CHAPTER II

CURVATURES OF THE CONGRUENCE LINE ASSOCIATED WITH ONE PARAMETER FAMILY OF SURFACES AND THE CONDITIONS FOR ESTABLISHING AN AREA PRESERVING REPRESENTATION

2.1 Introduction

A study of rectilinear congruences associated with focal surfaces has been done by various authors: Grigorios Tsagas [4], [5], [6], K.S. Amur [1], [2].

K.S. Amur in the paper [2] has formulated the general definitions of the curvatures, torsion associated with a rectilinear congruence and has obtained expressions for them.
explicitly in terms of the elements of the congruence and these quantities are not in general based on any specific curve on the surface of reference. But when the surface of reference is taken to be one of the focal surfaces of the congruence these quantities will correspond to a family of the edges of regression of the developables of the congruence. These quantities are closely connected with some invariants of the surface of reference of the congruence.

The concept of focal points, focal surfaces and the coefficients of their fundamental forms have been mentioned in the first chapter in articles 4 and 5.

In this chapter we consider a family of surfaces by introducing a parameter $t$ in the equation (1.5.1) and we study this family of surfaces in some detail. We derive expressions for the curvatures, torsion of the congruence line with reference to the family of surfaces mentioned above, so that the results which are obtained earlier for focal surfaces will follow as particular cases. Later we obtain the conditions for the rectilinear congruence to establish an area preserving representation between the two family of surfaces with parameter $t$. Using these conditions we show that the
expressions for the curvatures of the congruence line with respect to the family of surfaces with parameter \( t \) take very simple and elegant form.

It is known [5] that the Guichard congruence is an area preserving representation. For this congruence, using the conditions for preserving area obtained by us, we show how to determine \( \rho \), half the distance between the focal points.

2.2 Expressions for the fundamental quantities of a family of surfaces with parameter \( t \).

The family of surfaces \( S_t \) and \( S_{-t} \) are defined by

\[
\begin{align*}
(2.2.1) \quad & \tilde{x}^i = x^i + t\rho \lambda^i \\
& \text{and} \\
(2.2.2) \quad & \tilde{x}^i = x^i - t\rho \lambda^i, \quad t \in \mathbb{R}
\end{align*}
\]

and \( \rho \) is half the distance between focal points of the ray \( \lambda^i \).

These give us

\[
\begin{align*}
\tilde{x}^i_{,1} &= x^i_{,1} + t(f \lambda^i_{,1} + \lambda^i \rho_{,1}) \\
\tilde{x}^i_{,2} &= x^i_{,2} + t(f \lambda^i_{,2} + \lambda^i \rho_{,2})
\end{align*}
\]

Using (1.5.2) and (1.5.3) we get
(2.2.3) \[ \ddot{x}_1^1 = \partial (t - 1) \dot{\lambda}_1 + [(t + 1) \partial_1 + 2\ddot{\partial} \{ \frac{2}{12} \}] \dot{\lambda}_1 \]
and

(2.2.4) \[ \ddot{x}_2^1 = \partial (t + 1) \dot{\lambda}_2 + [(t - 1) \partial_2 - 2\ddot{\partial} \{ \frac{1}{12} \}] \dot{\lambda}_1 . \]

Similarly for the other family \( S_t \) we get

(2.2.5) \[ \ddot{x}_1^1 = -\partial (t + 1) \dot{\lambda}_1 - [(t - 1) \partial_1 - 2\ddot{\partial} \{ \frac{2}{12} \}] \dot{\lambda}_2 \]
and

(2.2.6) \[ \ddot{x}_2^1 = -\partial (t - 1) \dot{\lambda}_2 - [(t + 1) \partial_2 + 2\ddot{\partial} \{ \frac{1}{12} \}] \dot{\lambda}_2 . \]

For convenience let us denote

(2.2.7) \[ A = (t + 1) \partial_1 + 2\ddot{\partial} \{ \frac{2}{12} \}, \]
\[ A_1 = (t - 1) \partial_1 - 2\ddot{\partial} \{ \frac{2}{12} \}, \]
and

(2.2.8) \[ B = (t - 1) \partial_2 - 2\ddot{\partial} \{ \frac{1}{12} \}, \]
\[ B_1 = (t + 1) \partial_2 + 2\ddot{\partial} \{ \frac{1}{12} \} . \]

Let \( E_t, F_t, G_t \) denote the coefficients of the first fundamental form of the surface \( S_t \) and \( E_{-t}, F_{-t}, G_{-t} \) those of \( S_{-t} \). Using (2.2.3), (2.2.4), (2.2.7) and (2.2.8) we have
\[
\begin{aligned}
E_t &= \sum_i (\chi^i_1, \chi^i_1) = f^2 e(t - 1)^2 + A^2 \\
F_t &= \sum_i (\chi^i_1, \chi^i_2) = f^2 f(t^2 - 1) + AB \\
G_t &= \sum_i (\chi^i_2, \chi^i_2) = f^2 g(t + 1)^2 + B^2
\end{aligned}
\]

where we have also made use of the fact that

\[
\sum_i (\lambda^i) (\lambda^i) = 1, \quad \sum_i (\lambda^i, \lambda^i, \alpha) = 0
\]

and equation (1.2.1).

