CHAPTER V

ON A LINEAR TRANSFORMATION FOR A LINE CONGRUENCE

5.1 Introduction

In the study of a line congruence ($\mathcal{L}$) in a three dimensional Euclidean space, Kummer's quadratic form (1.2.3) plays a vital role. We define a linear transformation of the tangent space to the spherical representation of ($\mathcal{L}$) based on the symmetric part of the tensor defined by the Kummer's quadratic form. This linear transformation is analogous to the Weingarten map (Chapter I, article 8) and so can be used to define a normal curvature and geodesic torsion of the congruence. These curvatures of the
congruence are related to the parameter of distribution and 
the distance of the central point of a ray of the congruence 
from the director surface. We obtain several results based 
on these curvatures.

5.2 Preliminaries

Let \( S \) be a surface in a three dimensional Euclidean 
space and \( \Sigma \) -a unit sphere on which we have the spherical 
representation of the congruence \( (\lambda) \). Let \( ds, d\sigma \) 
respectively denote the arc lengths on \( S \) and \( \Sigma \). The 
rectilinear congruence \( (\lambda) \) is defined by \( (1.1.1) \) and the 
coefficients of the Kummer's quadratic form is given by 
\( (1.2.4) \). We set

\[
(5.2.1) \quad \mu(\alpha\beta) = \frac{1}{2} (\mu_{\alpha\beta} + \mu_{\beta\alpha})
\]

and

\[
(5.2.2) \quad \mu[\alpha\beta] = \frac{1}{2} (\mu_{\alpha\beta} - \mu_{\beta\alpha})
\]

The fundamental tensor \( G_{\alpha\beta} \) of the spherical representation 
is also given by

\[
(5.2.3) \quad G_{\alpha\beta} = g^{\gamma\delta} \mu_{\gamma\alpha} \mu_{\delta\beta} + v_{\alpha} v_{\beta}
\]

where we have used [3]
(5.2.4) \[ \lambda_{i}^{\alpha} = \mu_{\alpha} x_{i}^{\delta} + v_{\alpha} x_{i} \]

with

(5.2.5) \[ \mu_{\alpha} g_{\delta\beta} = \mu_{\alpha\beta} \]

and \( v_{\alpha} \) is given by (2.3.14).

Suppose the parametric curves on the spherical representation of the congruence correspond to the principal surfaces of the congruence then,

(5.2.6) \[ \mu_{12} + \mu_{21} = 0 ; G_{12} = 0 \]

or

(5.2.7) \[ \mu_{(12)} = 0 ; f = 0 \]

Let \( K_{m} \) and \( K \) respectively denote the 'mean curvature' and the 'Gaussian curvature' of \( S \). Let \( \gamma_{g} \) and \( k_{n} \) respectively be the 'geodesic torsion' and 'normal curvature' of a curve \( C \) on \( S \) which determines a ruled surface of the congruence.

We have [1]

(5.2.8) \[ G_{\alpha\beta} = K_{m} d_{\alpha\beta} - K g_{\alpha\beta} \]

(5.2.9) \[ \frac{d\sigma^{2}}{d\sigma^{2}} = K_{m} k_{n} - K \]

(5.2.10) \[ \gamma_{g} = \frac{e^{\gamma\delta}}{\sqrt{g}} d_{\delta\alpha} g_{\gamma\beta} \frac{du^{\alpha}}{d\sigma} \frac{du^{\beta}}{d\sigma} \]

where \( \gamma_{g} = \text{det}(g_{\alpha\beta}) \)
5.3 Curvatures of the congruence in terms of a linear transformation

