3.0 Introduction

If we suspect possible bias in estimators from sequential procedures, then, it would be useful to have a method of constructing unbiased estimators. Girshick, Mosteller and Savage (1946) gave a method of obtaining unbiased estimators of $\theta$ for certain binomial sampling problems. However, the method does not claim any optimal property.

Blackwell (1947) has generalized the Girshick-Mosteller-Savage estimate. If the observations have a distribution admitting a sufficient statistic $V_n$ for an unknown parameter $\theta$ when sample size $n$ is fixed and if $X$ is any unbiased estimate of $\theta$ then the statistic $E(X \mid V_n)$ is an unbiased estimate of $\theta$ with variance smaller than that of $X$. The interesting fact is that the estimate $E(X \mid V_n)$ is always a better estimate than $X$ in the sense of having a smaller variance, unless $X$ is already function of $V_n$ only, in which case the two estimates $X$ and $E(X \mid V_n)$ clearly coincide. Under certain conditions, if the sampling is governed by stopping rule depending on the
values taken by \( V_n \) as \( n \) increases, it is possible to use this result to obtain an unbiased estimate of \( \theta \) for the sequential procedure; the estimate can be expressed in the form
\[
E(X | N, V_n) \tag{3.0.1}
\]
Conditions under which (3.0.1) is the unique unbiased estimate of \( \theta \), having uniformly minimum variance, have been given by Lehmann and Stein (1950).

A considerable amount of effort has been devoted to investigating the theoretical possibilities of sequential estimation in general. Much of this work is of pure mathematical rather than ordinary statistical interest and a very brief survey of it is as follows.

Let \( Z_N \) denote the sum of \( N \) independent observations whose common distribution has finite expectation equal to \( \theta \), where the sample size \( N \) is determined by any sequential rule such that \( E(N) < \infty \). Blackwell (1946), by a simple application of the strong law of large numbers, showed that
\[
E(Z_N) = \theta E(N) \tag{3.0.2}
\]
a result previously obtained by Wald for his likelihood-ratio test. Another proof was given by Wolfowitz (1947), together with similar results for variances and third moments, all proved under very weak conditions. In particular Wolfowitz (1947) used them to extend the
Frechet-Darmois-Rao-Cramer inequality of sequential estimates. Let $Y_N$ be an estimate of $\theta$ derived from $N$ observations whose distribution contains only the one unknown parameter $\theta$, and let

$$E(Y_N) = \theta + b(\theta) \quad (3.0.3)$$

so that $b(\theta)$ is the bias of $Y_N$. Then under certain regularity conditions, Wolfowitz showed that

$$\text{Var}(Y_N) \geq \frac{(1 + \frac{db}{d\theta})^2}{E(N) E \left( \frac{\partial}{\partial \theta} \right)^2} \quad (3.0.4)$$

where $L$ denotes the log-likelihood function of a single observation. A similar result holds for the simultaneous estimation of several unknown parameters, variance being replaced by Dermond-Crâmer concept of a "Concentration ellipsoid". A result closely related to (3.0.4) was also proved by Blackwell's statistic (3.0.1), under different regularity conditions. Seth (1949) has slightly extended Wolfowitz's results. Related to this work there is existence theorem by Stein (1950) for unbiased estimates having minimum variance, in which expressions for the minimum variance are given (the attained minimum, not, nearly a lower bound like (3.0.4)).

The method of obtaining an unbiased estimator of a parameter subject to an upper bound on the variance of the
estimator in the independent and identically distributed case have been discussed by Cox (1952). He obtained an unbiased estimate with assigned accuracy using as few observations as possible. As a special case he considered the variance \( \sigma^2 \) corresponding to a given fractional standard error \( \sqrt{\sigma} \). The method suggested are sequential estimation based on two-stage sampling. Cox (1952) also uses the data of both the stages as Samuel (1966) for constructions of unbiased estimates. The variance bound suggested by Cox (1952) is a known function of parameter, whereas that of Samuel is independent of the parameter. The important applications of the procedure of Cox (1952) are to the estimation of means of Poisson, binomial and normal (when the population variance is known). Estimates given by Cox (1952), in his paper, are simple when compared to the estimates given by other.