Denoting

\[
(2.2.10) \quad \Delta_t = (E_t G_t - F_t^2)^{1/2}
\]

we have

\[
(2.2.11) \quad \Delta_t^2 = f^4 (eg - f^2) (t^2 - 1)^2 \\
+ f^2 [eB^2 (t - 1)^2 + gA^2 (t + 1)^2] \\
- 2 f^2 f(t^2 - 1) AB
\]

Also when \( t = 1 \) we get,

\[
(2.2.12) \quad \Delta_1^2 = 16 f^2 g(f_1, + f_{12})^2.
\]

Similarly the coefficients of the first fundamental form of the surface \( S_{-t} \) are given by
Let us denote,

\[(2.2.14) \Delta_{-t} = (E_{-t} G_{-t} - F_{-t}^2)^{1/2}\]

and we have,

\[(2.2.15) \Delta_{-t}^2 = f^4(\varepsilon g - f^2)(t^2 - 1)^2 \]
\[+ f^2[\varepsilon B_1^2(t + 1)^2 + gA_1^2(t - 1)^2] \]
\[- 2 f^2f(t^2 - 1)A_1B_1 \]

Further when \( t = 1 \) we get,

\[(2.2.16) \Delta_{-1}^2 = 16 f^2e(\varphi_{1}^2 + \varphi_{12}^1)^2 \]

2.3 Expressions for the curvatures of the congruence line associated with one parameter family of surfaces.

(i) Geodesic curvature:

It has been shown \([2]\) that the geodesic curvature of the congruence line with respect to \( S \) as defined by \((1.7.5)\) can
be put in the form

\begin{equation}
(2.3.1) \quad K_g(l) = e^{\alpha\beta} \mu_{\alpha\beta}
\end{equation}

Let \( \tilde{K}_g(l) \) and \( \tilde{K}_g(l) \) respectively denote the geodesic curvature of the congruence line with respect to \( S_t \) and \( S_{-t} \) and the coefficients of the Kummer's quadratic form for \( S_t \) and \( S_{-t} \) be denoted by \( \tilde{\mu}_{\alpha\beta}, \tilde{\mu}_{\alpha\beta} \) respectively. Similarly \( \tilde{e}^{\alpha\beta} \) and \( \tilde{e}^{\alpha\beta} \) are associated with \( S_t \) and \( S_{-t} \) respectively.

Therefore from (2.3.1) we have,

\[
\tilde{K}_g(l) = \tilde{e}^{\alpha\beta} \tilde{\mu}_{\alpha\beta} = \tilde{e}^{12} \tilde{\mu}_{12} + \tilde{e}^{21} \tilde{\mu}_{21}.
\]

Using (1.3.1) we get

\[
\tilde{K}_g(l) = \frac{\tilde{\mu}_{12} - \tilde{\mu}_{21}}{(E_t G_t - F_t^2)^{1/2}}
\]

Similarly we get,

\[
\tilde{K}_g(l) = \frac{\tilde{\mu}_{12} - \tilde{\mu}_{21}}{(E_{-t} G_{-t} - F_{-t}^2)^{1/2}}
\]

From (2.2.10) and (2.2.14) the above expressions may be written as

\begin{equation}
(2.3.2) \quad \tilde{K}_g(l) = \frac{\tilde{\mu}_{12} - \tilde{\mu}_{21}}{\Delta_t} ; \quad \tilde{K}_g(l) = \frac{\tilde{\mu}_{12} - \tilde{\mu}_{21}}{\Delta_{-t}}
\end{equation}
We also have from (1.2.1), (1.2.4), (2.2.3) and (2.2.4)

\[ \tilde{\mu}_{12} = \sum_{i} \lambda_{1}^{i} \tilde{x}_{2}^{i} = \varphi(t + 1)f \]

\[ \tilde{\mu}_{21} = \sum_{i} \lambda_{2}^{i} \tilde{x}_{1}^{i} = \varphi(t - 1)f \]

Similarly from (1.2.1), (1.2.4), (2.2.5) and (2.2.6) we get,

\[ \tilde{\mu}_{12} = \sum_{i} \lambda_{1}^{i} \tilde{x}_{2}^{i} = -\varphi(t - 1)f \]

\[ \tilde{\mu}_{21} = \sum_{i} \lambda_{2}^{i} \tilde{x}_{1}^{i} = -\varphi(t + 1)f \]

With these (2.3.2) takes the form,

\[ (2.3.3) \quad \tilde{K}_{g(1)} = \frac{2\varphi f}{\Delta_{t}} ; \quad \tilde{K}_{g(1)} = \frac{2\varphi f}{\Delta_{-t}} \]

These results lead to the following theorem.

**Theorem 3.1**

A necessary and sufficient condition for the congruence \( (\lambda) \) to be normal is that the curvatures \( \tilde{K}_{g(1)}, \tilde{K}_{g(1)} \) are identically zero.

**Proof:** From (1.6.1), the condition for a normal congruence is:

\[ b = b' \]

But we also have from (1.4.12)
\[ b = -b' = \dot{\rho} f, \quad \ddot{\rho} \neq 0 \quad \text{in general.} \]

This implies that \( f = 0 \) and from (2.3.3) we get
\[ \dot{\kappa}_g(1) = 0 = \ddot{\kappa}_g(1) \]

Conversely, when the two curvatures vanish we must have \( f = 0 \), since \( \ddot{\rho} \neq 0 \) in general. Hence the congruence must be a normal congruence. \( \text{q.e.d.} \)

**Expressions for the curvatures in case of focal surfaces.**

When \( t = 1 \), \( \Delta_1 \) is given by (2.2.12) and \( \Delta_{-t} \) by (2.2.16). Hence (2.3.3) as a consequence of these become
\[ \ddot{\kappa}_g(1) = \frac{f}{2 \sqrt{g} \left( \dot{\rho},_1 + \dot{\rho}_{\{12\}} \right)} \]
\[ \ddot{\kappa}_g(1) = \frac{f}{2 \sqrt{\epsilon} \left( \dot{\rho},_2 + \dot{\rho}_{\{12\}} \right)} \]

where \( \ddot{\kappa}_g(1), \ddot{\kappa}_g(1) \) respectively denote the curvatures corresponding to the focal surfaces, which agree with the results already obtained in [2].

**Remark:** We have from the above theorem that the geodesic curvature of the congruence line with respect to any surface
vanishes identically and in particular with respect to a focal surface. But the geodesic curvature of the congruence line with respect to a focal surface is the geodesic curvature of a curve in which a developable of the congruence meets the focal surface. Hence the developables of a normal congruence meet the focal surface in geodesics, a result which is otherwise well known.