Since $\Sigma$ is a unit sphere on which we have the spherical representation of $(\lambda^i)$, the point $P'(\lambda^i)$ on $\Sigma$ corresponds to the point $P(x^i)$ on $S$. Since $\lambda^i$ is a unit vector, $\frac{d\lambda^i}{d\sigma}$ is orthogonal to $\lambda^i$ and lies in the tangent space to $\Sigma$ at $P'(\lambda^i)$. The vector $\frac{d\lambda^i}{d\sigma}$ can also be considered as a vector in the tangent space to $E^3$ at $P(x^i)$ since $d\sigma$ is defined at $P(x^i)$ by

$$d\sigma^2 = (g^{\gamma\delta}) \mu^\gamma \mu^\delta + v^\alpha v^\beta d\alpha d\beta$$

Lemma 3.1 In the tangent space to $\Sigma$ at $P'(\lambda^i)$, let $N^i$ be the unit vector orthogonal to $\frac{d\lambda^i}{d\sigma}$ such that \(\left(\frac{d\lambda^i}{d\sigma}, N^i, \lambda^i\right)\) form a right handed system. Then

$$N^i = \rho^\gamma \lambda^i \gamma$$

where

$$\rho^\gamma = E^{\gamma\delta} G_{\delta\gamma} \frac{d\omega}{d\sigma}$$

is a unit vector.
Proof:

\[
\sum_{i=1}^{3} N_i = \left( f^\gamma \lambda_{i,\gamma} \right) \left( f^\omega \lambda_{i,\omega} \right) = G_{\gamma\omega} E^\delta E^\mu G_{\omega,\delta} G_{\mu,\gamma} \frac{d\xi}{d\sigma} \frac{d\eta}{d\sigma} = G^{\delta\mu} G_{\omega,\delta} G_{\mu,\gamma} \frac{d\xi}{d\sigma} \frac{d\eta}{d\sigma} = G_{\alpha,\gamma} \frac{d\xi}{d\sigma} \frac{d\eta}{d\sigma} = 1,
\]

where we have used (1.3.4)(ii). \( \text{q.e.d.} \)

We consider a linear transformation \( L^* \) of the tangent space \( \Sigma \) at \( P'(\lambda^i) \) induced by the matrix \( \left( G^{\alpha\beta} \mu(\delta\beta) \right) \) and set,

\[
(5.3.4) \quad L^* \left( \frac{d\lambda^i}{d\sigma} \right) = G^{\alpha\beta} \mu(\delta\beta) \frac{d\xi}{d\sigma} \frac{d\eta}{d\sigma} \lambda^i, \alpha
\]

Note that \( L^* \) is similar to the Weingarten map.

**Definition 3.1:** We set

\[
(5.3.5) \quad \lambda^K_{\alpha} = \sum_{i=1}^{3} L^* \left( \frac{d\lambda^i}{d\sigma} \right) \frac{d\lambda^i}{d\sigma}
\]

and

\[
(5.3.6) \quad \lambda^C_{\alpha} = \sum_{i=1}^{3} L^* \left( \frac{d\lambda^i}{d\sigma} \right) N_i
\]
where \( N^i \) is given by (5.3.2) and \( L^* \) is defined by (5.3.4).

We call \( \lambda^k_n^* \) the 'normal curvature' of the congruence \( \lambda \) and \( \lambda^g^* \), the 'geodesic torsion' of the congruence \( \lambda \).

**Lemma 3.2** We have

\[
\lambda^k_n^* = -r
\]

and

\[
\lambda^g^* = E^\gamma^\delta \mu(\delta^\alpha) G^\gamma^\beta \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma}
\]

where \( r \) is the distance of the central point of the line of congruence at \( P(x^i) \) from the director surface \( S \).

