CHAPTER IV

RECTILINEAR CONGRUENCE WHOSE RAYS ESTABLISH AN AREA MAGNIFYING REPRESENTATION

4.1 Introduction

In the second and third chapters we studied rectilinear congruences whose rays establish an area preserving representation. In this chapter we study a rectilinear congruence whose lines establish an area magnifying representation. We regard the area element $\Delta_t$ of $S_t$ as a multiple of a suitable function of $t$ with the area element $\Delta$ of $S$, the middle surface of the congruence, and get a consistent system of
equations which help us to determine the congruence. Our construction is such that $S_t$ and $S_{-t}$ will have the same area magnification, this leads us naturally to relate the congruences studied in the last two chapters with the one we study in the present chapter. We give an example of a congruence which allows such a magnification of area.

4.2 The conditions for area magnifying representation

Let $(\mathcal{X}^i)$ be a rectilinear congruence and the family of surfaces $S_t$ be defined by

\[(4.2.1) \quad x^i = x^i + tf^i \]

where $t$ is a real parameter and $f^i$ denotes half the distance between focal points. Let $E_t, F_t, G_t$ denote the first fundamental magnitudes of $S_t$ and $E,F,G$ denote the same for $S$ at $t = 0$. We have,

\[(4.2.2) \quad \Delta_t = (E_t G_t - F_t^2)^{1/2} \]

represents the elemental area of $S_t$ and

\[(4.2.3) \quad \Delta = (EG - F^2)^{1/2} \]

represents the elemental area of $S$. We set,

\[(4.2.4) \quad (E_t G_t - F_t^2) = (t^2 + 1)^2(EG - F^2) \]
that is,
\[ \Delta_t^2 = (t^2 + 1)^2 \Delta^2 \]

We observe that when \( t = 0 \) the two areas coincide and \( (t^2 + 1)^2 \) is well defined for all values of \( t \). From (2.2.7), (2.2.8), (2.2.9) the first fundamental magnitudes of \( S_t \) are given by

\[
\begin{align*}
E_t &= \beta^2 e(t - 1)^2 + A^2 \\
F_t &= \beta^2 f(t^2 - 1) + AB \\
G_t &= \beta^2 (t + 1)^2 + B^2
\end{align*}
\]

where,

\[
(4.2.6) \quad A = (t+1) \int_{1}^{2} + 2 \int \frac{2}{12}, \quad B = (t-1) \int_{2}^{1} - 2 \int \frac{1}{12}
\]

Let us denote,

\[
(4.2.7) \quad P = \int_{1}^{2} + 2 \int \frac{2}{12}, \quad Q = \int_{2}^{1} + 2 \int \frac{1}{12}
\]

which will give us,

\[
(4.2.8) \quad A = t \int_{1} + P = P_t \text{ (say)}, \quad B = t \int_{2} - Q = Q_t \text{ (say)}
\]

Further we write,

\[
(4.2.9) \quad L_t = f(t - 1) \quad \text{and} \quad M_t = f(t + 1)
\]
Thus (4.2.5) as a consequence of (4.2.8), (4.2.9) takes the form,

\[ E_t = e + P_t^2 \]
\[ F_t = f L_t M_t + P_t Q_t \]
\[ G_t = g M_t^2 + Q_t^2 \]

These will give us,

\[ \Delta_t^2 = (eg-f^2)L_t^2 M_t^2 + e L_t Q_t^2 + g M_t^2 P_t^2 - 2f L_t M_t P_t Q_t \]

Further when \( t = 0 \),

\[ \Delta^2 = (eg-f^2) \rho^4 + e \rho^2 Q^2 + g \rho^2 P^2 - 2f \rho^2 P Q \]

Hence with these (4.2.4) takes the form,

\[ (eg-f^2) \rho^4 (t^2 - 1)^2 + e \rho^2 (t-1)^2 (t \rho, 2 - Q)^2 \]
\[ + g \rho^2 (t+1)^2 (t \rho, 1 + P)^2 - 2f \rho^2 (t^2-1) (t \rho, 1 + P) (t \rho, 2 - Q) \]
\[ = (t^4 + 2t^2 + 1) [(eg-f^2) \rho^4 + e \rho^2 Q^2 + g \rho^2 P^2 \]
\[ - 2f \rho^2 P Q] \]