(ii) Geodesic torsion:

It has been shown that [2] the geodesic torsion of the congruence line with respect to $S$ as defined by (1.7.6) can be put in the form

$$
(2.3.4) \quad \mathcal{C}_g(1) = -g^{\alpha\beta} \xi_{\alpha\beta}
$$

Let $\tilde{\mathcal{C}}_g(1)$, $\tilde{\mathcal{C}}_{g(1)}$ respectively denote the geodesic torsion of the congruence line with respect to $S_t$ and $S_{-t}$ and $\tilde{\xi}_{\alpha\beta}$, $\bar{\xi}_{\alpha\beta}$ be the corresponding coefficients of Sannia's quadratic form. $\bar{g}^{\alpha\beta}$, $\bar{g}^{\alpha\beta}$ are associated metric tensors of $S_t$, $S_{-t}$ respectively. Hence we have

$$
(2.3.5) \quad \tilde{\mathcal{C}}_g(1) = -\bar{g}^{\alpha\beta} \bar{\xi}_{\alpha\beta}
$$

Here it is necessary to evaluate the coefficients of Sannia's
quadratic form first. \( \xi_{\alpha\beta} \) as defined by (1.2.6) can also be put in the form

\[
\xi_{\alpha\beta} = [x^i, \lambda^i, \lambda^j_{,\beta}]
\]

where '[ ]' indicates the scalar triple product. Therefore we have

\[
\xi_{\alpha\beta} = [x^i, \lambda^i, \lambda^i_{,\beta}]
\]

and we know

\[
\lambda^i_{,\alpha} x \lambda^i_{,\beta} = E_{\alpha\beta} \lambda^i
\]

Therefore

\[
\lambda^i_{,1} x \lambda^i_{,2} = E_{12} \lambda^i.
\]

Using (1.3.2) we get

(2.3.6) \[ \lambda^i = \frac{\lambda^i_{,1} x \lambda^i_{,2}}{h} \]

Now consider

\[
\lambda^i x (\lambda^i_{,\beta}) = \frac{1}{h} (\lambda^i_{,1} x \lambda^i_{,2}) x \lambda^i_{,\beta}
\]

\[
= \frac{1}{h} \sum_{\beta} (\lambda^i_{,\beta} \cdot \lambda^i_{,1}) \lambda^i_{,2} - (\lambda^i_{,\beta} \cdot \lambda^i_{,2}) \lambda^i_{,1}
\]

\[
= \frac{1}{h} \left\{ G_{\beta 1} \lambda^i_{,2} - G_{\beta 2} \lambda^i_{,1} \right\}
\]
where we have used (1.2.1).

Hence we have,

\[ \lambda_1 \times (\lambda_1') = \frac{1}{h} (e \lambda_2' - f \lambda_1') \]

and

\[ \lambda_1 \times (\lambda_2') = \frac{1}{h} (f \lambda_2' - g \lambda_1') \]

Now using (1.2.1), (2.2.3) and (2.2.4) we have

\[ \tilde{\zeta}_{11} = \sum_{i} \tilde{\lambda}_1' \times \tilde{\lambda}_1' = \frac{\mathcal{P}(t - 1)}{h} (ef - fe) = 0 \]

\[ \tilde{\zeta}_{22} = \sum_{i} \tilde{\lambda}_2' \times \tilde{\lambda}_2' = \frac{\mathcal{P}(t + 1)}{h} (fg - gf) = 0 \]

\[ \tilde{\zeta}_{12} = \sum_{i} \tilde{\lambda}_1' \times \tilde{\lambda}_2' = \frac{\mathcal{P}(t - 1)}{h} (f^2 - eg) \]

\[ \tilde{\zeta}_{21} = \sum_{i} \tilde{\lambda}_2' \times \tilde{\lambda}_1' = \frac{\mathcal{P}(t + 1)}{h} (eg - f^2) \]

Thus the coefficients of the Sannia's quadratic form with respect to \( S_t \) are given by

\[ (2.3.7) \begin{aligned}
    \tilde{\zeta}_{11} &= 0, \\
    \tilde{\zeta}_{22} &= 0, \\
    \tilde{\zeta}_{12} &= -\mathcal{P}(t - 1)h, \\
    \tilde{\zeta}_{21} &= \mathcal{P}(t + 1)h
\end{aligned} \]
Similarly for the other surface $S_{-t}$ we have,

$$
(2.3.8) \begin{align*}
\tilde{\xi}_{11} &= 0, \quad \tilde{\xi}_{22} = 0, \quad \tilde{\xi}_{12} = \gamma(t + 1)h, \\
\tilde{\xi}_{21} &= -\gamma(t - 1)h.
\end{align*}
$$

Now the expression for the geodesic torsion (2.3.5) as a consequence of (2.3.7) takes the form

$$
(2.3.9) \quad \tilde{\tau}_g(1) = -\{ \tilde{\gamma}^{12} \tilde{\xi}_{12} + \tilde{\gamma}^{21} \tilde{\xi}_{21} \}.
$$

But we know,

$$
(2.3.10) \quad \tilde{\gamma}^{12} = \tilde{\gamma}^{21} = -\frac{\tilde{\gamma}_{12}}{\Delta_t^2} = -\frac{F_t}{\Delta_t^2}
$$

where $\tilde{\gamma}^{\alpha\beta}$ denotes the contravariant components of the first fundamental tensor $\tilde{\gamma}^{\alpha\beta}$ of $S_t$. Thus (2.3.9) as a consequence of (2.3.7) and (2.3.10) becomes

$$
(2.3.11) \quad \tilde{\tau}_g(1) = \frac{(2\gamma h)F_t}{\Delta_t^2}
$$

Similarly it could be shown that,

$$
(2.3.12) \quad \tilde{\tau}_g(1) = \frac{(2\gamma h)F_{-t}}{\Delta_{-t}^2}
$$

**Theorem 3.2:** When the congruence is that of Guichard,

$$
F_t = F_{-t} \quad \text{for all} \quad t.
$$
Proof: Since \( \left\{ \begin{array}{l} l_1 \\ l_2 \end{array} \right\} = 0 = \left\{ \begin{array}{l} 2l_1 \\ l_2 \end{array} \right\} \) for a Guichard congruence we have from (2.2.7) and (2.2.8)

\[
A = (t + 1) \phi_{,1} ; \quad A_1 = (t - 1) \phi_{,1}
\]
\[
B = (t - 1) \phi_{,2} ; \quad B_1 = (t + 1) \phi_{,2}
\]

which gives us the result that,

\[ AB = A_1 B_1 . \]