The extension of the results of Samuel (1966) to certain class of independent increment processes has been discussed in the previous chapter. However the estimators suggested in that chapter are not easy to compute. Nevertheless the method of choosing the random variable \( T \) is quite easy.
In this chapter under certain assumptions we obtain an unbiased estimate of the parameter for the exponential class of independent increment processes using as minimum period as possible and with the property that the variance of the estimate is uniformly bounded above by a pre-assigned upper bound $B$, that is,

$$\text{Var}\left[Z(T)\right] \leq B \quad \text{for all } \theta, \quad (3.0.5)$$

The need for such an estimator arises in the following situations. It is known that for such a class of processes the variance of an unbiased estimate is a function of the parameter and if such a variance attains the lower bound of Cramer-Rao inequality, the estimates are said to be Uniformly Minimum Variance Unbiased Estimate (UMVUE). Since the variance is a function of the parameter value increases and it may tend to infinity even though it attains the lower bound. In such situations it is necessary to have some upper bound for the variance. So that even if the value of the parameter $\theta$ increases the variance will not increase beyond a certain value. This is achieved through the sequential procedure.

The general procedure of obtaining an unbiased estimator with required property has been discussed in the section 3.1 and some properties are given. In section 3.2 some particular cases of the procedure have been discussed.
With obvious modifications of some stochastic processes, as a special case of the procedure described in section 3.1, has been discussed in section 3.3.

3.1. General procedure

Let \( \{ X(t), \ t \geq 0 \} \) be a Koopman-Darmois family of processes having the probability mass function (p.m.f).

\[
f_t(x; \theta) = g(x; t) \left[ C(\theta) \right]^t \exp \left[ Q(\theta)x \right], \ x \in \mathbb{R},
\]

with \( f_0(0; \theta) = 1, \ f_0(x; \theta) = 0 \) for \( x \neq 0 \), where \( R \) is discrete or continuous set of points on the real line, \( C(\theta) > 0, \ g(x, t) > 0 \), and \( Q(\theta) \) is strictly monotone in \( \theta \). Assume that the mean of \( X(t) \) per unit time is equal to \( \theta \) and hence the variance of \( X(t) \) per unit time will be \( \sigma_\theta^2 = \left[ Q'(\theta) \right]^{-1} \), where \( Q'(\theta) \) is the first derivative of \( Q(\theta) \) with respect to \( \theta \).

3.1.1. One Parameter Koopman-Darmois family of Stochastic Processes

For this family of stochastic processes it is known that \( X(t) \) is transitive sufficient statistic for the parameter \( \theta \) and \( \left[ X(t)/t \right] \) is an unbiased estimator of \( \theta \), and the variance of the estimate will be a function of \( \theta \). For such process
is unbiased estimate of \( g \) and the variance of \( z(T) \) will be uniformly bounded by \( B \) provided \( T \) is choosen to satisfy

\[
E(1/T) \leq B/\sigma'(\theta)^{-1}
\]

**Theorem 1**: Suppose the process \( \{X(t), t \geq 0\} \) whose p.d.f. (or p.m.f.) is defined by (3.1) has been observed initially for a known finite period of time \( \tau \). If the total period of observation is defined by the random variable,

\[
T = h(x(\tau), \tau)/B 1(\tau)
\]

where \( B \) is a known preassigned constant and \( h \) and \( l \) arc choosen such that

\[
E(1/T) \leq \sigma'(\theta) B \quad \forall \theta
\]

then the estimate

\[
z(T) = \begin{cases} 
    x(\tau)/\tau & \text{if } T \leq \tau \\
    [x(\tau) + x(T - \tau)]/T & \text{if } T > \tau
\end{cases}
\]

is unbiased and its variance is uniformly bounded above by \( B \).