**Proof:** Since 

\[
\frac{d\lambda^i}{d\sigma} = \lambda^i,^\omega \frac{du^\omega}{d\sigma}
\]

from (5.3.4) and (5.3.5) we have,

\[
\lambda^k_n^* = \left( G^\sigma^\delta \mu(\delta^\beta) \frac{du^\beta}{d\sigma} \lambda^i,^\alpha \right) \lambda^i,^\omega \frac{du^\omega}{d\sigma}
\]

\[
= G^\sigma^\delta \mu(\delta^\beta) G^\alpha^\omega \frac{du^\beta}{d\sigma} \frac{du^\omega}{d\sigma}
\]

\[
= \mu^\beta^\omega \frac{du^\beta}{d\sigma} \frac{du^\omega}{d\sigma}
\]

and from (1.4.2) we have,

\[
\lambda^k_n^* = -r
\]
Also from (5.3.2), (5.3.4) and (5.3.6) we have,

\[ \mathcal{C}_g^* = \frac{3}{\Sigma} \left( G^\delta_{\alpha\delta} \mu_{(\delta\beta)} \frac{du^\beta}{d\sigma} \lambda^i_{\gamma} \right) \left( \int \gamma^i \lambda^i_{\gamma} \right) \]

\[ = E^\gamma \omega G_{\alpha\gamma} G_{\omega\beta} G^\alpha_{\delta} \mu_{(\delta\beta)} \frac{du^\beta}{d\sigma} \frac{du^\Theta}{d\omega} \]

\[ = E^\gamma \omega G_{\omega\beta} \mu_{(\gamma\beta)} \frac{du^\beta}{d\sigma} \frac{du^\Theta}{d\omega} . \]

q.e.d.

**Theorem 3.1:** A curve \( C' \) on \( \Sigma \) with tangent vector \( \frac{du^\alpha}{d\sigma} \) at \( P'(\lambda^i) \) corresponds to a principal surface of the congruence passing through the ray at \( P(x^i) \) if and only if \( \mathcal{C}_g^* = 0 \).

**Proof:** Consider

\[ E^\gamma \delta \mu_{(\delta\alpha)} G_{\gamma\beta} du^\alpha du^\beta \]

This can be expanded as,

\[ \left\{ E^{12} \mu_{(2\alpha)} G_{1\beta} + E^{21} \mu_{(1\alpha)} G_{2\beta} \right\} du^\alpha du^\beta \]

using (1.3.3) we have,

\[ \frac{1}{h} \left[ (\mu_{(21)} G_{11} - \mu_{(11)} G_{21}) (du^1)^2 + \left\{ (\mu_{(21)} G_{12} - \mu_{(11)} G_{22}) + (\mu_{(22)} G_{11} - \mu_{(12)} G_{21}) \right\} du^1 du^2 + (\mu_{(22)} G_{12} - \mu_{(12)} G_{22}) (du^2)^2 \right] \]
We have from (5.2.1)

\[ \mu(11) = \mu_{11}, \quad \mu(22) = \mu_{22}, \quad \mu(12) = \mu(21) \]

and also making use of (1.2.1), (1.2.4) we have,

\[
\frac{1}{h} \left( \left( \frac{e(b + b')}{2} - fa \right) (du^1)^2 + \left\{ \frac{f(b + b')}{2} - ga + ec \right\} du^1 du^2 + \left( fc - \frac{g(b + b')}{2} \right) (du^2)^2 \right)
\]

that is,

\[
-\frac{1}{2h} \left\{ \left( 2fa - e(b + b') \right) (du^1)^2 + 2(ga - ec) du^1 du^2 + \left[ g(b + b') - 2fc \right] (du^2)^2 \right\}
\]

The equation which determines the two principal ruled surfaces through the ray of the congruence at \( P(x^1) \) is given by (1.4.3). In view of the expression (5.3.8) for \( \lambda^*_g \) and comparing (1.4.3) with the above expression the theorem follows immediately. q.e.d.