Hence considering the expressions for \( F_t \) and \( F_{-t} \) in (2.2.9) and (2.2.13) we get

\[ F_t = F_{-t} \text{ for all } t . \quad \text{q.e.d.} \]

**Expressions for the torsions in the case of focal surfaces.**

We have from (2.2.9) when \( t = 1 \),

\[ F_1 = -4 \phi \left\{ \begin{array}{l} l_1 \\ l_2 \end{array} \right\} ( \phi_{,1} + \phi \left\{ \begin{array}{l} 2l_1 \\ l_2 \end{array} \right\} ) \]

and \( \Delta_1^2 \) is given by (2.2.12). Let \( \bar{\xi}_g(1) \), \( \bar{\xi}_g(1) \) denote the torsions corresponding to the focal surfaces. Then we have,

\[
\bar{\xi}_g(1) = \frac{8 \phi^2 h \left\{ \begin{array}{l} l_1 \\ l_2 \end{array} \right\} ( \phi_{,1} + \phi \left\{ \begin{array}{l} 2l_1 \\ l_2 \end{array} \right\} )}{16 \phi^2 g( \phi_{,1} + \phi \left\{ \begin{array}{l} 2l_1 \\ l_2 \end{array} \right\} )^2}
\]
\[
= \frac{h \left\{ \begin{array}{l} l_1 \\ l_2 \end{array} \right\}}{2g( \phi_{,1} + \phi \left\{ \begin{array}{l} 2l_1 \\ l_2 \end{array} \right\} )}.
\]
From (2.2.12) we have,

\[ \frac{\Delta_1}{4 F \sqrt{g}} = \left( \varphi,_{1} + \varphi,^{2}_{2} \right) \]

Thus the expression for \( \overline{c}_g(1) \) takes the form

\[ \overline{c}_g(1) = - \frac{2 \varphi h \left\{ \begin{array}{c} 1 \\ 12 \end{array} \right\} \Delta_1}{\sqrt{g}} \]

Similarly we can deduce,

\[ \overline{c}_g(1) = - \frac{2 \varphi h \left\{ \begin{array}{c} 2 \\ 12 \end{array} \right\} \Delta_{-1}}{\sqrt{e}} \]

The above expressions for the torsions corresponding to the focal surfaces agree with the results already obtained in [2].

**Remark:** If the geodesic torsion of the congruence line with respect to a focal surface vanishes identically, the developables of the congruence meet the focal surface in its lines of curvature and the congruence is a Guichard congruence.

(For details [2] may be referred).

**(iii) Normal curvature**

It has been shown [2] that the normal curvature of the congruence line with respect to \( S \) as defined by (1.7.7) can be put in the form
\[(2.3.13)\quad K_n(1) = g^{\alpha\beta} p_\beta v_\alpha\]

where

\[(2.3.14)\quad v_\alpha = \Sigma \lambda^i_\alpha x^i\]

and

\[(2.3.15)\quad p_\alpha = \Sigma \lambda^i_\alpha x^i\]

we have,

\[x^i = \frac{x^i_{1} \times x^i_{2}}{\sqrt{g}}\]

Let \(\tilde{K}_n(1)\), \(\tilde{K}_n(1)\) respectively denote the normal curvatures of the congruence line with respect to \(S_t\) and \(S_{-t}\). Similarly

\[\tilde{v}_\alpha, \tilde{v}_\alpha; \tilde{p}_\alpha, \tilde{p}_\alpha; \tilde{g}^{\alpha\beta}, \tilde{g}^{\alpha\beta}\] corresponds to \(S_t\) and \(S_{-t}\) respectively. Therefore we have,

\[(2.3.16)\quad \tilde{K}_n(1) = \tilde{g}^{\alpha\beta} \tilde{p}_\beta \tilde{v}_\alpha\]

that is,

\[\tilde{K}_n(1) = \tilde{g}^{11} \tilde{p}_1 \tilde{v}_1 + \tilde{g}^{12} \tilde{p}_2 \tilde{v}_1 + \tilde{g}^{21} \tilde{p}_1 \tilde{v}_2 + \tilde{g}^{22} \tilde{p}_2 \tilde{v}_2\]

We have from \((2.3.15)\), \((2.2.3)\) and \((2.2.4)\)

\[(2.3.17)\quad \tilde{p}_1 = \Sigma \lambda^i_1 x^i_{1} = A\]

\[(2.3.18)\quad \tilde{p}_2 = \Sigma \lambda^i_2 x^i_{2} = B\]

Also we have from \((2.3.14)\)
(2.3.19) \( \tilde{v}_\alpha = \sum_i \chi^i,\alpha \chi^i \) with \( \tilde{\chi}^i = \frac{\tilde{\chi}^i_1 \times \tilde{\chi}^i_2}{\Delta t} \)

Now from (2.2.3) and (2.2.4) we have,

\[
\tilde{\chi}^i_1 \times \tilde{\chi}^i_2 = \rho^2(t^2 - 1)(\lambda^1,1 \times \lambda^1,2) + \rho(t - 1)B(\lambda^1,1 \times \lambda^1) + \rho(t + 1)A(\lambda^i \times \lambda^i,2)
\]

