CHAPTER I

FUNCTIONS $f(z)$ FOR WHICH $zf'(z)$ IS $\alpha$ -SPIRAL OF ORDER $\beta$

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the unit disk $E = \{ z : |z| < 1 \}$ and satisfy the conditions $f(0) = 0$ and $f'(0) = 1$. For each real number $\alpha \in (-\pi/2, \pi/2)$, let $P(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ in $\mathcal{A}$ which satisfy the condition

$$\text{Re} \left\{ \frac{e^{i\alpha} f'(z)}{zf(z)} \right\} > 0 \text{ for } z \in E$$

and let $\mathcal{F} = \bigcup_{|\alpha| < \pi/2} P(\alpha)$. Functions of the class $\mathcal{F}$ are called spirallike functions in $E$ and Špáček showed that all spirallike functions are univalent in $E$. For each $\alpha \in (-\pi/2, \pi/2)$, we call functions of $P(\alpha)$ as $\alpha$ -spirallike functions.
For $a \in (-\pi/2, \pi/2)$ and $\beta \in [0,1)$,
let $F(a, \beta)$ denote the class of functions $f(z)$
belonging to $\mathcal{A}$ which satisfy the condition

$$\Re \left\{ e^{ia} \frac{f'(z)}{f(z)} \right\} > \beta \cos a \quad \text{for } z \in E.$$ 

Clearly $F(a, \beta) \subseteq F(a)$ and $F(a, 0) \equiv F(a)$.
Also, $F(0, \beta)$ is the well-known class $S^*(\beta)$ of
starlike functions of order $\beta$. We call functions of
$F(a, \beta)$ as $a$-spiral functions of order $\beta$.

The class $F(a, \beta)$ was introduced and studied
by Libera [13].

For $a \in (-\pi/2, \pi/2)$ and $\beta \in [0,1)$, we
denote by $G(a, \beta)$ the class of functions $f(z)$
belonging to $\mathcal{A}$ which satisfy the condition

$$\Re \left\{ e^{ia} \left( 1 + z \frac{f''(z)}{f'(z)} \right) \right\} > \beta \cos a \quad \text{for } z \in E.$$
It is clear that \( f(z) \in G(\alpha, \beta) \) if and only if \( zf'(z) \in F(\alpha, \beta) \). The class \( G(\alpha, 0) \) was studied by Robertson [26]. \( G(0, \beta) \) is the well-known class \( K(\beta) \) of convex functions of order \( \beta \).

In this chapter we investigate the class \( G(\alpha, \beta) \) and amongst other results, we obtain the sharp radius of close-to-convexity of the class \( G(\alpha, \beta) \). We first obtain certain transformations which preserve memberships in \( F(\alpha, \beta) \) and \( G(\alpha, \beta) \).

**Lemma 1:** Let \( x \in E \). For \( f(z) \in F(\alpha, \beta) \), let

\[
 f'_1(z) = \frac{xf\left( \frac{z + x}{1 + xz} \right)}{f(x)(z + x)(1 + xz)^\gamma} \quad \text{for } z \in E, \quad (1)
\]

where \( \gamma = 2(1 - \beta) \cos \alpha \quad e^{-i\alpha} - 1 \).

Then \( f'_1(z) \in F(\alpha, \beta) \).
PROOF: Let \( \rho \) be a real number, \( 0 < \rho < 1 \), and let

\[
 f_\rho(z) = \frac{xzf\left(\rho \frac{z + x}{1 + \bar{z}z}\right)}{f(\rho x)(z + x)(1 + \bar{z}z)^c}, \tag{2}
\]

for \( |z| \leq 1 \). Clearly \( f_\rho(z) \) is analytic in \( |z| \leq 1 \) and \( f_\rho(z) \in \mathcal{A} \).

Logarithmic differentiation yields

\[
 \frac{f_\rho'(z)}{f_\rho(z)} = 1 + \frac{f'(\rho \frac{z + x}{1 + \bar{z}z})}{f(\rho \frac{z + x}{1 + \bar{z}z})} \rho \bar{z} \frac{1 - |x|^2}{(1 + \bar{z}z)^2} - \frac{z}{z + x} - \frac{\bar{\alpha}xz}{1 + \bar{z}z}.
\]

Thus,

\[
 \frac{f_\rho'(z)}{f_\rho(z)} = \frac{z(1 - |x|^2)}{(z + x)(1 + \bar{z}z)} + \frac{\bar{\alpha}xz}{f(w)} + \frac{x}{f(z + x) - \frac{\bar{\alpha}xz}{1 + \bar{z}z}}. \tag{3}
\]
where \( w = \frac{z + x}{1 + xz} \).