**Proof**: Since we observe the process for a period of length of \( \tau \) and \( x(\tau) \) being the value of \( X(t) \) at \( \tau \) in the first phase, \( [x(\tau)/\tau] \) is an unbiased estmimator of \( \theta \) viz.,

\[
E\left[\frac{x(\tau)}{\tau}\right] = \theta
\]
and
\[ \text{Var}\left[\frac{x(T)}{T}\right] = \left[\frac{Q'(\theta)}{T}\right]^{-1} \]

Suppose \( x(T - \tau) \) be the value of \( X(t) \) at \( (T - \tau) \) in the second phase and \( F_T \) denotes the probability distribution function of the random variable \( T \).

Now,
\[ E\left[Z(T)\right] = E\left\{ E\left[Z(T)/T\right]\right\} \]
\[ = \int_{0}^{T} E\left[\frac{x(T)}{T}\right]dF_T + \int \frac{\omega}{T} E\left[x(T)\right] \]
\[ + \int \frac{\omega}{T} E\left[x(T - \tau)\right]dF_T \]
\[ = \theta \int_{0}^{T} dF_T + \omega \int dF_T \]
\[ = \theta. \quad (3.7) \]

Hence \( Z(T) \) is an unbiased estimate of \( \theta \). Now
\[ \text{Var}\left[Z(T)\right] = E\left\{ \text{Var}\left[Z(T)/T\right]\right\} + \text{Var}\left\{ E\left[Z(T)/T\right]\right\} \]
\[ = E\left\{ \text{Var}\left[Z(T)/T\right]\right\} \]
\[ = \int \text{Var}\left[\frac{x(T)}{T}\right]dF_T + \int \frac{\omega}{T} \text{Var}\left\{ x(T) + x(T - \tau) \right\} \]
\[ = \frac{1}{\Theta(\theta)} \int_{0}^{T} \frac{1}{T} dF_T + \frac{1}{\Theta(\theta)} \int \frac{1}{T} dF_T \]
\[ = \frac{1}{\Theta(\theta)} E(1/T) + \frac{1}{\Theta(\theta)} \int_{0}^{T} \left(\frac{1}{T} - \frac{1}{T}\right) dF_T \]

Since \( \left(\frac{1}{T} - \frac{1}{T}\right) \) is negative, for \( 0 < T < \tau \), we have
\[ \text{Var}\left[Z(T)\right] \leq \left[\frac{1}{\Theta(\theta)}\right]^{-1} E(1/T) \quad (3.8) \]
From (3.5) and (3.8)
\[ \text{Var} \left( \hat{Z}(T) \right) \leq B, \text{ for all } \theta \]  \hspace{1cm} (3.9)

Hence we have an unbiased estimate \( \hat{\theta}(T) \) whose variance is bounded by \( B \), the pre-assigned quantity.

**Theorem 2**: For \( T \) defined in (3.4) and (3.5) and \( x(T) \) defined by (3.6)
\[ E(T) \geq \left[ \mathcal{O}'(\partial) B \right]^{-1} \] \hspace{1cm} (3.10)
and
\[ \left[ E(T) \right]^{-1} \leq \left\{ \text{Var} \left( Z(T) \right) / \text{Var} \left( X(T) \right) \right\} \leq E(1/T) \] \hspace{1cm} (3.11)

**Proof**: Wolfowitz (1947) generalised the C-R inequality and using its extension (c.f. Lipster and Shiryayev (1978)) to the continuous time stochastic processes, the sequential estimation procedure giving an unbiased estimate of \( \theta \), the parameter of the family of p.m.f. or p.d.f. \( \left\{ f(x(t), \theta) \right\}, \theta \in \Theta \)
\[ \text{Var} \left( \hat{Z}(T) \right) \geq \left\{ E(T) E \left( \frac{\partial \log f}{\partial \theta} \right)^2 \right\}^{-1} \]

For K-D family of processes whose p.d.f is given by (3.1)
\[ E \left( \frac{\log f}{\partial \theta} \right)^2 = \mathcal{O}'(\partial), \text{ per unit time} \]
and hence
\[ \text{Var} \left( \hat{Z}(T) \right) \geq \left\{ E(T) \mathcal{O}'(\partial) \right\}^{-1} \] \hspace{1cm} (3.13)

From (3.9) and (3.13)
\[ E(T) \geq \left[ \mathbb{E} Q(\theta) \right]^{-1} \] (3.14)

and from (3.8) and (3.13) we have

\[ \left[ \frac{\text{Var} \ Z(T)}{\text{Var} \ X(1)} \right] \leq E(1/T) \] (3.15)

Hence the proof.

