may also note that the exact slopes and approximate slopes of these likelihood ratio tests the same.

5. Bahadur efficiency of tests of Markov process $O$ observed under Scheme 2.

In this section we consider a Markov process observed under Scheme 2. In this case total number of transitions $n$ is a r.v. and total time of observation $T$ is fixed. Let there be $n$ transitions in $(0,T)$, starting initially from state $i_0$. During $(t_j, t_{j+1})$, let the process be in state $i_j$ ($j = 1, 2, \ldots, n-1$) and during $(t_n, T)$ let the process be in state $i_n$. In this section we consider all the hypotheses specified in Chapter II and III and show that they are Bahadur optimal in the sense of exact slope and also in the sense of approximate slope.

Adke and Manjunath (1984) have shown that asymptotically as $T \to \infty$

\begin{equation}
\frac{n_{ij}}{T} \to \pi_j \lambda_{ij}
\end{equation}

where $\pi_j$'s ($\pi_j > 0$, $\Sigma \pi_j = 1$) are the stationary probabilities. Thus as $T \to \infty$, $n \to \infty$. 

As explained in section 4 of Chapter II we get the likelihood function for the Markov process given by (2.4.3) since contribution \((T - t_n)\) becomes negligible as \(T \to \infty\). We discuss the Bahadur optimality of the tests for the process observed under scheme 2 in the following theorem.

**THEOREM 5.1.** The likelihood ratio test for testing the hypotheses specified in Theorems (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5) and (2.2.6) for testing the Markov process observed under scheme 2 is optimal in the sense of exact slope.

**PROOF.** Consider testing the hypothesis \(H_0\) against \(H_1\) of Theorem (2.2.1)(a). Since we consider the information up to \(t_n\) only, the contribution of \((T - t_n)\) becomes negligible as \(T \to \infty\). The likelihood ratio criterion is given by (2.2.6) and \(-2 \log \Lambda\) is given by (2.2.7).

From (5.1) we have

\[
\frac{n}{T} = \sum_{i,j=1}^{m} \pi_{ij} = \bar{\lambda} \quad \text{as} \quad (5.2)
\]

Since \(T\) is fixed and \(n\) is random, in this case we use \(T\) as the norm and hence the generalized Kullback-Leibler number is given by

\[
K^T(\lambda^%, \lambda^0) = \bar{\lambda} K(\lambda^%, \lambda^0) \quad \text{where} \quad K(\lambda^%, \lambda^0) \quad \text{is given by} \quad (3.6)
\]

and \(\bar{\lambda}\) is given by (5.2). Also from (2.5) we get

\[
J^T(\lambda^%) = \bar{\lambda} J(\lambda^%) \quad \text{with} \quad J(\lambda^%) \quad \text{given by} \quad (3.7)
\]

Now, we obtain the exact slope of \(T_n = -2 \log \Lambda\). Let \(U_T\) be as defined in (3.8), i.e.

\[
U_T = T^{-1} \log \Lambda \quad , \quad (5.3)
\]

where \(\Lambda\) is given by (2.2.6). \(U_T\) is equivalent to \(T_n\). To get the
optimality of $T_n$ in the sense of exact slope, we must show that

$$T^{-1} \log L_T \rightarrow J^f(\hat{\lambda}) \quad \text{a.s.}, \quad (5.4)$$

when a non null $\lambda^f$ obtains. $L_T$ is the level attained by $U_T$ and also by $T_n$. We note that $L_T$ is the same as $L_n$ obtained as in case (i)(a) of Theorem (3.1) because $L_T$ is the level attained by $T_n$ given by (2.2.7). As shown in case (i)(a) of Theorem (3.1) we have as $(T \rightarrow \infty)$

$$n^{-1} \log L_T \rightarrow -J(\hat{\lambda}) \quad \text{a.s.}, \quad (5.5)$$

when a non null $\lambda^f$ obtains. Hence from (5.2), (5.3), (5.4) and (5.5) we have as $T \rightarrow \infty$

$$T^{-1} \log L_T \rightarrow -\hat{c} J^f(\hat{\lambda})$$

$$= -J^f(\hat{\lambda}) \quad , \quad (5.6)$$

when a non null $\lambda^f$ obtains. Thus $T_n$ given by (2.2.7) is optimal in the sense of exact slope.

Berk and Brown (1978) have used a random norm for the convergence of the level attained by a test statistic. Hence the result (5.5) holds even if $n$ is random as shown by Berk and Brown (1978).

The proof of the remaining results follows along the same lines as in case (i)(a).

By the same argument as in case (i)(a) of Theorem (5.1), Bahadur optimality of the likelihood ratio tests for testing the hypotheses specified in Chapter III follows.

Similarly all these likelihood ratio tests for testing hypotheses specified in Chapter II and III when the process is observed
under Scheme 2 have asymptotic chi-square distribution as shown in section 4 of Chapter II and section 7 of Chapter III. Their Bahadur optimality in the sense of exact slope follows along the same lines as in section 4 and also as in the Theorem (4.1).

6. Concluding remark

In this chapter we have obtained the approximate and also the exact slope of the likelihood ratio tests for testing the hypotheses about continuous time finite state Markov chains when it is observed under both the schemes 1 and 2. As remarked by Bahadur (1967) we see that for these likelihood ratio tests the approximate slope and the exact slope are equal.

Bahadur relative efficiency is defined by the ratio of the slopes. In other words, let \( c_1 \) and \( c_2 \) denote the slopes of two test statistics \( T_1 \) and \( T_2 \). Then Bahadur relative efficiency is defined as

\[
E_{12} = \frac{c_1}{c_2}
\]

If \( E_{12} = 1 \), then \( T_1 \) and \( T_2 \) are said to be equally efficient. If \( E_{12} < 1 \), then \( T_2 \) is said to be more efficient than \( T_1 \) and if \( E_{12} > 1 \), then \( T_1 \) is said to be more efficient than \( T_2 \). If \( c_1 \) and \( c_2 \) are strong exact slopes, then \( E_{12} \) is called strong Bahadur relative efficiency. If \( c_1 \) and \( c_2 \) are weak exact slopes then \( E_{12} \) is called weak Bahadur relative efficiency.

Here we note that the exact and the approximate slope of these tests for Markov process observed under scheme 1 is the same as the exact and the approximate slope of these tests for the process observed under scheme 2. Hence we conclude that the two tests are equally efficient.