Putting \( z = e^{i\theta} \) in (3), multiplying by \( e^{i\alpha} \) and substituting \( c = 2(1 - \beta) \cos \alpha \, e^{-i\alpha} - 1 \), we obtain for \(|z| = 1\),

\[
\text{Re} \left\{ e^{i\alpha} \frac{f'(z)}{f(z)} \right\} = \frac{1 - |x|^2}{|1 + xe^{-i\theta}|^2} \text{Re} \left\{ e^{i\alpha} w \frac{f'(w)}{f(w)} \right\}
\]

\[
+ \text{Re} \left\{ e^{i\alpha} \frac{xe^{-i\theta}}{1 + xe^{-i\theta}} \left\{ \frac{2(1 - \beta) \cos \alpha \, e^{-i\alpha} - 1}{1 + xe^{-i\theta}} \right\} \right\}
\]

\[
\geq \frac{1 - |x|^2}{|1 + xe^{-i\theta}|^2} \beta \cos \alpha
\]

\[
+ \text{Re} \left\{ \frac{(xe^{-i\theta} + xe^{i\theta} + 2 |x|^2) e^{i\alpha} - 2(1 - \beta) \cos \alpha \, (xe^{i\theta} + |x|^2)}{|1 + xe^{-i\theta}|^2} \right\}
\]

since \( f(z) \in F(\alpha, \beta) \).
So, for $|z| = 1$,

$$\text{Re}\left\{ e^{i\alpha} \frac{f'_\rho(z)}{f_\rho(z)} \right\} >$$

$$\text{Re} \left( (1+|x|^2) \beta \cos \alpha + 2\beta \cos \alpha \bar{xe}^{-i\theta} + 2|x|^2(e^{-i\alpha} - \cos \alpha) \right)$$

$$\geq \frac{-2\cos \alpha \bar{xe}^{-i\theta} + 2\left\{ \text{Re}(\bar{xe}^{i\theta}) \right\} e^{i\alpha} - i\theta}{|1 + xe^{-i\theta}|^2}$$

$$= \frac{1 + 2\text{Re}(\bar{xe}^{i\theta}) + |x|^2}{|1 + xe^{-i\theta}|^2} \beta \cos \alpha$$

$$= \beta \cos \alpha .$$

Therefore, $f_\rho(z) \in F(\alpha, \beta)$. From the compactness of $F(\alpha, \beta)$, it follows that $f_1(z) = \lim_{\rho \to 1} f_\rho(z)$ is in $F(\alpha, \beta)$.

This proves Lemma 1.
THEOREM 1: If $f(z) \in F(a, \beta)$, then, for $\gamma \in (-\pi/2, \pi/2)$,

$$\Re \left\{ e^{i\gamma} \frac{f'(z)}{f(z)} \right\} > 0 \text{ for } |z| < r_\gamma,$$

where $r_\gamma$ is the smallest positive root of the equation

$$\cos \gamma - 2(1-\beta) \cos \alpha x$$

$$+ \sum 2(1-\beta) \cos \alpha \cos (\gamma - \alpha) - \cos \gamma \int r^2 = 0. \quad (4)$$

This value of $r_\gamma$ is sharp.

Remark: We call $r_\gamma$ the $\gamma$-spiral radius of the class $F(a, \beta)$.

PROOF: By Lemma 1, $f_1(z)$ defined by (1) is also in $F(a, \beta)$. Let $f_1(z) = z + A_2 z^2 + \ldots$.

Then, by $\int 13, (3.10)$, we have

$$|A_2| \leq 2(1-\beta) \cos \alpha. \quad (5)$$
On the other hand,

\[ A_2 = \frac{f''(0)}{2} = (1 - \left| x \right|^2) \frac{f'(x)}{f(x)} - \left( \frac{1}{x} + \frac{\alpha}{x} \right). \]  

(6)

For \( z \in E \), choosing \( z \) in place of \( x \) and using (5), we obtain, from (6),

\[ \left| \frac{f'(z)}{f(z)} - \frac{1 + \alpha r^2}{1 - r^2} \right| \leq \frac{2(1 - \beta) \cos \alpha r}{1 - r^2} \]

where \( r = |z| \).

So,

\[ \text{Re} \left\{ e^{i\gamma} \frac{f'(z)}{f(z)} \right\} \geq \text{Re} \left\{ \frac{1 + \alpha r^2}{1 - r^2} e^{i\gamma} \right\} - \frac{2(1 - \beta) \cos \alpha r}{1 - r^2} \]

\[ = \frac{\cos \gamma - 2(1 - \beta) \cos \alpha r + \sqrt{2(1 - \beta) \cos \alpha \cos (\gamma - \alpha) - \cos \gamma} r^2}{1 - r^2} \]

Thus, \( \text{Re} \left\{ e^{i\gamma} \frac{f'(z)}{f(z)} \right\} > 0 \) for \( |z| < r_\gamma \),

where \( r_\gamma \) is the smallest positive root of (4).
To show that the result is sharp, consider the function

\[ f_0(z) = \frac{z}{(1 - z)^\alpha + 1} \]  \hspace{0.5cm} (7)

which is in \( F(\alpha, \beta) \). For \( r \in [0,1) \), we have

\[ \text{Re} \left\{ e^{i\gamma} w \frac{f'_0(w)}{f_0(w)} \right\} = \frac{M(r)}{1 - r^2} \]

where \( w = \frac{r(r - e^{i(\alpha - \gamma)})}{1 - re^{i(\alpha - \gamma)}} \) and \( M(r) \) is the left member of (4).

So, \( \text{Re} \left\{ e^{i\gamma} w \frac{f'_0(w)}{f_0(w)} \right\} = 0 \) at \( r = r_\gamma \).

This completes the proof of Theorem 1.