From theorem 2, it is clear that for \( T \) defined in (3.4) and (3.5) and \( M(T) \) defined in (3.6) the equation (3.14) must hold. If the equality in (3.14) holds then the estimate may be called as Minimum Variance Unbiased Estimates (UMVUE) subject to an upper bound for the variance of the estimator.

3.2 Particular cases

In this section the method of estimation for some particular cases of K-D family has been discussed.

**Theorem 3**: Let \( \{X(t), \ t \geq 0\} \) be the Poisson process with parameter \( \theta \). If the total period of observation is defined by

\[ T = \left[ x(T) + 1 \right] / \tau B \] (3.16)

where \( B \) is the pre-assigned upper bound, then the estimate \( z(T) \) defined by (3.2) is unbiased and its variance is uniformly bounded above.

**Proof**: The p.m.f. of the Poisson process is given by

\[ f_T(x; \theta) = (\theta t)^x e^{-\theta t} / x! , \quad x = 0, 1, 2, \ldots \]

\[ \theta > 0 \] (3.17)
Comparing (3.1) and (3.17)

\[ \Omega(\theta) = \ln(\theta) \]

Now by (3.16)

\[ T = \frac{x(t^*) + 1}{T^* B} \]

then

\[
E(1/T) = B \left[ 1 - e^{\theta T^*} \right] / \theta
\]

\[
\leq B/\theta
\]

\[ = B \Omega'(\theta) \quad (3.18) \]

and it can be easily verified that \( \mathbf{z}(T) \) with \( T \) defined by (3.16) satisfies (3.7) and (3.9).

**Theorem 4**: Let \( \{X(t), \quad t \geq 0\} \) be the gamma process and suppose \( x(t^*) \) is the value of \( X(t) \) at \( t^* \). If the random variable \( T \) is defined by

\[ T = \frac{x^2(t^*)}{(\xi - 1)(\xi - 2)B} \quad (3.19) \]

where \( B \) is a known pre-assigned constant, then the estimate \( \mathbf{z}(T) \) defined by (3.2) is unbiased and its variance is bounded above by \( B \).

**Proof**: The p.d.f. of the gamma process is

\[ f_t(x ; \theta) = \frac{x^{t-1} e^{-x/\theta}}{\theta^t \Gamma(t), \quad x \geq 0, \quad \theta > 0,} \quad (3.20) \]

Comparing (3.1) and (3.20), we have

\[ \Omega(\theta) = -1/\theta. \]

Now from (3.19),

\[ T = \frac{x^2(t^*)}{(\xi - 1)(\xi - 2)B}. \]
then

\[ E(T) = \frac{B}{\theta} = B Q(\theta) \]

and hence the estimate \( z(T) \) satisfies (3.7) and (3.9). Hence it is unbiased and its variance is bounded above by \( B \).

**Theorem 5.** Suppose the process \( \{ X(t), t \geq 0 \} \) be the negative binomial process and suppose \( (x(T) \) is the value of \( X(t) \) at \( T \) \( T > 2 \). If the random variable \( T \) is defined by

\[ T = (x(T) + 1)(x(T) + (T - 1))/(T - 1)(T - 2)B. \]  

(3.21)

where \( B \) is the known pre-assigned constant, then the estimate \( z(T) \) defined by (3.2) satisfies (3.7) and (3.9).