Suppose the parametric curves on the spherical representation \( \Sigma \) of the congruence \( \lambda \) correspond to the principal surfaces of the congruence. Then we have from (5.2.6) \( b + b' = 0, \ f = 0 \). If \( r_1, r_2 \) are the limits
as defined in (1.4.5), then we have [3]

\begin{equation}
(5.3.9) \quad -r_1 = \frac{a}{e}, \quad -r_2 = \frac{c}{g}
\end{equation}

**Lemma 3.3** Let \( e_1^i \) and \( e_2^i \) be the unit tangent vectors to the curves \( u^2 = \text{const.} \) and \( u^1 = \text{const.} \) in \( \Sigma \) respectively. Then \( e_1^i \) and \( e_2^i \) are orthogonal and

\begin{equation}
(5.3.10) \quad L^*(e_1^i) = -r_1 e_1^i; \quad L^*(e_2^i) = -r_2 e_2^i
\end{equation}

that is \( r_1 \) and \( r_2 \) are the characteristic values of the linear transformation \( L^* \).

**Proof:** Along the curve \( u^2 = \text{const.} \)

\[
d\sigma^2 = e(du^1)^2
\]

and so

\begin{equation}
(5.3.11) \quad e_1^i = \left( \lambda_1^1 \right) \left( \frac{du^1}{d\sigma} \right) u^2 = \text{const.} = \frac{\lambda_1^1}{\sqrt{e}}
\end{equation}

Similarly we have,

\begin{equation}
(5.3.12) \quad e_2^i = \frac{\lambda_2^1}{\sqrt{g}}
\end{equation}

Since \( f = 0 \), \( e_1^i, e_2^i \) are orthogonal. From (5.3.4) and (5.3.11) we get
\[ L^*(e_1^i) = G^\delta \mu(\delta\alpha) \left( \frac{du^\alpha}{d\sigma} \right) u^2 = \text{const.} \]
\[ = G^{11} \mu(11) \left( \frac{du^{11}}{d\sigma} \right) u^2 = \text{const.} \]
\[ = \left( \frac{\mu_{11}}{G_{11}} \right) \left( \frac{du^{11}}{d\sigma} \right) u^2 = \text{const.} \]
\[ = \left( \frac{a}{e} \right) \frac{\lambda^1,1}{\sqrt{e}} \]

Using (5.3.9), (5.3.11) we get,

\[ L^*(e_1^i) = -r_1 e_1^i \]

Similarly we can show that,

\[ L^*(e_2^i) = -r_2 e_2^i \]

q.e.d.

**Theorem 3.2:** We have,

\[ \lambda^*_n = r_1 \cos^2 \theta + r_2 \sin^2 \theta \]  

(Hamilton's formula)

and

\[ \lambda^*_g = (r_1 - r_2) \sin \theta \cos \theta \]

where \( \theta \) is the angle made by the tangent vector \( \frac{d\lambda^i}{d\sigma} \) to a
curve through \( P'(\lambda^i) \) on \( \Sigma \) representing a ruled surface through the line of congruence \((\lambda^i)\) at \( P(x^i) \) and the curve \( u^2 = \text{const.} \).

**Proof:** By hypothesis,

\[
\frac{d\lambda^i}{d\sigma} = (\cos \theta) e^i_1 + (\sin \theta) e^i_2
\]

Then since \( N^i \) is orthogonal to \( \frac{d\lambda^i}{d\sigma} \) and \( (\frac{d\lambda^i}{d\sigma}, N^i, \lambda^i) \) form a right handed system at \( P'(\lambda^i) \) we have,

\[
N^i = (-\sin \theta) e^i_1 + (\cos \theta) e^i_2
\]

Since \( L^* \) is a linear transformation of the tangent space to \( \Sigma \) at \( P'(\lambda^i) \) we have from (5.3.10)

\[
L^* \left( \frac{d\lambda^i}{d\sigma} \right) = -r_1 \cos \theta e^i_1 - r_2 \sin \theta e^i_2
\]

and so from (5.3.5) we get,

\[
\lambda^*_n = \frac{3}{1=1} (-r_1 \cos \theta e^i_1 - r_2 \sin \theta e^i_2)(\cos \theta e^1_1 + \sin \theta e^1_2)
\]

that is,

\[
-\lambda^*_n = r_1 \cos^2 \theta + r_2 \sin^2 \theta
\]