REMARK : Theorem 1 was also obtained using different
methods by Libera

For \( \gamma = 0 \), Theorem 1 reduces to the following

**COROLLARY 1**: The radius of starlikeness of \( F(\alpha, \beta) \) is the smallest positive root of the equation

\[
1 - 2(1 - \beta) \cos \alpha + \sqrt{2(1 - \beta) \cos^2 \alpha - 1} r^2 = 0.
\]

For \( \beta = 0 \), Corollary 1 yields the following result which was first obtained by Robertson

**COROLLARY 2**: The radius of starlikeness of the class \( F(\alpha) \) is \( (\cos \alpha + |\sin \alpha|)^{-1} \)

Choosing \( \alpha = 0 \) in Theorem 1, we obtain

**COROLLARY 3**: The \( \gamma \)-spiral radius of the class \( S^*(\beta) \) of starlike functions of order \( \beta \) is the smallest positive root of the equation

\[
\cos \gamma - 2(1 - \beta) r + (1 - 2\beta \cos \gamma) r^2 = 0.
\]
To obtain the $\gamma$-spiral radius of the class $K(\beta)$ of convex functions of order $\beta$, we use the following result of MacGregor: 

**THEOREM A:** $K(\beta) \subseteq S^* (\sigma(\beta))$, where 

$$
\sigma(\beta) = \frac{-4\beta (2\beta-1)}{4 - 2\ \frac{1}{4\beta}} \quad \text{if} \quad \beta \neq \frac{1}{2}
$$

and 

$$
\sigma(1/2) = \frac{1}{\log 4}
$$

Corollary 3 now yields 

**COROLLARY 4:** The $\gamma$-spiral radius of $K(\beta)$ is the smallest positive root of the equation

$$
\cos \gamma - 2(1 - \sigma(\beta)) x + (1 - 2 \sigma(\beta)) \cos \gamma x^2 = 0,
$$

where $\sigma(\beta)$ is as defined in Theorem A.

The particular case of Corollary 4 for the case $\beta = 0$ was obtained by Libera, Corollary 5.
LEMMA 2: Let \( x \in E \). For \( f(z) \in G(\alpha, \beta) \), define
\[ f_2(z) \] by
\[
\frac{f'(\frac{z + x}{1 + xz})}{f'(x)(1 + xz)^{c+1}} \text{ for } z \in E, \tag{8}
\]
and \( f_2(0) = 0 \), where \( c \) is as defined in Lemma 1. Then \( f_2(z) \in G(\alpha, \beta) \).

PROOF: Since \( f(z) \in G(\alpha, \beta) \), it follows that
\[ g(z) = zf'(z) \in F(\alpha, \beta) \]. By Lemma 1, \( g_1(z) \in F(\alpha, \beta) \).
But \( g_1(z) = zf_2(z) \).
Thus \( zf_2(z) \in F(\alpha, \beta) \). Hence \( f_2(z) \in G(\alpha, \beta) \).

The following theorem can be deduced similarly from Theorem 1.

THEOREM 2: If \( f(z) \in G(\alpha, \beta) \), then
for \( \gamma \in (-\pi/2, \pi/2) \),
\[ \text{Re} \left\{ e^{i\gamma} \left( 1 + z \frac{f''(z)}{f'(z)} \right) \right\} > 0 \quad \text{for} \quad |z| < r_\gamma, \]

where \( r_\gamma \) is as defined in Theorem 1.

The result is sharp.

Choosing \( \gamma = 0 \) in Theorem 2, we obtain

COROLLARY 5: The radius of convexity of \( G(\alpha, \beta) \) is the smallest positive root of the equation

\[ 1 - 2(1 - \beta)\cos \alpha r + \int_0^{2\pi} (1 - \beta)\cos^2 \alpha - 1 \int_0^r x^2 = 0. \]

This result was also obtained using different methods by Pinchuk. In fact, in Pinchuk considered the class \( C_\beta(\alpha) \) of functions \( f(z) \) belonging to \( A \) which satisfy the condition

\[ \text{Re} \left\{ e^{i\alpha} \left( 1 + z \frac{f''(z)}{f'(z)} \right) \right\} > \beta \quad \text{for all} \quad z \in E, \]

where \( \alpha \in (-\pi/2, \pi/2) \) and \( \beta \in [0,1) \). Clearly
C \beta \cos a (a) \equiv G(a, \beta). \text{ Pinchuk showed Lemma 5.1 that the radius of convexity of } C \beta (a) \text{ is}

\sqrt{(\cos a - \beta) + \sqrt{\beta^2 + \sin^2 a}}^{-1}.

Replacing \beta by \beta \cos a we obtain Corollary 5.

We now obtain certain values of \alpha and \beta for which the functions of \ G(a, \beta) \ are univalent in \ E.

The following Lemma is a result of Decker.

**LEMMA 3**: If \ f(z) \in A \text{ and} \quad \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{\alpha}{1 - |z|^2}

for \ z \in E, \text{ then } f(z) \text{ is univalent in } E.

**THEOREM 3**: Functions of the class \ G(a, \beta) \ are
univalent in \ E \text{ whenever}

$$4(1 - \beta) \cos \alpha \leq 1 \quad (9)$$
PROOF: Let \( f(z) \in G(a, \beta) \). Then \( f_2(z) = z + B_2z^2 + \ldots \), as defined by (3) is also in \( G(a, \beta) \).