**Proof:** The p.d.f. of the negative binomial process is defined by

\[ f_t(x; \theta) = \frac{x+t}{x!} \frac{\theta^x}{(1+\theta)^{x+t}}, \quad x = 0, 1, 2, \ldots \] 

\[ t > 0 \]  

(3.22)

Comparing (3.1) and (3.22) we have

\[ Q(\theta) = \ln \left[ \theta / (1 + \theta) \right] \]

From (3.21),

\[ E(1/T) = B \left[ 1 - 1/(1 + \theta)^{T-2} \right] / 0 (1 + \theta) \]

\[ \leq B / \theta (1 + \theta) = B Q(\theta). \]

which satisfies (3.5) and the estimator \( z(T) \) with \( T \) defined by (3.21) can be shown to be an unbiased estimate of \( \theta \) and its variance is bounded above by \( B \).
3.3. Special Cases

In this section we consider the processes which do not belong to the K-D family and have shown that the procedure suggested in section 3.1 works with certain obvious modifications.

3.3.1. Birth and Death Processes

Let \( \{X(t), t \geq 0\} \) be a birth and death process with birth and death parameters \( \lambda \) and \( \mu \) respectively. Suppose this process is observed continuously until \( n_0 > 2 \) events (births and deaths) have been recorded. Let the successive transitions occur at the epochs \( 0 < t_1 < t_2 < \ldots < t_{n_0} \) and let \( T_k = t_k - t_{k-1} \), \( k = 1, 2, \ldots, n_0 \), with \( t_0 = 0 \) are independently distributed as exponential random variables with parameter \( \lambda \) when \( X_{t_{k-1}} = x_{k-1} \). Then Basawa and Prakasa Rao (1980) have shown that

\[
\sum_{k=1}^{n_0} T_k x_{t_{k-1}}
\]

is sufficient for \( \theta \) and an unbiased estimate of \( \theta \) given by

\[
\hat{\theta} = \left( n_0 - 1 \right) / \sum_{k=1}^{n_0} T_k x_{t_{k-1}}
\]

is UMVUE and has variance.

\[
\text{Var} (\hat{\theta}) = \frac{\theta^2}{(n_0 - 2)} , \quad n_0 > 2
\]

Define the statistic
\[ n_0 = \begin{cases} \frac{(n_0-1)}{\sum_{k=1}^{n_0} t_k x_{t_k}} & \text{if } n_0 \leq n_0 \\ \frac{N_0}{\sum_{k=1}^{n_0} t_k x_{t_k}} & \text{if } n_0 > n_0 \end{cases} \]

and \( N_0 = \left[ \frac{n_0(n_0+1)}{2(\sum_{k=1}^{n_0} t_k x_{t_k})^2} \right] + 2 \). Since \( \sum_{k=1}^{n_0} t_k x_{t_k} \) is distributed as a gamma variate with parameter \( \theta \) and index \( n_0 \). It can be verified that \( z(N_0) \) is bounded variance unbiased estimate of \( \theta \).

### 3.3.2. Stable Process

Let \( \{X(t), t \geq 0\} \) be a stable process with parameters \( \alpha \) and \( \beta \). Let \( t_k \) denote the interval between the \( k^{th} \) and \( (k-1)^{th} \) jumps of magnitude \( \geq \varrho \), where \( \varrho \) is some fixed number \( > 0 \), and \( U_k(\varrho) \) be the magnitude of the \( k^{th} \) jump, assuming that we are only observing jumps of size \( \geq \varrho \). The density of \( U_k(\varrho) \) for the stable process with index \( \beta \), \((0 < \beta < 1)\), and scale parameter \( \alpha > 0 \), is given by

\[ f_{U_k}(\varrho)(u) = \varrho^{\beta} u^{1-\beta}, \quad u \geq \varrho, \]

which is free from \( \alpha \). The likelihood function (conditional on \( N = n \)) which is based only on \( \{u_1, \ldots, u_n\} \) is given by

\[ L(\beta) = \beta^n \sum_{k=1}^{n} u_k^{\beta} \frac{\varrho}{\sum_{k=1}^{n} u_k^{1-\beta}} \]