Since \( \rho^2 \neq 0 \) in general, we have

\[ \rho^2(eg - f^2)(t^4 - 2t^2 + 1) + e(t^2 - 2t + 1)(t^2 \rho, 2 - 2t \rho, 2 Q + Q^2) \]
\[ + g(t^2 + 2t + 1)(t^2 \rho, 1 + 2t \rho, 1 P + P^2) \]
\[ - 2f(t^2 - 1) (t^2 \rho, 1 \rho, 2 + t(P \rho, 2 - Q \rho, 1) - PQ) \]
\[ = (t^4 + 2t^2 + 1) [(eg - f^2) \rho^2 + eQ^2 + gP^2 - 2fPQ] \]
Collecting the coefficients of various powers of $t$ we get,

$$
\begin{align*}
&t^4 [ (eg-f^2) + e \mathcal{P}_2^2 + g \mathcal{P}_1^2 - 2f \mathcal{P}_1 \mathcal{P}_2 - f^2(eg-f^2) \\
&- eQ^2 - gp^2 + 2fPQ ] + t^3 [ e(-2 \mathcal{P}_2 Q - 2 \mathcal{P}_1^2) \\
&+ (2 \mathcal{P}_1 P + 2 \mathcal{P}_1^2) - 2f(P \mathcal{P}_2 - Q \mathcal{P}_1) ] \\
&+ t^2 [-2 \mathcal{P}^2(eg-f^2) + e(Q^2 + 4 \mathcal{P}_2 Q + \mathcal{P}_1^2) \\
&+ g(P^2 + 4 \mathcal{P}_1 P + \mathcal{P}_1^2) - 2f(-PQ - \mathcal{P}_1 \mathcal{P}_2) \\
&- 2 \mathcal{P}^2(eg-f^2) - 2eQ^2 - 2gp^2 + 4fPQ] \\
&+ t[e(-2Q^2 - 2 \mathcal{P}_2 Q) + g(2P^2 + 2 \mathcal{P}_1 P) \\
&+ 2f(P \mathcal{P}_2 - Q \mathcal{P}_1) ] = 0
\end{align*}
$$

that is,

$$
\begin{align*}
&t^4 [ (e \mathcal{P}_2^2 + g \mathcal{P}_1^2 - 2f \mathcal{P}_1 \mathcal{P}_2) - (eQ^2 + gp^2 - 2fPQ) ] \\
&+ 2t^3 [-e \mathcal{P}_2 (\mathcal{P}_2 + Q) + g \mathcal{P}_1 (\mathcal{P}_1 + P) - f(P \mathcal{P}_2 - Q \mathcal{P}_1) ] \\
&+ t^2 [-4 \mathcal{P}^2(eg-f^2) - (eQ^2 + gp^2) + (e \mathcal{P}_2^2 + g \mathcal{P}_1^2) \\
&+ 4(e \mathcal{P}_2 Q + g \mathcal{P}_1 P) + 2f(PQ + \mathcal{P}_1 \mathcal{P}_2) + 4fPQ] \\
&+ 2t[-eQ(\mathcal{P}_2 + Q) + gP(\mathcal{P}_1 + P) + f(P \mathcal{P}_2 - Q \mathcal{P}_1) ] = 0
\end{align*}
$$

The above will be identically satisfied for all $t$ when the
coefficients of various powers of \( t \) are zero. So we have a system of conditions given by,