Making use of (2.3.6) and the corresponding expressions for \((\lambda^1,1 \times \lambda^1)\) and \((\lambda^1 \times \lambda^i,2)\) the above expression takes the form,

\[
\tilde{\chi}^i_1 \times \tilde{\chi}^i_2 = \rho^2(t^2 - 1)h \lambda^i + \frac{\rho(t - 1)B}{h} (\lambda^1,1 - e \lambda^1,2)
\]

\[+ \frac{\rho(t + 1)A}{h} (f \lambda^1,2 - g \lambda^1,1)\]

Hence (2.3.19) becomes

\[
\tilde{v}_\alpha = \frac{1}{\Delta t} \left[ \frac{\rho(t - 1)B}{h} (f G_{\alpha 1} - e G_{\alpha 2}) + \frac{\rho(t + 1)A}{h} (f G_{\alpha 2} - g G_{\alpha 1}) \right]
\]

This gives us,

\[
\tilde{v}_1 = \frac{\rho(t + 1)A(f^2 - eg)}{(\Delta t)h}
\]

or,

(2.3.20) \( \tilde{v}_1 = -\frac{\rho(t + 1)Ah}{\Delta t} \)
Similarly,

\[ \tilde{v}_2 = -\frac{p(t - 1)Bh}{\Delta_t} \]

We also know that,

\[ \tilde{g}^{11} = \frac{G_t}{\Delta_t}, \quad \tilde{g}^{12} = \tilde{g}^{21} = -\frac{F_t}{\Delta_t}, \quad \tilde{g}^{22} = \frac{E_t}{\Delta_t} \]

Thus the expanded form of (2.3.16) as a consequence of (2.3.17), (2.3.18), (2.3.20), (2.3.21) and (2.3.22) takes the form,

\[ \tilde{K}_n(1) = -\frac{1}{(\Delta_t)^3} \left[-\frac{p(t + 1)A^2hG_t - F_t}{\Delta_t} - \frac{p(t - 1)ABh}{\Delta_t} - (t - 1)B^2hE_t \right] \]

that is,

\[ \tilde{K}_n(1) = -\frac{p_h}{(\Delta_t)^3} \left[(t + 1)G_tA^2 - 2tF_tA^B + (t - 1)E_tA^2 \right] \]

Similarly it could be shown that,

\[ \tilde{K}_n(1) = -\frac{p_h}{(\Delta_{-t})^3} \left[-(t - 1)G_{-t}A^2 + 2tF_{-t}A^B \right] \]

Expression for the normal curvatures in the case of focal surfaces.

Let \( \tilde{K}_n(1), \bar{K}_n(1) \) denote the normal curvatures corresponding to the forcal surfaces. When \( t = 1, \)
(2.3.23) takes the form

\[ (2.3.25) \quad \bar{K}_n(1) = -\frac{2\wp h}{(\Delta_1)^3} [G_tA^2 - F_tAB]_{t=1} \]

we have from (2.2.7)

\[ (A)_{t=1} = 2(\wp,_{11} + \wp,_{12}^2) \]

and from (2.2.12)

\[ \Delta_1 = 4\wp g (\wp,_{11} + \wp,_{12}^2) , \]

that is,

\[ (2.3.26) \quad \Delta_1 = 2\wp g (A)_{t=1} \]

We also have from (2.2.9)

\[ (2.3.27) \quad G_1 = 4\wp^2 g + (B^2)_{t=1} ; \quad F_1 = (AB)_{t=1} \]

Thus (2.3.25) as a consequence of (2.3.26) and (2.3.27) takes the form

\[
\bar{K}_n(1) = \frac{2\wp h}{(2\wp g A)^3} [(4\wp^2 g + B^2)A^2 - (AB)^2]_{t=1} \\
= -\frac{2\wp h}{8\wp g^{3/2}(A)^3} (4\wp^2 g)(A^2)_{t=1} \\
= -\frac{h}{\sqrt{g} (A)_{t=1}}
\]
Substituting the value of \((A)_{t=1}\) from \((2.3.26)\) the above expression for \(\bar{K}_{n(1)}\) takes the form

\[ \bar{K}_{n(1)} = -\frac{2\rho h}{\Delta_1} \]

Similarly we can deduce that,

\[ K_{n(1)} = \frac{2\gamma h}{\Delta_{-1}} \]

The above two results agree with the results already obtained in \([2]\).

**Remark:** If the middle surface is coincident with the focal surfaces, the normal curvature of the congruence line with respect to the middle surface vanishes identically and the congruence is such that the focal surfaces are met by the developables in asymptotic lines. Conversely also if the developables of the congruence meet the focal surfaces in asymptotic lines, the focal surface coincide with the middle surface.

(For details \([2]\) may be referred).

**Theorem 3.1**

When the congruence \((\mathcal{A})\) is normal the geodesics on \(S_1\) in which the lines of the congruence meet \(S_1\) are
cylindrical helices if \( \sqrt{\frac{g}{T_{12}}} \) is a function of \( v \) only and those on \( S_2 \) are cylindrical helices if \( \sqrt{\frac{e}{T_{21}}} \) is a function of \( u \) only.

**Proof:** It is known that (see 1.6.1) if the congruence (\( \mathcal{A} \)) is normal, \( b = b' \). If the surface of reference be a middle surface of the congruence and the parametric curves on the sphere represent the developables of the congruence we have (1.4.12)

\[
b = -b' = \mathcal{P} f
\]

where \( \mathcal{P} \) is half the distance between focal points. This implies \( f = 0 \) for a normal congruence, since \( \mathcal{P} \neq 0 \) in general. The geodesic torsions corresponding to the focal surfaces \( S_1, S_2 \) are given by

\[
\vec{\kappa}_g(1) = -\frac{2\mathcal{P} h \{12\}}{\sqrt{g} \Delta_1} \quad ; \quad \vec{\kappa}_g(1) = -\frac{2\mathcal{P} h \{21\}}{\sqrt{e} \Delta_{-1}}
\]

and the normal curvatures are given by

\[
\vec{K}_n(1) = -\frac{2\mathcal{P} h}{\Delta_1} \quad ; \quad \vec{K}_n(1) = \frac{2\mathcal{P} h}{\Delta_{-1}}
\]