So, \( zf_2(z) = z + 2B_2z^2 + \ldots \), is in \( F(a, \beta) \).

Therefore, by (13), (3.10),

\[
|B_2| \leq (1 - \beta) \cos \alpha \quad (10)
\]

On the other hand,

\[
B_2 = \frac{f_2''(0)}{2} = \frac{1}{2} \left\{ (1 - |x|^2) \frac{f''(x)}{f'(x)} - (e + 1) \right\} \quad (11)
\]

For \( z \in \mathbb{E} \), choosing \( z \) in place of \( x \) and using (10), we obtain, from (11),

\[
\left| \frac{f''(z)}{f'(z)} - \frac{2(1 - \beta) \cos \alpha e^{-i\alpha z}}{1 - |z|^2} \right| \leq \frac{2(1 - \beta) \cos \alpha}{1 - |z|^2}.
\]

Thus, for \( z \in \mathbb{E} \),

\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{2(1 - \beta) \cos \alpha (1 + |z|)}{1 - |z|^2} < \frac{4(1 - \beta) \cos \alpha}{1 - |z|^2}.
\]
By Lemma 3, it now follows that \( f(z) \) is univalent in \( E \) if (9) holds, which proves Theorem 3.

REMARK: It follows from Theorem 3 (choosing \( \beta = 0 \)) that functions of \( G(a, 0) \) are univalent in \( E \) if \( \cos a \leq 1/4 = 0.25 \). Robertson has shown \( \int_{26} \) that functions of \( G(a, 0) \) are univalent in \( E \) whenever \( \cos a \leq x_1 \), where \( 0.231 < x_1 < 0.232 \).

We now obtain a bound for the modulus of the Schwarzian derivative of functions in \( G(a, \beta) \). We recall that if \( f(z) \in \mathcal{A} \) and \( f'(z) \neq 0 \) for \( z \in E \), then the Schwarzian derivative \( \{ f, z \} \) is defined by

\[
\{ f, z \} = \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \quad \text{for } z \in E.
\]

Nehari \( \int_{17} \) has shown that \( f(z) \) is univalent in \( E \) whenever

\[
\left| \{ f, z \} \right| \leq \frac{2}{(1 - |z|^2)^2} \quad \text{for } z \in E.
\]
Lemma 4: If \( f(z) = z + az^2 + \ldots, \in G(a, \beta) \) and \( \psi \) is any complex number then

\[
|a_3 - \psi a_2^2| \leq \frac{1}{3}(1 - \beta) \cos \alpha \max \left\{ 1, |1 - (3\psi - 2)e^{-i\alpha}(1 - \beta)\cos\alpha| \right\}
\]  
(12)

Proof: \( zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n \) is in \( G(a, \beta) \).

So, as was shown by Keogh and Merkes [8],

\[
|3a_3 - 4 \mu a_2^2| \leq (1 - \beta) \cos \alpha \max \left\{ 1, |1 - 2(1 - \beta) \cos \alpha e^{-i\alpha}(2\mu - 1)| \right\}
\]

Substituting \( \frac{3}{4} \psi \) for \( \mu \) we obtain (12).

Theorem 4: If \( f(z) \in G(a, \beta) \) and \( |z| = r < 1 \), then

\[
\left| \left\{ f, z \right\} \right| \leq \frac{2(1 - \beta) \cos \alpha \left\{ 1 + \left( \beta^2 \cos^2 \alpha + \sin^2 \alpha \right)^{1/2} \right\} \left(2r + r^2\right)}{(1 - r^2)^2}
\]  
(13)
and \( f(z) \) is univalent in \( E \) if
\[
(1 - \beta) \cos \alpha \left\{ 1 + 3(\beta^2 \cos^2 \alpha + \sin^2 \alpha)^{1/2} \right\} \leq 1.
\]

**PROOF:** By Lemma 2, \( f_2(z) \), as defined in Lemma 2 is in \( F(a, \beta) \).

Let \( f_2(z) = z + B_2 z^2 + B_3 z^3 + \ldots, z \in E. \)

Then by Lemma 4, choosing \( \psi = 1 \), we have
\[
|B_3 - B_2^2| \leq \frac{1}{3} (1 - \beta) \cos \alpha \tag{14}
\]

Now,
\[
B_2 = \frac{f''_2(0)}{2} = \frac{1}{2} \left\{ (1 - |x|^2) \frac{f''(x)}{f'(x)} - (c + 1) \frac{x}{f'(x)} \right\}
\]

and
\[
B_3 = \frac{f'''_2(0)}{6} = \frac{1}{6} \left\{ (1 - |x|^2)^2 \frac{f''(x)}{f'(x)} - 2x(c+2)(1-|x|^2) \frac{x}{f'(x)} \right. \]
\[
+ \left. (c + 1)(c + 2) \frac{x^2}{f'(x)} \right\}.
\]
So, \[ B_3 - B_2^2 = \frac{1}{6} (1 - |x|^2)^2 \left\{ \frac{f''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \right\} + \]

\[ + \frac{1}{6} (c-1)(1 - |x|^2) \frac{f''(x)}{f'(x)} - \frac{1}{12} (c+1)(c-1) \bar{z}^2 \]

\[ = \frac{1}{6} (1 - |x|^2)^2 \left\{ f, x \right\} + \frac{1}{6} (c-1) \bar{z} \left\{ 2B_2 + (c+1) \bar{z} \right\} - \frac{1}{12} (c^2-1) \bar{z}^2 \]

\[ = \frac{1}{6} (1 - |x|^2)^2 \left\{ f, x \right\} + \frac{1}{3} (c-1) B_2 \bar{z} + \frac{1}{12} (c^2-1) \bar{z}^2 . \]