\begin{align*}
(4.2.10) \quad & e \mathfrak{p}_{2}^{2} + g \mathfrak{p}_{1}^{2} - 2f \mathfrak{p}_{1} \mathfrak{p}_{2} = eQ^{2} + gP^{2} - 2fPQ \\
(4.2.11) \quad & g \mathfrak{p}_{1}( \mathfrak{p}_{1} + P) + fQ \mathfrak{p}_{1} = e \mathfrak{p}_{2}( \mathfrak{p}_{2} + Q) + fP \mathfrak{p}_{2} \\
(4.2.12) \quad & -4P^{2}(eg - f^{2}) - (eQ^{2} + gP^{2}) + (e \mathfrak{p}_{2}^{2} + g \mathfrak{p}_{1}^{2}) \\
& \quad + 4(e \mathfrak{p}_{2}Q + g \mathfrak{p}_{1}P) + 2f(PQ + \mathfrak{p}_{1} \mathfrak{p}_{2}) \\
& \quad + 4fPQ = 0 \\
(4.2.13) \quad & gP( \mathfrak{p}_{1} + P) - fQ \mathfrak{p}_{1} = eQ( \mathfrak{p}_{2} + Q) - fP \mathfrak{p}_{2}
\end{align*}

These four conditions lead to the following lemma.

**Lemma 2.1** We have,

\begin{align*}
(4.2.14) \quad & \sqrt{g}( \mathfrak{p}_{1} + P) = \pm \sqrt{e}( \mathfrak{p}_{2} + Q) \\
(4.2.15) \quad & PQ = \mathfrak{p}_{1} \mathfrak{p}_{2} \\
(4.2.16) \quad & \sqrt{\frac{g}{e}} = \sqrt{\frac{17}{12}} = \frac{Q}{\mathfrak{p}_{1}} = \frac{\mathfrak{p}_{2}}{P}
\end{align*}

**Proof:** Adding (4.2.11) and (4.2.13) we get

\[
g(\mathfrak{p}_{1} + P)^{2} = e(\mathfrak{p}_{2} + Q)^{2}
\]
This gives us,
\[ \sqrt{g} ( \rho_1 + P) = \pm \sqrt{e} ( \rho_2 + Q) \]
which establishes (4.2.14).

Now (4.2.10) can also be written as,
\[ e( \rho_{1,2}^2 - Q^2) + g( \rho_{1,2}^2 - P^2) = 2f( \rho_{1,2}^2 - PQ) \]
that is,
\[ \sqrt{e} ( \rho_{1,2} + Q) \sqrt{e} ( \rho_{2,1} - Q) + \sqrt{g} ( \rho_{1,1} + P) \sqrt{g} ( \rho_{1,1} - P) \]
\[ = 2f( \rho_{1,2}^2 - PQ) \]
Using (4.2.14) we get,
\[ \sqrt{eg} ( \rho_{1,1} + P)( \rho_{2,1} - Q) + \sqrt{eg} ( \rho_{1,1} + Q)( \rho_{1,2} - P) \]
\[ = 2f( \rho_{1,2}^2 - PQ) \]
that is,
\[ \sqrt{eg} ( \rho_{1,1} \rho_{2,2} + P \rho_{1,2} - Q \rho_{1,1} - PQ + \rho_{1,1} \rho_{1,2} \]
\[ + Q \rho_{1,1} - P \rho_{1,2} - PQ) = 2f( \rho_{1,2}^2 - PQ) \]
Therefore,
\[ -2 \sqrt{eg} ( \rho_{1,1} \rho_{2,2} - PQ) = 2f( \rho_{1,2}^2 - PQ) \]
Hence we get,

\[(\sqrt{eg - f})(\frac{\mathcal{P}_1}{\mathcal{P}_1} \frac{\mathcal{P}_2}{\mathcal{P}_2} - PQ) = 0\]

Since \(\sqrt{eg - f} \neq 0\) we get

\[PQ = \frac{\mathcal{P}_1}{\mathcal{P}_1} \frac{\mathcal{P}_2}{\mathcal{P}_2}\]

which establishes (4.2.15).

From (4.2.14) and (4.2.15) we have

\[\sqrt{\frac{g}{e}} = \frac{\mathcal{P}_2 + Q}{\mathcal{P}_1 + P} \quad \frac{Q}{\mathcal{P}_1} = \frac{\mathcal{P}_2}{P}\]

We also have,

\[
\frac{Q}{\mathcal{P}_1} = \frac{\mathcal{P}_2}{P} = \frac{Q + \mathcal{P}_2}{\mathcal{P}_1 + P} = \sqrt{\frac{g}{e}}
\]

\[Q = \frac{\mathcal{P}_2}{P} = \frac{Q - \mathcal{P}_2}{\mathcal{P}_1 - P} = \frac{2\mathcal{P} \frac{1}{12}}{-2\mathcal{P} \frac{2}{12}} = -\frac{\{1\}}{\{12\}}\]