Similarly from (5.3.6) we get,
Therefore we get

\[ \lambda^*_g = (r_1 - r_2) \sin \theta \cos \theta. \quad \text{q.e.d.} \]

5.4 Parameter of distribution as a sum of two torsions of the congruence

The parameter of distribution \( \beta \) has been defined by (1.4.7), is

\[
\beta = \frac{1}{\Sigma} \left( \epsilon_{ijk} \frac{dx^i}{ds} \frac{d\lambda^j}{ds} \lambda^k \right) \frac{ds^2}{d\sigma^2}
\]

\[
= \left( \epsilon_{ijk} \lambda^k \frac{d\lambda^3}{d\sigma} \right) \frac{dx^i}{ds} \frac{ds}{d\sigma}
\]

Since \( \left( \frac{d\lambda^i}{d\sigma}, \ N^i, \lambda^i \right) \) form a right handed system \( \beta \) takes the form,

\[
\beta = \frac{3}{\Sigma} \left( N^i \frac{dx^i}{ds} \right) \frac{ds}{d\sigma}
\]

Now using (5.3.2) we have,

\[
\beta = \left( \epsilon^\delta_{\gamma \omega} \frac{du^\omega}{d\sigma} \lambda^i, \gamma \right) \frac{dx^i}{ds} \frac{ds}{d\sigma}
\]
that is,

\[ \beta = \sum_{\gamma} \frac{d\gamma^\omega}{d\sigma} \lambda_{\gamma} x_{\alpha} \frac{du^\alpha}{d\sigma} \]

Hence we have,

\[ (5.4.1) \quad \beta = \sum_{\gamma} \frac{d\gamma^\omega}{d\sigma} \mu_{\gamma} \frac{du^\alpha}{d\sigma} \frac{du^\omega}{d\sigma} \]

**Definition 4.1:** Let

\[ (5.4.2) \quad \Gamma_{g}^{**} = \sum_{\gamma} \frac{d\gamma^\omega}{d\sigma} \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \]

and we call \( \Gamma_{g}^{**} \), the 'Torsion' of the congruence \((\lambda)\).

The equations (5.2.1) and (5.2.2) gives us

\[ (a \beta) + [a \beta] = \mu_{a \beta} \]

Hence the parameter of distribution \( \beta \), along with (5.3.8) and (5.4.2) takes the form

\[ (5.4.3) \quad \beta = \Gamma_{g}^{*} + \Gamma_{g}^{**} \]

**Theorem 4.1:** The torsion \( \Gamma_{g}^{**} \) of the congruence \((\lambda)\) is a point invariant and is given by

\[ (5.4.4) \quad \Gamma_{g}^{**} = \frac{1}{2} \epsilon^{\beta \alpha} \mu_{\alpha \beta} \]

**Proof:** Now (5.4.2) be expanded as,
Also (5.2.2) gives us,

\[ \mu_{[11]} = 0 = \mu_{[22]} \text{ and } \mu_{[12]} = \mu_{[21]} \]

Hence,

\[
\lambda^{**}_g = E^{12} \left\{ \mu_{[21]} \left[ G_{11} \left( \frac{du^1}{d\sigma} \right)^2 + G_{12} \left( \frac{du^1}{d\sigma} \right) \left( \frac{du^2}{d\sigma} \right) \right] \right. \\
- \left. \mu_{[12]} \left[ G_{21} \left( \frac{du^2}{d\sigma} \right) \left( \frac{du^1}{d\sigma} \right) + G_{22} \left( \frac{du^2}{d\sigma} \right)^2 \right] \right\}
\]

\[
= E^{12} \mu_{[21]} \left\{ G_{11} \left( \frac{du^1}{d\sigma} \right)^2 + 2G_{12} \left( \frac{du^1}{d\sigma} \right) \left( \frac{du^2}{d\sigma} \right) \\
+ G_{22} \left( \frac{du^2}{d\sigma} \right)^2 \right\}
\]

Since \( G_{\alpha\beta} \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} = 1 \) we have,

\[
\lambda^{**}_g = E^{12} \mu_{[21]} = \frac{1}{2} E^{\beta\alpha} \mu_{\alpha\beta}
\]

Thus the torsion of the congruence (\( \lambda \)) is a point invariant.

q.e.d.