For a geodesic, since the geodesic curvature is zero, its curvature is the normal curvature and its torsion is the...
geodesic torsion. So we consider

\[
\frac{\kappa_n(1)}{\kappa_g(1)} = \frac{\sqrt{g}}{\sqrt{\gamma}} \quad \text{and} \quad \frac{\xi}{\xi_g} = -\frac{\sqrt{e}}{\sqrt{\gamma}}
\]

Suppose \(\sqrt{g}\) is a function of \(v\) only and \(\sqrt{e}\) is a function of \(u\) only, then since \(v = \text{const.}\) are geodesics on \(S_1\) and \(u = \text{const.}\) are geodesics on \(S_2\) the ratios in (2.3.28) are constants and the geodesics on \(S_1\) and \(S_2\) are cylindrical helices.

q.e.d.

To illustrate this theorem, we have when \(u,v\) are isometric parameters of the isometric net \((u,v)\) on the sphere \(S\) the coefficients of the linear element are ([3] P.301)

\[
e = g = \frac{4}{(u^2 + v^2 + 1)^2} \quad \text{and} \quad f = 0
\]

The Christoffel symbols \(\{1\}^{12}_{12}\) and \(\{2\}^{12}_{12}\) as given by (1.4.15), (1.4.16) in view of the values given above become,

\[
\{1\}^{12}_{12} = \frac{g \frac{\partial e}{\partial v}}{2 eg} = \frac{\delta}{\delta v} \log \sqrt{e} = -\frac{2v}{u^2 + v^2 + 1}
\]

\[
\{2\}^{12}_{12} = \frac{e \frac{\partial g}{\partial u}}{2 eg} = \frac{\delta}{\delta u} \log \sqrt{g} = -\frac{2u}{u^2 + v^2 + 1}
\]
So we have,
\[
\frac{\bar{K}_n(1)}{\bar{\mathcal{C}}_g(1)} = \sqrt{g} \begin{bmatrix} 1 \\ 12 \end{bmatrix} = \frac{-2}{u^2 + v^2 + 1} = \frac{1}{u^2 + v^2 + 1} = \frac{1}{v}
\]

and
\[
\frac{\bar{K}_n(1)}{\bar{\mathcal{C}}_g(1)} = -\sqrt{e} \begin{bmatrix} 2 \\ 12 \end{bmatrix} = \frac{2}{u^2 + v^2 + 1} = -\frac{1}{u^2 + v^2 + 1}
\]

Remark: The above two results have been obtained by us without using the condition of area preserving representation whereas the same results are obtained in [4] using the condition of area preserving representation.

2.4 Condition for the rectilinear congruence to establish an area preserving representation between a pair of one parameter family of surfaces.

We know that a necessary and sufficient condition for the lines of the congruence to establish an area preserving representation between the family of surfaces $S_t$ and $S_{-t}$ is
that

\[ \Delta_t = \Delta_{-t} \]

Squaring and using (2.2.11) and (2.2.15) we have,

\[
\begin{align*}
&\hat{\phi}^4(eg - f^2)(t^2 - 1)^2 + \hat{\phi}^2[eB^2(t - 1)^2 + gA^2(t + 1)^2] \\
&\quad - 2 \hat{\phi}^2 f(t^2 - 1)AB \\
&= \hat{\phi}^4(eg - f^2)(t^2 - 1)^2 + \hat{\phi}^2[eB^2(t + 1)^2 + gA^2(t - 1)^2] \\
&\quad - 2 \hat{\phi}^2 f(t^2 - 1)A_1B_1
\end{align*}
\]

that is,

\[
(t + 1)^2(gA_2^2 - eB_1^2) - 2f(t^2 - 1)(AB - A_1B_1)
\]

\[
= (t - 1)^2(gA_1^2 - eB^2)
\]

Using (2.2.7), (2.2.8) and collecting the terms with various powers of \((t + 1), (t - 1)\) and \(t(t^2 - 1)\) we get,

\[
(t + 1)^4(g_{,1}^2 - e_{,2}^2) + 4 \hat{\phi} (t + 1)^3(g_{,1} \left\{ \begin{array}{c} 2 \\ \{12\} \end{array} \right\} - e_{,2} \left\{ \begin{array}{c} 1 \\ \{12\} \end{array} \right\} )
\]

\[
+ 4 \hat{\phi}^2(t + 1)^2(g_{,1} \left\{ \begin{array}{c} 2 \\ \{12\} \end{array} \right\} ^2 - e_{,1} \left\{ \begin{array}{c} 1 \\ \{12\} \end{array} \right\} )
\]

\[
- 8 \hat{\phi} f (t^2 - 1)( \hat{\phi}_{,2} \left\{ \begin{array}{c} 2 \\ \{12\} \end{array} \right\} - \hat{\phi}_{,1} \left\{ \begin{array}{c} 1 \\ \{12\} \end{array} \right\} )
\]

\[
= (t - 1)^4(g_{,1}^2 - e_{,2}^2)
\]

\[
- 4 \hat{\phi} (t - 1)^3(g_{,1} \left\{ \begin{array}{c} 2 \\ \{12\} \end{array} \right\} - e_{,2} \left\{ \begin{array}{c} 1 \\ \{12\} \end{array} \right\} )
\]

\[
+ 4 \hat{\phi}^2(t - 1)^2(g_{,1} \left\{ \begin{array}{c} 2 \\ \{12\} \end{array} \right\} ^2 - e_{,1} \left\{ \begin{array}{c} 1 \\ \{12\} \end{array} \right\} )
\]
The above equation may be written as
\[(2.4.1) \quad \alpha(t + 1)^4 + \beta(t + 1)^3 + \gamma(t+1)^2 + \eta t(t^2 - 1) = \alpha(t - 1)^4 + \beta(1 - t)^3 + \gamma(t - 1)^2\]

where,
\[\alpha = g_{1,1}^2 - e_{1,2}^2; \quad \beta = 4f_1 \left( g_{1,2}^{21} - e_{1,2}^{11} \right); \]
\[\gamma = 4f_2 \left( g_{12}^{21} - e_{12}^{11} \right); \quad \eta = 8f_1 \left( g_{1,1}^{11} - e_{1,2}^{12} \right)\]