For \( z \in B \), choosing \( z \) for \( x \), we obtain

\[ \{ f, z \} = \frac{12(B_3-B_2^2) - 4(c-1)B_2 \bar{z} - (c^2-1)\bar{z}^2}{2(1 - r^2)^2} \quad (15) \]

where \( r = |z| \).

Using (10) and (14) and substituting the value
of $\phi(=2(1-\beta)\cos \alpha e^{-i\alpha}-1)$, we obtain (13) from (15).

By Nehari's test, it follows, from (13), that $f(z)$ is univalent in $E$ if

$$(1-\beta)\cos \alpha \left\{1+3(\beta^2\cos^2 \alpha + \sin^2 \alpha)^{1/2}\right\} \leq 1.$$ 

For $\beta = 0$, Theorem 4 yields the following result of Libera and Ziegler [14, Theorem 1].

**COROLLARY 6**: If $f(z) \in G(a,0)$ and $|z| = r < 1$, then

$$\left|\left\{f, z\right\}\right| \leq \frac{2\cos \alpha \left\{1+3|\sin \alpha|\right\}}{(1-r^2)^2}$$

and $f(z)$ is univalent in $E$ if

$$\cos \alpha \left\{1+3|\sin \alpha|\right\} \leq 1.$$ 

Let $0 \leq \rho \leq 1$. A function $f(z)$ belonging to $A$ is said to be close-to-convex of order $\rho$ in
If there exists a function $h(z)$ belonging to $A$ and convex in $E$ such that

$$\left| \arg \frac{f'(z)}{h'(z)} \right| \leq \rho \pi /2 \text{ for all } z \in E.$$ 

We denote by $C(\rho)$ the class of functions of $A$ which are close-to-convex of order $\rho$ in $E$. The class $C(\rho)$ was introduced by Pommerenke. The functions of the class $C(1)$ are simply called close-to-convex functions and were introduced by Kaplan. We denote the class $C(1)$ simply by $C$. Kaplan showed that close-to-convex functions are univalent in $E$. It follows that for every $\rho \in [0, 1]$, functions of $C(\rho)$ are univalent in $E$. $C(0)$ is precisely the class $K$ of convex functions in $E$.

It is known that if $f(z) \in A$ and $f(z)$ has a nonvanishing derivative in $E$ then $f(z) \in C(\rho)$, $0 \leq \rho \leq 1$, if and only if
arg \{z_2^f(z_2)\} - arg \{z_1^f(z_1)\} \geq - \rho \pi \quad (16)

for all \(z_1\) and \(z_2\) in \(E\) satisfying \(z_2 = e^{i\theta}z_1\), \(0 < \theta < 2\pi\).

The sharp value of \(r\) for which (16) holds for all \(z_1, z_2\) in \(|z| < r\) for every function \(f(z)\) in \(G(\alpha, \beta)\) is called the radius of close-to-convexity of order \(\rho\) of \(G(\alpha, \beta)\). We determine this radius using techniques similar to those employed by Krzyz \(\int_{10}^J\) to obtain the radius of close-to-convexity for the class of univalent functions in \(E\).

The following lemma is a result of Kulshrestha \(\int_{11}^J\).

**Lemma 5**: If \(f(z) \in F(\alpha, \beta)\), then for \(|z| = r < 1\),

\[N_1(r) \leq \arg \frac{f(z)}{z} \leq N_2(r),\]

where

\[N_1(r) = -2(1-\beta)\cos^2 \alpha \sin^{-1}(r \cos \alpha)\]

\[+ (1-\beta)\sin 2\alpha \log \int(1-r^2 \cos^2 \alpha)^{1/2} - r \sin \alpha \]

\( \int_{17}^J \)
and \( N_2(r) = 2(1- \beta) \cos^2 \alpha \sin^{-1}(r \cos \alpha) \)

\[ + (1- \beta) \sin 2 \alpha \log \sqrt{(1-r^2 \cos^2 \alpha)^{1/2} + r \sin \alpha} \]  

(18)

These bounds are sharp.

Using the fact that if \( f(z) \in G(\alpha, \beta) \) then \( zf'(z) \in F(\alpha, \beta) \) we obtain

**Lemma 6:** If \( f(z) \in G(\alpha, \beta) \), then for \( |z| = r < 1 \),

\[ N_1(r) \leq \arg f'(z) \leq N_2(r) \]

where \( N_1(r) \), \( N_2(r) \) are given by (17) and (18).

These bounds are sharp.