**Corollary 4.1**: For a normal congruence the parameter of distribution reduces to the geodesic torsion of the congruence.
Proof: Since $\mu_{[12]} = 0$ for a normal congruence the result follows immediately from (5.4.3). That is,

$$\beta = \frac{C_\gamma}{\lambda^g}$$

Corollary 4.2: The parameter of distribution for the congruence of normals to $S$ is given by

(5.4.5) 

$$\beta = \frac{C_\gamma}{K - K_m k_n}$$

where $C_\gamma$ and $k_n$ are the geodesic torsion and normal curvature of the curve on $S$ which determines the ruled surface of the congruence and $K, K_m$ respectively are the Gaussian curvature and mean curvature of $S$.

Proof: Since the congruence of normals to $S$ is a normal congruence, from corollary 4.1 we have,

$$\beta = \frac{C_\gamma}{\lambda^g}$$

Since $\chi^i = x^i$, $\mu_{\alpha\beta} = \sum_{i=1}^{3} x^i,\alpha x^i,\beta = -d_{\alpha\beta}$

and

$$G_{\alpha\beta} = \sum_{i=1}^{3} x^i,\alpha x^i,\beta$$

From (5.2.8)

$$G_{\alpha\beta} = \{K_m d_{\alpha\beta} - K g_{\alpha\beta}\}$$
Also,
\[ G = \det(G_{\alpha\beta}) = k^2 \gamma^r = h^2 \]
where we have,
\[ \gamma^r = \det(g_{\alpha\beta}) \]

Hence from (5.3.8) we have,
\[ \beta = E^\gamma d_\delta (K_m d_\gamma - K g_{\gamma\beta}) \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \]
\[ = K_m (E^\gamma d_\delta d_\gamma g_{\gamma\beta}) \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \]
\[ - K \left( E^\gamma d_\delta g_{\gamma\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right) \left( \frac{ds}{d\sigma} \right)^2 \]

The first bracket vanish and we have,
\[ \beta = -\frac{K}{h} \left( E^\gamma d_\delta g_{\gamma\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} \right) \left( \frac{ds}{d\sigma} \right)^2 \]

Hence from (5.2.8), (5.2.9) we get
\[ \beta = \frac{-K}{K \sqrt{\gamma}} \frac{1}{\gamma_g} \frac{1}{K_m k_n - K} \]

which will give us,
\[ \beta = \frac{\gamma_g}{K - K_m k_n} \]
q.e.d.
5.5 **Invariants associated with one parameter family of surfaces**

Suppose \((\tilde{\mu}_\alpha^\beta, \tilde{\nu}_\alpha^\beta)\); \((\tilde{\kappa}_n^*, \tilde{\chi}_n^*)\); \((\tilde{L}^*, \tilde{L}^*)\); \((\tilde{\kappa}_g^*, \tilde{\chi}_g^*)\) respectively represent the coefficients of Kummer's quadratic form, the normal curvature of the congruence, the linear transformation of the tangent space to \(\Sigma\). The geodesic torsion of the congruence corresponding to the family of surfaces \(S_t\) and \(S_{-t}\), then the following results are valid.