The equation (2.4.1) is identically satisfied for all \( t \) only when,
\[\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \eta = 0\]

which will lead to the required conditions, namely,
\[(2.4.2) \quad g_{1,1}^2 = e_{1,2}^2\]
\[(2.4.3) \quad g_{1,1}^{21} = e_{1,2}^{11}\]
\[(2.4.4) \quad g_{12}^{21} = e_{12}^{11}\]
\[(2.4.5) \quad e_{1,2}^{21} = e_{1,2}^{11}\]

It can be easily verified that the above set of four conditions for area preserving representation are consistent.
We give an example of a rectilinear congruence with area preserving representation in the next article.

2.5 An example and theorems associated with the curvatures of the congruence line having an area preserving representation.

The Guichard congruence has been defined in article 6 of the first chapter. A necessary and sufficient condition for a Guichard congruence is that [3]

\[(2.5.1) \quad \begin{bmatrix} 1 \\ 12 \end{bmatrix} \begin{bmatrix} 1 \\ 12 \end{bmatrix} = 0\]

The only real solution of the above equation is [3]

\[e = 1, \quad g = 1, \quad f = -\cos \omega\]

where \(\omega\) is the angle between the focal planes passing through a ray of the congruence. Let \(\varphi = F(\omega)\) and we have,

\[(2.5.2) \quad \begin{bmatrix} \varphi, 1 \\ \varphi, 2 \end{bmatrix} = \begin{bmatrix} \frac{\delta \varphi}{\delta u} = F'(\omega) \frac{\delta \omega}{\delta u} \\ \frac{\delta \varphi}{\delta v} = F'(\omega) \frac{\delta \omega}{\delta v} \end{bmatrix}\]

Of the four conditions for the rectilinear congruence to
establish an area preserving representation between the family of surfaces $S_t$ and $S_{-t}$, three conditions are immediately satisfied in view of (2.5.1) and the only other condition is,

$$g^2 \varphi_{1,1} = e^2 \varphi_{2,2}$$

This with $e = 1$, $g = 1$ leads to $\varphi_{1,1} = \varphi_{2,2}$.

Hence (2.5.2) imply that,

$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial v}$$

Let us take $\omega$ to be solution of

(2.5.3) \quad $\frac{\partial \omega}{\partial u} = 2 \sin(\omega/2) = \frac{\partial \omega}{\partial v}$

This gives,

$$\frac{\partial^2 \omega}{\partial u \partial v} = \cos(\omega/2) \frac{\partial \omega}{\partial u} = 2 \sin(\omega/2) \cos(\omega/2)$$

that is,

$$\frac{\partial^2 \omega}{\partial u \partial v} = \sin \omega$$

which in fact verifies the Gauss equation of spherical representation (1.4.13) for the values $e = 1$, $g = 1$, $f = -\cos \omega$.

Now, from (2.5.2)

$$\frac{\partial^2 \rho}{\partial u \partial v} = F'(\omega) \frac{\partial^2 \omega}{\partial u \partial v} + \frac{\partial \omega}{\partial u} F''(\omega) \frac{\partial \omega}{\partial v}$$
that is,
\[ \frac{\partial^2 \rho}{\partial u \partial v} = F'(\omega) \sin \omega + 4 \sin^2(\omega/2)F''(\omega) \]

Therefore, we have,
\[ \frac{\partial^2 \rho}{\partial u \partial v} = F'(\omega) \sin \omega + 2(1 - \cos \omega)F''(\omega) \]

Further \( f \) should satisfy (1.4.14), which with (2.5.1) reduces to,
\[ \frac{\partial^2 \rho}{\partial u \partial v} + f = 0 \]

that is,
\[ F'(\omega) \sin \omega + 2(1 - \cos \omega)F''(\omega) - \cos \omega F(\omega) = 0 \]

that is,
\[ [2(1 - \cos \omega)F'(\omega) + 2F'(\omega) \sin \omega] - [F'(\omega) \sin \omega + \cos \omega F(\omega)] = 0 \]

Hence,
\[ 2 \frac{d}{d\omega} [(1 - \cos \omega)F'(\omega)] = \frac{d}{d\omega} [F(\omega) \sin \omega] \]

Integrating,
\[ 2(1 - \cos \omega)F'(\omega) = F(\omega) \sin \omega + c \]
Separating the variables with \( \omega = 0 \) when \( c = 0 \),

\[
\frac{2F'(\omega)}{F(\omega)} = \frac{\sin \omega}{1 - \cos \omega}
\]

so that we have,

\[
2 \int \frac{F'(\omega)}{F(\omega)} \, d\omega = \int \frac{\sin \omega}{1 - \cos \omega} \, d\omega + c'
\]

that is,

\[
\log F(\omega)^2 = \log(1 - \cos \omega) + \log k
\]

Therefore,

\[
F(\omega)^2 = 2k \sin^2(\omega/2)
\]

Thus we have determined,

\[
F(\omega) = \int = \alpha \sin(\omega/2), \quad \alpha \text{ being a constant.}
\]

We now proceed to prove a few theorems associated with the curvatures of one parameter family of surfaces when the rectilinear congruence establish an area preserving representation between \( S_t \) and \( S_{-t} \).

**Theorem 5.1**

A necessary and the sufficient condition for the rectilinear congruence to establish an area preserving
representation between the family of surfaces $S_t$ and $S_{-t}$ is that their corresponding geodesic curvatures of the congruence line are numerically equal.