**Theorem 5:** Let \( \alpha \neq 0 \), \( r_0 \) be the radius of convexity of \( G(\alpha, \beta) \),

\[ y_0 = \frac{1- \sqrt{2(1-\beta)\cos^2 \alpha + 1} - x^2}{2r \sqrt{1-(1-\beta^2) \cos^2 \alpha}} \] for \( r \in [r_0, 1) \),

\[ \theta_0 = 2 \cos^{-1} y_0, \ 0 \leq \theta_0 < \pi \]
and

\[
\Delta(r) = \theta_0 + 2(1-\beta) \cos^2 \alpha \tan^{-1}\left(\frac{r^2 \sin \theta_0}{1-r^2 \cos \theta_0}\right) - 2(1-\beta) \cos^2 \alpha \sin^{-1}\left(r \cos \alpha \left\{\frac{2(1-\cos \theta_0)}{1-2r^2 \cos \theta_0 + r^4}\right\}^{1/2}\right) - (1-\beta) \sin 2\alpha \log \left\{1-2r^2(\sin^2 \alpha \cos \theta_0 + \cos^2 \alpha) + r^4\right\}^{1/2} - r \sin \alpha \left\{2(1-\cos \theta_0)\right\}^{1/2} - (1-\beta) \sin 2\alpha \log (1 - r^2). \quad (19)
\]

Then the radius of close-to-convexity of order $\rho$ of $G(\alpha, \beta)$ is the unique root of the equation

\[
\Delta(r) = -\rho \pi \quad \text{in the interval } (r_0, 1).
\]

**PROOF**: Let $\Delta(r, \theta) = \inf_{f(z) \in G(\alpha, \beta)} \left\{\frac{z_2^2 f''(z_2)}{z_1^2 f'(z_1)}\right\}$,

where $z_1, z_2$ are any two points satisfying
\[ |z_1| = r < 1, \ z_2 = z_1 e^{i\theta}, \ 0 \leq \theta < 2\pi \text{ and the argument is so chosen as to vary continuously from an initial value of zero.} \]

Let \( f(z) \in G(\alpha, \beta) \) and \( f_2(z) \) be defined as in Lemma 2 with \( z_1 \) in place of \( x \).

\[
\frac{f'(\frac{z + z_1}{1 + z_1 z})}{f'(z_1)(1 + z_1 z)^{c+1}} \quad \text{for } s \in \mathbb{C}, \ f_2(0) = 0.
\]

Then, by Lemma 2, \( f_2(z) \in G(\alpha, \beta) \). Let \( w_0 = \frac{z_2 - z_1}{1 - z_1 z_2} \).

\[
\text{Then } f_2'(w_0) = \frac{f'_2(z_2)}{f'(z_1)} \left\{ \frac{1 - z_1 z_2}{1 - |z_1|^2} \right\}^{c+1},
\]

whence it follows that

\[
\text{arg} \left( \frac{z_2 f'(z_2)}{z_1 f'(z_1)} \right) = \text{arg} \left\{ \frac{z_2 \left(1 - |z_1|^2 \right)^{c+1}}{z_1 \left(1 - z_1 z_2 \right)} \right\} + \text{arg} \left\{ f_2'(w_0) \right\}.
\]

(20)
We have \[ |w_0| = r \left\{ \frac{2(1 - \cos \theta)}{1 - 2r^2 \cos \theta + r^4} \right\}^{1/2} \tag{21} \]

Since \( f_2(z) \in G(\alpha, \beta) \) it now follows from Lemma 6 that

\[ \arg f_2'(w_0) \geq -2(1 - \beta) \cos^2 \alpha \sin^{-1}\left\{ \frac{r \cos \alpha \left\{ \frac{2(1 - \cos \theta)}{1 - 2r^2 \cos \theta + r^4} \right\}^{1/2} \right\} \]

\[ + (1 - \beta) \sin 2\alpha \log \left[ \left\{ 1 - 2r^2 (\cos \theta \sin^2 \alpha + \cos^2 \alpha) + r^4 \right\}^{1/2} \right. \]

\[ - r \sin \alpha \left\{ 2(1 - \cos \theta) \right\}^{1/2} \]

\[ - (1 - \beta) \sin 2\alpha \log (1 - 2r^2 \cos \theta + r^4)^{1/2} \tag{22} \]

Also

\[ \arg \left\{ \frac{z_2}{z_1} \left\{ 1 - \frac{|z_1|^2}{1 - z_1 \bar{z}_2} \right\}^{c + 1} \right\} = \theta \]

\[ + 2(1 - \beta) \cos^2 \alpha \tan^{-1}\left\{ \frac{r^2 \sin \theta}{1 - r^2 \cos \theta} \right\} \]
- \( (1 - \beta) \sin 2\alpha \log \left\{ \frac{1 - r^2}{(1 - 2r^2 \cos \theta + r^4)^{1/2}} \right\} \) \hspace{1cm} (23)