The above two equations will give us

\[\sqrt{\frac{g}{e}} = -\frac{\{1\}}{\{12\}} = \frac{Q}{\mathcal{P}_1} = \frac{\mathcal{P}_2}{P}\]

which establishes (4.2.16). \(\text{q.e.d.}\)
As a consequence of lemma 2.1 the condition (4.2.10) becomes

$$ (4.2.17) \quad e p_{2}^{2} + g p_{1}^{2} = e q^{2} + g p^{2} $$

Using (4.2.17) along with the lemma 2.1 the condition (4.2.12) becomes

$$ (4.2.18) \quad -4 p^{2}(eg-f^{2}) + 4(e p_{2}q + g p_{1}p) + 8f p_{1} p_{2} = 0 $$

This equation leads to the following lemma, where we have a simple formula for determining $P$.

**Lemma 2.2** We have,

$$ (4.2.19) \quad P^{2} = \frac{2 p_{1} p_{2}}{\sqrt{e g - f}} $$

**Proof:** We have (4.2.18) taking the form,

$$ -P^{2}(eg-f^{2}) + (e p_{2}q + g p_{1}p) + 2f p_{1} p_{2} = 0 $$

that is,

$$ -P^{2}(eg-f^{2}) + (\sqrt{e} p_{2} \cdot \sqrt{q} q + \sqrt{g} p_{1} \cdot \sqrt{p} p) + 2f p_{1} p_{2} = 0 $$

But,

$$ \frac{Q}{P} = \frac{p_{2}}{P} = \frac{\sqrt{g}}{\sqrt{e}} $$

Hence we have,
\[- \rho^2(eg - f^2) + (\int eg \ p_{1,1} p_{1,2} + \int eg \ p_{1,1} p_{1,2}) + 2f \ p_{1,1} p_{1,2} = 0\]

that is,

\[- \rho^2(\int eg - f)(\int eg + f) + 2(\int eg + f) \ p_{1,1} p_{1,2} = 0\]

Since \( \int eg + f \neq 0 \) we get (4.2.19). \( \text{q.e.d.} \)

4.3 A differential equation for the congruence with area magnifying representation

The condition given by,

\[\sqrt{\frac{g}{e}} = - \left\{ \begin{array}{c} 1 \\ 12 \\ 2 \\ 12 \end{array} \right\} \]

is similar to the one obtained in the third chapter. If we take

Case 1

\[\sqrt{\frac{g}{e}} = - \left\{ \begin{array}{c} 1 \\ 12 \\ 2 \\ 12 \end{array} \right\} = \left( \frac{u}{v} \right)\]

we will have,

\[\sqrt{g} \left\{ \begin{array}{c} 2 \\ 12 \end{array} \right\} = - \sqrt{e} \left\{ \begin{array}{c} 1 \\ 12 \end{array} \right\} \]

that is,

\[\sqrt{g} \left( e \frac{\partial g}{\partial u} - f \frac{\partial e}{\partial v} \right) = - \sqrt{e} \left( g \frac{\partial e}{\partial v} - f \frac{\partial g}{\partial u} \right)\]
or,

\[(e \sqrt{g} - f \sqrt{e}) \frac{dg}{du} = (f \sqrt{g} - g \sqrt{e}) \frac{de}{dv}\]

Hence,

\[\sqrt{e} (\sqrt{eg} - f) \frac{dg}{du} = \sqrt{g} (f - \sqrt{eg}) \frac{de}{dv}\]

with \(\sqrt{eg} - f \neq 0\) we get,

\[\sqrt{e} \frac{dg}{du} + \sqrt{g} \frac{de}{dv} = 0\]

or,

\[\frac{dg}{du} = -\left[\frac{g}{\sqrt{e}} \frac{de}{dv} = \left(\frac{u}{v}\right) \frac{de}{dv}\right]\]