**Proof:** We have from (2.3.3)

$$\tilde{K}_g(1) = \frac{2f}{\Delta_t}, \quad \tilde{K}'_g(1) = \frac{2f}{\Delta_{-t}}$$

When the rectilinear congruence establish an area preserving representation between $S_t$ and $S_{-t}$ we must have, $\Delta_t = \Delta_{-t}$.

This gives us

$$\tilde{K}_g(1) = \tilde{K}'_g(1)$$

Conversely, if the two curvatures are equal we get

$$\Delta_t = \Delta_{-t} \quad \text{q.e.d.}$$

**Lemma 5.1**

When the rectilinear congruence establish an area preserving representation between the family of surfaces $S_t$ and $S_{-t}$ we have,

(2.5.5) \quad AB = A_1B_1 \\
(2.5.6) \quad F_t = F_{-t}
Proof: From (2.2.7) and (2.2.8) we have,

\[ AB - A_1B_1 = [(t^2 - 1) \mathcal{S}_{t,1} \mathcal{S}_{t,2} - 2f(t + 1) \mathcal{S}_{t,1} \{ \frac{1}{12} \} \]
\[ + 2f(t - 1) \mathcal{S}_{t,2} \{ \frac{2}{12} \} - 4f^2 \{ \frac{1}{12} \} \{ \frac{2}{12} \} ] \]
\[ - [(t^2 - 1) \mathcal{S}_{t,1} \mathcal{S}_{t,2} - 2f(t + 1) \mathcal{S}_{t,2} \{ \frac{2}{12} \} \]
\[ + 2f(t - 1) \mathcal{S}_{t,1} \{ \frac{1}{12} \} - 4f^2 \{ \frac{1}{12} \} \{ \frac{2}{12} \} ] \]

that is,

\[ AB - A_1B_1 = 4f(t \{ \frac{2}{12} \} - \mathcal{S}_{t,1} \{ \frac{1}{12} \}) \]

In view of (2.4.5) the right hand side drops out and we get (2.5.5). 

Also from (2.2.9) and (2.2.13) we have

\[ F_t = f^2(t^2 - 1) + AB \]
\[ F_{-t} = f^2(t^2 - 1) + A_1B_1 \]

As a consequence of (2.5.5) we get (2.5.6). q.e.d.

We use this lemma to establish the following theorem.

**Theorem 5.2**

When the rectilinear congruence establish an area preserving representation between the family of surfaces \( S_t \)
and \( S_t \) , the corresponding geodesic torsions are numerically equal.

**Proof:** We have from (2.3.11) and (2.3.12)

\[
\tilde{\tau}_g(1) = \frac{(2 \rho h)F_t}{\Delta_t^2}; \quad \check{\tau}_g(1) = \frac{(2 \rho h)F_{-t}}{\Delta_{-t}^2}
\]

and we also have by hypothesis \( \Delta_t = \Delta_{-t} \). Also by lemma 5.1, \( F_t = F_{-t} \). So we get,

\[
\tilde{\tau}_g(1) = \check{\tau}_g(1)
\]

q.e.d.

**Theorem 5.3**

When the rectilinear congruence establish an area preserving representation between the family of surfaces \( S_t \) and \( S_{-t} \), the sum of the corresponding normal curvatures is identically zero.

**Proof:** In the expressions for \( \tilde{\kappa}_n(1) \) and \( \check{\kappa}_n(1) \) given in (2.3.23), (2.3.24) we use \( \Delta_t = \Delta_{-t} \) and lemma 5.1 to get

\[
\tilde{\kappa}_n(1) + \check{\kappa}_n(1) = -\frac{2h}{(\Delta_t)^3} \left[ (t + 1)(G_tA^2 - E_{-t}B_1^2) \right. \\
\left. + (t - 1)(E_tB^2 - G_{-t}A_1^2) \right]
\]

Now using (2.2.7) and (2.2.8) we get
\( \widetilde{K}_n(1) + \tilde{K}_n(1) = -\frac{p^h}{(\Delta_t)^3} [(t + 1) \{ (\rho^2 g(t + 1)^2 + B^2)A^2 \\
- (\rho^2 e(t + 1)^2 + A^2 B^2) \} \\
+ (t - 1) \{ (\rho^2 e(t - 1)^2 + A^2 B^2) \} \\
- (\rho^2 g(t - 1)^2 + B^2)A^2 \} ] \)

Again using (2.5.5) the right hand side of the above equation reduces to

\[-\frac{j^3 h}{(\Delta_t)^3} [(t + 1)^3(gA^2 - eB^2) + (t - 1)^3(eB^2 - gA^2)] \]

Using the values of \( A, A_1, B, B_1 \) from (2.2.7) and (2.2.8) and collecting terms with various powers of \( (t + 1) \) and \( (t - 1) \) we have,

\( \widetilde{K}_n(1) + \tilde{K}_n(1) = -\frac{j^3 h}{(\Delta_t)^3} [(t + 1)^5(g \rho_{1,1}^2 - e \rho_{2,1}^2) \\
+ 4 \rho (t + 1)^4(g \rho_{1,1} \{ 2 \}_{12} - e \rho_{2,1} \{ 1 \}_{12}) \\
+ 4 \rho^2 (t + 1)^3(g \{ 2 \}_{12}^2 - e \{ 1 \}_{12}^2) \\
+ (t - 1)^5(e \rho_{1,2}^2 - g \rho_{1,1}^2) \\
+ 4 \rho (t - 1)^4(e \rho_{1,2} \{ 1 \}_{12} - g \rho_{1,1} \{ 2 \}_{12}) \\
+ 4 \rho^2 (t - 1)^3(e \{ 1 \}_{12}^2 - g \{ 2 \}_{12}^2) ] \)
Using the conditions for area preserving representation (2.4.2), (2.4.3), (2.4.4) we get

\[ \tilde{\kappa}_n(1) + \tilde{\kappa}_n(1) = 0. \]

q.e.d.
REFERENCES