Using (22) and (23), we obtain from (20),

\[
\text{arg} \frac{z_2 f'(z_2)}{z_1 f'(z_1)} \geq \theta + 2(1 - \beta) \cos^2 \alpha \tan^{-1} \left\{ \frac{r^2 \sin \theta}{1 - r^2 \cos \theta} \right\}
\]

\[- 2(1 - \beta) \cos^2 \alpha \sin^{-1} \left\{ \frac{r^2 \cos \alpha}{(1 - 2r^2 \cos \theta + r^4)^{1/2}} \right\}
\]

\[+ (1 - \beta) \sin 2\alpha \log \left[ \left\{ 1 - 2r^2 (\cos \theta \sin^2 \alpha + \cos^2 \alpha) + r^4 \right\}^{1/2}
\]

\[\left. - r \sin \alpha \left\{ 2(1 - \cos \theta) \right\}^{1/2} \right\}
\]

\[- (1 - \beta) \sin 2\alpha \log (1 - r^2). \] \hspace{1cm} (24)

Further, there exists a function \( f(z) \) in \( G(\alpha, \beta) \)

for which equality holds in (24), for fixed \( z_1 \) and \( z_2 \).
letting \( g(z) \) be the function in \( G(\alpha, \beta) \) for which equality holds on the left in Lemma 6 at the point \( w_0 \) and letting \( f(z) \) be defined in \( E \) by

\[
    f'(z) = \frac{g'(\frac{z - z_1}{1 - \overline{z}_1 z})}{g'(-z_1)(1 - \overline{z}_1 z)^{c+1}}, \quad f(0) = 0,
\]

it follows by Lemma 2 that \( f(z) \in G(\alpha, \beta) \) and equality holds in (24) for this \( f(z) \).

Hence, it follows that

\[
    \Delta(r, \theta) = \text{the expression on the right of (24)}.
\]

Let \( \Delta(r) = \inf \Delta(r, \theta) \). Then \( \Delta(r) \) is a decreasing function of \( r \) and it follows from (16) that the radius of close-to-convexity of order \( \rho \) of \( G(\alpha, \beta) \) is the root \( r_1 \) of the equation \( \Delta(r) = -\rho \pi \), provided such a root exists. We now show that \( \Delta(r) \) is given by (19)
and establish the existence of $r_1$.

If $r_0$ is the radius of convexity of $G(\alpha, \beta)$, then $\Delta(r) \geq 0$ for $r \leq r_0$ and so $r_1 > r_0$. We assume from now on that $r \in \mathbb{R}_0^1$.

From the expression on the right of (24) for $\Delta(r, \theta)$, we have, differentiating with respect to $\theta$,

$$\frac{\partial \Delta(r, \theta)}{\partial \theta} = 1 + \frac{2(1 - \beta) \cos^2 \alpha \cdot r^2(\cos \theta - r^2)}{1 - 2r^2 \cos \theta + r^4}$$

Putting $y = \cos \theta/2$, $0 \leq \theta < 2\pi$ so that $\cos \theta = 2y^2 - 1$

and $y = \frac{\sin \theta}{\sqrt{2(1 - \cos \theta)}}^{1/2}$, we obtain
\[
\frac{\partial \Delta(r, \theta)}{\partial \theta} = \frac{p(y) - 2(1 - \beta) \, r \cos \alpha \, q(y)}{g(y)}
\]

where

\[
p(y) = (1 + r^2) \left \{ 1 + r^2 - 2(1 - \beta) r^2 \cos^2 \alpha \right \} - 4r^2 y^2 \left \{ 1 - (1 - \beta) \cos^2 \alpha \right \},
\]

\[
q(y) = \left \{ 1 - 2r^2 \cos 2\alpha + r^4 - 4r^2 y^2 \sin^2 \alpha \right \}^{1/2},
\]

and \( g(y) = (1 + r^2)^2 - 4r^2 y^2 \).

Clearly \( g(y) > 0 \) for \( r \in [-1, 1) \) and \( y \in (-1, 1) \) and so the zeros of \( \frac{\partial \Delta(r, \theta)}{\partial \theta} \) will precisely be the zeros of \( p(y) = p(y) - 2(1 - \beta) \, r \cos \alpha \, q(y) \).

Also, for \( y \in (-1, 1) \),

\[
p(y) = (1 + r^2) \left \{ 1 + r^2 - 2(1 - \beta) r^2 \cos^2 \alpha \right \} - 4r^2 \left \{ 1 - (1 - \beta) \cos^2 \alpha \right \}
\]


\[ p(y) \geq (1 - r^2)^2 > 0 \]

and \( q(y) > 0 \). Hence, for \( y \in (-1, 0) \), \( p_1(y) > 0 \)
so that \( p_1(y) \) has no zeros in \((-1, 0)\). For \( y \in (0, 1) \),
equivalently, for \( \theta \in (0, \pi) \), we see that the zeros
of \( p_1(y) \) are precisely the zeros of \( p_2(y) \) where

\[
p_2(y) = \left\{ \frac{p(y)}{q(x)} \right\}^2 - 4(1 - \beta)^2 r^2 y^2 \cos^2 \alpha \left\{ q(x) \right\}^2
\]

\[ = a_0 - a_1 y^2 + a_2 y^4 \]

where

\[ a_0 = (1 + r^2)^2 \int 1 + r^2 - 2(1 - \beta) \cos^2 \alpha r^2 \, d\theta, \]

\[ a_1 = 4r^2 \int \left\{ 1 + r^2 - 2(1 - \beta) \cos^2 \alpha r^2 \right\}^2 + (1 + r^2)^2 \left\{ 1 - (1 - \beta^2) \cos^2 \alpha \right\} \, d\theta, \]

and \( a_2 = 16 r^4 \int 1 - (1 - \beta^2) \cos^2 \alpha \, d\theta. \)

The positive zeros of \( p_2(y) \) are

\[
y_0 = \frac{1 - \int 2(1 - \beta) \cos^2 \alpha - 1 \, d\theta}{2r \int 1 - (1 - \beta^2) \cos^2 \alpha \, d\theta}^{1/2}
\]
and $y_1 = \frac{1 + r^2}{2r}$.