\[
(5.5.1) \quad \tilde{\mu}_\alpha^\beta = \mu_\alpha^\beta + t^\sigma G_\alpha^\beta; \quad \tilde{\nu}_\alpha^\beta = \mu_\alpha^\beta - t^\sigma G_\alpha^\beta
\]

\[
(5.5.2) \quad \tilde{\mu}_\alpha^\beta - \tilde{\nu}_\beta^\alpha = \tilde{\mu}_\alpha^\beta - \tilde{\nu}_\beta^\alpha
\]

\[
(5.5.3) \quad \tilde{\kappa}_n^* = \kappa_n^* + t^\sigma; \quad \tilde{\chi}_n^* = \kappa_n^* - t^\sigma
\]

\[
(5.5.4) \quad \tilde{\kappa}_n^* + \tilde{\chi}_n^* = 2\kappa_n^*
\]

\[
(5.5.5) \quad \tilde{L}^* \left( \frac{d\lambda^i}{d\sigma} \right) = L^* \left( \frac{d\lambda^i}{d\sigma} \right) + t^\sigma \left( \frac{d\lambda^1}{d\sigma} \right)
\]

\[
(5.5.6) \quad \tilde{\chi}_g^* \left( \frac{d\lambda^i}{d\sigma} \right) = L^* \left( \frac{d\lambda^i}{d\sigma} \right) - t^\sigma \left( \frac{d\lambda^1}{d\sigma} \right)
\]

\[
(5.5.7) \quad \tilde{\chi}_g^* = \chi_g^* = \chi_g^*
\]
Proof: We have,

\[ \bar{\Gamma}_{\alpha\beta} = \sum \lambda_{i,\alpha} \bar{x}_{i,\beta} \quad ; \quad \bar{\mu}_{\alpha\beta} = \sum \lambda_{i,\alpha} \bar{x}_{i,\beta} \]

where, \( \bar{x}^i = x^i + t \gamma^i \) ; \( \bar{x}^i = x^i - t \gamma^i \) represent the family of surfaces \( S_t \) and \( S_{-t} \). This will give us

\[ \bar{\mu}_{\alpha\beta} = \sum \lambda_{i,\alpha} \left\{ x_{i,\beta} + t(\gamma^i,\beta) + \gamma_i \right\} \]

and

\[ \bar{\mu}_{\alpha\beta} = \sum \lambda_{i,\alpha} \left\{ x_{i,\beta} - t(\gamma^i,\beta) + \gamma_i \right\} \]

Making use of (1.2.1), (1.2.3) we have the above expressions taking the form

\[ \tilde{\mu}_{\alpha\beta} = \mu_{\alpha\beta} + t \gamma G_{\alpha\beta} \quad ; \quad \tilde{\mu}_{\alpha\beta} = \mu_{\alpha\beta} - t \gamma G_{\alpha\beta} \]

which establishes (5.5.1). Also it could be easily seen that

\[ \tilde{\mu}_{\alpha\beta} = \mu_{\alpha\beta} + t \gamma G_{\alpha\beta} \quad ; \quad \tilde{\mu}_{\alpha\beta} = \mu_{\alpha\beta} - t \gamma G_{\alpha\beta} \]

which along with (5.5.1) will give us

\[ \tilde{\mu}_{\alpha\beta} - \tilde{\mu}_{\beta\alpha} = \mu_{\alpha\beta} - \mu_{\beta\alpha} \]

and

\[ \tilde{\mu}_{\alpha\beta} - \tilde{\mu}_{\beta\alpha} = \mu_{\alpha\beta} - \mu_{\beta\alpha} \]

Hence the above two equations lead to,
\[ \tilde{\mu}_{\alpha\beta} - \tilde{\mu}_{\beta\alpha} = \tilde{\mu}_{\alpha\beta} - \tilde{\mu}_{\beta\alpha} \]

which establishes (5.5.2).