Now,

\[\sqrt{g} = -(\frac{u}{v})\sqrt{e}\]

gives us,

\[\frac{dg}{du} = -\left[2 \frac{\sqrt{eg}}{v} + \sqrt{g} \left(\frac{u}{v}\right) \frac{de}{du}\right]\]

that is,

\[\frac{u}{v} \frac{de}{dv} = -\left[-\frac{2ue}{v^2} - \frac{u^2}{v^2} \frac{de}{du}\right]\]

The above will lead us to the equation

\[\frac{v}{\frac{de}{dv}} - u \frac{de}{du} = 2e\]
which is the same as (3.2.9) and we will have \( e = u^{k-1} v^{k+1} \) and \( g = u^{k+1} v^{k-1} \) as solutions. The congruence is determined as in Chapter III. Further when \( k = -1 \) we get the Guichard congruence.

**Case-2:** If

\[
\sqrt{e} = - \begin{bmatrix} 1 \\ 2 \\ 12 \end{bmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}
\]

we get,

\[
\sqrt{g} = \begin{pmatrix} u \\ -v \end{pmatrix} \sqrt{e}
\]

and

\[
\frac{dg}{du} = 2 \sqrt{eg} \frac{e}{v} + \left( \frac{u}{v} \right) \sqrt{g} \frac{de}{du}
\]

As before,

\[
\sqrt{g} \begin{bmatrix} 2 \\ 12 \end{bmatrix} = - \sqrt{e} \begin{bmatrix} 1 \\ 12 \end{bmatrix}
\]

will give us,

\[
\frac{dg}{du} = - \sqrt{g} \frac{de}{v} = - \left( \frac{u}{v} \right) \frac{de}{dv}
\]

Hence we get,

\[
\frac{u}{v} \frac{de}{dv} + \frac{u}{v} \frac{de}{du} = 2ue + \frac{u^2}{v^2} \frac{de}{dv}
\]

that is,

\[
\frac{de}{dv} + u \frac{de}{du} = -2e
\]
The above can also be solved by the method of separation of variables which will give us,

\[ e = u^{-(k+1)} v^{k-1} \quad ; \quad g = u^{-k+1} v^{k-3} \]

when \( k = 1 \) we get the Guichard congruence.

4.4 An example of a rectilinear congruence with area magnifying representation

We have seen in Chapter III article 3, that the Guichard congruence establishes an area preserving representation. We shall consider the same example where we have

\[
(4.4.1) \quad e = \frac{1}{u^2}, \quad g = \frac{1}{v^2}, \quad f = \frac{\cos \omega}{uv}
\]

and

\[
(4.4.2) \quad \varphi = u \frac{\delta \omega}{\delta u} = 2 \cos(\omega/2) = v \frac{\delta \omega}{\delta v}
\]

It would be enough if we verify the following conditions for area magnifying representation namely,

\[
(4.4.3) \quad g( \varphi_1 + p)^2 = e( \varphi_2 + q)^2
\]

\[
(4.4.4) \quad PQ = \varphi_1 \varphi_2
\]

\[
(4.4.5) \quad f^2 = \frac{2 \varphi_1 \varphi_2}{\frac{1}{eg} - f}
\]
In case of a Guichard congruence we have from (4.2.7)

\[ P = \gamma_{1}^{2}, \quad Q = \gamma_{2}^{2} \]

so that (4.4.3) reduces to

(4.4.6)

\[ g \gamma_{1}^{2} = e \gamma_{2}^{2} \]

We have from (4.4.2),

\[
\begin{align*}
\gamma_{1} &= -\sin(\omega/2) \frac{\partial \omega}{\partial u} = -\frac{\sin \omega}{u} \\
\gamma_{2} &= -\sin(\omega/2) \frac{\partial \omega}{\partial v} = -\frac{\sin \omega}{v}
\end{align*}
\]

The above equations along with (4.4.1) verifies (4.4.6) and

(4.4.4) is also satisfied. Using (4.4.7) and (4.4.1) in the

right hand side of (4.4.5) we get

\[
\frac{2 \sin^{2} \omega}{1 - \cos \omega} = 2(1 + \cos \omega) = 4 \cos^{2}(\omega/2) = \varphi^{2}
\]

Hence all the required conditions for an area magnifying

representation are satisfied.