Here $y_1 > 1$ and so if $p_2(y)$ has a zero in $(0, 1]$, then it must be $y_0$, so that $y_0 \leq 1$.

We have

$$p_2(0) = a_0 > 0$$

and $p_2(1) = a_0 - a_1 + a_2$

$$= (1 - r^2)^2 \left[ 1 + r^2 - 2(1 - \beta) \cos^2 \alpha \right] r^2 - r^2 \cdot 2(1 - \beta) \cos \alpha \cdot r$$

$$= 4r^2(1 - r^2)^2 \left[ 1 - (1 - \beta^2) \cos^2 \alpha \right]$$

$$= (1 - r^2)^2 \left[ 1 - 2(1 - \beta) \cos \alpha \cdot r \right] + \left\{ 2(1 - \beta) \cos^2 \alpha - 1 \right\} r^2$$

Since $r \geq r_0$, we have, by Corollary 5,

$$1 - 2(1 - \beta) \cos \alpha \cdot r + \left\{ 2(1 - \beta) \cos^2 \alpha - 1 \right\} r^2 \leq 0.$$
Also,

\[ 1 + 2(1 - \beta) \cos \alpha r + \left\{ 2(1 - \beta) \cos^2 \alpha - 1 \right\} r^2 \leq (1 - r^2) > 0. \]

Hence \( p_2(1) \leq 0 \). So \( p_2(y) \) has a zero in \( (0, 1) \) which has to be \( y_0 \) so that \( y_0 \leq 1 \). Incidentally, for \( r = r_0 \), \( p_2(1) = 0 \) so that \( y_0 = 1 \).

Thus \( y_0 \) is the unique root of \( p_1(y) \) in \( (-1, 1) \) or, equivalently, \( \theta_0 = 2\cos^{-1} y_0 \) is the only zero of

\[ \frac{\partial \Delta(r, \theta)}{\partial \theta} \quad \text{for} \quad 0 \leq \theta < 2\pi. \]

It is easy to see that \( \frac{\partial^2 \Delta(r, \theta)}{\partial \theta^2} > 0 \) for \( \theta = \theta_0 \) so that \( \Delta(r, \theta) \) assumes its minimum value at \( \theta = \theta_0 \).

Hence

\[ \Delta(r) = \inf_{\theta} \Delta(r, \theta) = \Delta(r, \theta_0), \quad \text{which yields (19)}. \]

We have \( \Delta(r_0) = 0 \) (since \( y_1 = 1 \) for \( r = r_0 \) so that
\[ \theta_0 = 0 \text{ and } \Delta(r) \to -\infty \text{ as } r \to 1^- \text{. Since } \Delta(r) \text{ is a continuous, decreasing function it follows that there exists a unique root } r_1 \text{ of the equation } \Delta(r) = -\rho^2 \text{ in the interval } (r_0, 1) \text{ and this root } r_1 \text{ is the radius of close-to-convexity of } G(\alpha, \beta). \]

For \( \rho = 1 \) and \( \beta = 0 \), Theorem 5 reduces to the result of Libera and Ziegler \( \int 14 \), Theorem 2.

**Theorem 6:** If \( f(z) \in G(\alpha, \beta) \), then for \( |z| = r < 1 \),

\[ -2(1-\beta)\cos \alpha \log(1 + r) \leq \text{Re} \left\{ e^{ia} \log f'(z) \right\} \leq -2(1-\beta)\cos \alpha \log(1-r). \] \( (25) \)

These bounds are sharp.

**Proof:** As seen in the proof of Theorem 3,

\[ \left| \frac{f''(z)}{f'(z)} - \frac{2(1-\beta)\cos \alpha e^{-ia} \bar{z}}{1-|z|^2} \right| \leq \frac{2(1-\beta)\cos \alpha}{1-|z|^2} \]
That is,

\[ \left| \frac{e^{ia} z^2 f''(z)}{f'(z)} - \frac{2(1-\beta) \cos \alpha |z|^2}{1-|z|^2} \right| \leq \frac{2(1-\beta) \cos \alpha |z|^2}{1-|z|^2} \]

For \( z = re^{i\theta} \) this yields

\[ \frac{2(1-\beta) \cos \alpha (r^2 - r)}{1 - r^2} \leq \text{Re} \left\{ e^{ia} \frac{3}{r} \log f'(re^{i\theta}) \right\} \]

\[ \leq \frac{2(1-\beta) \cos \alpha (r^2 + r)}{1 - r^2} \]

Dividing by \( r \) and integrating with respect to \( r \) from 0 to \( r \) we obtain (25).

The bounds in (25) are attained by the function \( f_0(z) \) defined by

\[ f_0(z) = \frac{1}{(1-z)^{\alpha+1}} , \ f_0(0) = 0. \]