We have from (5.3.7) and (1.4.2)

\[ \mathcal{K}^*_n = \mu_{\alpha\beta} \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \]

\[ = (\mu_{\alpha\beta} + t^f G_{\alpha\beta}) \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} \]

\[ = \mu_{\alpha\beta} \frac{du^\alpha}{d\sigma} \frac{du^\beta}{d\sigma} + t^f \]

Hence we get

\[ \mathcal{K}^*_n = \mathcal{K}^*_n + t^f \]

Similarly,

\[ \mathcal{K}^*_n = \mathcal{K}^*_n - t^f \]

which establish (5.5.3). Adding the above two results we get,

\[ \mathcal{K}^*_n + \mathcal{K}^*_n = 2 \mathcal{K}^*_n \]

which proves (5.5.4).

Now from (5.3.4) we get,

\[ \tilde{L}^* \left( \frac{d\lambda^i}{d\sigma} \right) = \delta^{i\delta} \tilde{\mu}(\delta\beta) \frac{du^\beta}{d\sigma} \lambda^i_{,\alpha} \]
But, \[ \bar{\mu}(\delta \beta) = \frac{1}{2} (\bar{\mu}_\beta + \bar{\mu}_\delta) \]

Using (5.5.1) the above takes the form,
\[ \bar{\mu}(\delta \beta) = \frac{1}{2} \left\{ (\bar{\mu}_\beta + \bar{\mu}_\delta) + t_F (G_\beta + G_\delta) \right\} \]
\[ = \mu(\delta \beta) + (t_F) G_\delta \]

Hence we have,
\[ L^* \left( \frac{d \lambda^i}{d \sigma} \right) = \left\{ G^\alpha_\delta \frac{du^\beta}{d \sigma} \lambda^i_{\alpha, \alpha} \right\} + (t_F) \left\{ G^\alpha_\delta \frac{du^\beta}{d \sigma} \lambda^i_{\alpha, \alpha} \right\} \]
\[ = L^* \left( \frac{d \lambda^i}{d \sigma} \right) + (t_F) \lambda^i_{\alpha} \left( \frac{du^\alpha}{d \sigma} \right) \]

from which we get (5.5.5). Similarly we prove (5.5.6).

Further by adding (5.5.5) and (5.5.6) we get,
\[ \bar{\gamma}^* \left( \frac{d \lambda^i}{d \sigma} \right) + L^* \left( \frac{d \lambda^i}{d \sigma} \right) = 2 L^* \left( \frac{d \lambda^i}{d \sigma} \right) \]

Now from (5.3.6)
\[ \lambda^i_g = \sum_{i=1}^{3} \bar{L}^* \left( \frac{d \lambda^i}{d \sigma} \right) N^i \]

Using (5.5.5) we get
\[ \lambda^i_g = \sum_{i=1}^{3} L^* \left( \frac{d \lambda^i}{d \sigma} \right) N^i + (t_F) \sum_{i=1}^{3} \left( \frac{d \lambda^i}{d \sigma} \right) N^i \]
Therefore from (5.3.6) we get,

\[ \alpha^* g = \alpha^* g \]

Similarly we establish

\[ \alpha^* g = \alpha^* g \]

which proves (5.5.7).

From (5.5.4) and (5.5.7) we have the following theorem.

**Theorem 5.1:** Let \( S: x^i = x^i(u^\alpha) \) be the middle surface of a congruence \( (\lambda) \), and \( \rho \), half the distance between the focal points of a ray of \( (\lambda) \). Let \( S_t: \tilde{x}^i = x^i + t\rho\lambda^i \) define a family of surfaces for \( t \in \mathbb{R} \). Then the geodesic torsion of a ray of \( (\lambda) \) has the same value for all surfaces \( S_t \) and the normal curvature \( \lambda_n^* \) of the ray with respect to \( S \) is the mean of the normal curvatures \( \lambda_n^* \) of \( (\lambda) \) with respect to \( S_t \) and \( \lambda_n^* \) of \( (\lambda) \) with respect to \( S_{-t} \).
REFERENCES