Then the maximum likelihood estimate (conditional) of \( \beta \) is found to be
Letting $z_k = \log u_k - \log \theta$, we note that $z_k$ are independent negative exponential variates with mean $\beta^{-1}$ so that $(\hat{\beta} / n)$ is distributed as a reciprocal of a gamma $(n, \beta)$ variate, suppose we observe the process for the jumps of magnitude $> \theta$. Basawa and Prakasa Rao (1980) have shown that within the conditional set up $\sum_{k=1}^{n} z_k$ is complete sufficient statistic for $\beta$. For this process an unbiased estimate of $\beta$ is given by

$$\hat{\beta} = \frac{n-1}{\sum_{k=1}^{n} (\log u_k - \log \theta)}$$

where $z_k = \log u_k - \log \theta$, and is unbiased and has variance

$$\text{Var}(\hat{\beta}) = \frac{\beta^2}{n-2}, \quad n > 2.$$ 

which increases with $\beta$. Define the statistic

$$z(N) = \begin{cases} 
\frac{(n-1)}{\sum_{k=1}^{n} \log(u_k / \theta)} & \text{if } N \leq n \\
\frac{(N-1)}{\sum_{k=1}^{n} \log(u_k / \theta)} & \text{if } N > n.
\end{cases}$$

and

$$N = \left[ n(n+1) / B(\sum_{k=1}^{n} \log(u_k / \theta))^2 \right] + 2$$

Since within the conditional set up $\sum_{k=1}^{n} z_k$ is a complete sufficient statistic for $\beta$ and it can be verified that
3.3.3. Life-testing

Consider $n$ independent components (units) having identically distributed failure-times $\{T_1, T_2, \ldots\}$ with the common failure-time density

$$f(t/\theta) = \frac{1}{\theta} e^{-t/\theta}, \quad t > 0, \quad \theta > 0$$

which depends on a vector of unknown parameter $\theta$. Suppose that the failed units are not replaced. The likelihood function is

$$L = \frac{1}{\theta^n} \exp \left\{-\sum_{i=1}^{r} \frac{t_i + (n-r)t_r}{\theta} \right\}, \quad 0 \leq t_1 \leq \ldots \leq t_r.$$

The sufficient statistic for $\theta$ is $\sum_{i=1}^{r} \frac{t_i + (n-r)t_r}{r}$ and the maximum likelihood estimate of $\theta$ is

$$\hat{\theta} = \frac{1}{r} \sum_{i=1}^{r} \frac{t_i + (n-r)t_r}{r}.$$

The estimate $\hat{\theta}$ is unbiased estimate for $\theta$ and the variance is

$$\text{Var}(\hat{\theta}) = \frac{\theta^2}{r}, \quad r \leq n.$$

The reliability function for the exponential failure-time density is given by,

$$R(t/\theta) = e^{-t/\theta}$$

Since the reliability function $e^{-t/\theta}$ being a monotonic
function of \( \theta \) we can get the maximum likelihood estimate of \( R(t/\theta) \) for any given \( t \), as

\[
\hat{R}(t/\theta) = e^{-t/\hat{\theta}},
\]

where \( \hat{\theta} \) is the maximum likelihood estimate of \( \theta \). Define the statistic, for \( N \leq n \).

\[
Z(N) = \begin{cases} 
\frac{1}{r} \sum_{i=1}^{r} t_i + (n-r)t_r, & \text{if } r \leq N \\
\frac{N}{r} \sum_{i=1}^{N} t_i + (N-r), & \text{if } r > N.
\end{cases}
\]

and

\[
N = \frac{\left[ \sum_{i=1}^{r} t_i + (n-r)t_r \right]^2}{B(r-1)(r-2)}, \text{ for } N \leq n
\]

Since the distribution of \( \hat{\theta} \) is complete (Lehmann (1959)) it follows by the Rao-Blackwell Theorem that \( \hat{\theta} \) is UMVUE of \( \theta \). Then one can verify that \( Z(N) \) is an unbiased estimate of \( \theta \) and its variance is bounded above by the pre-assigned quantity.

3.4. Remarks

The procedure of obtaining an unbiased estimate of the parameter of a stochastic process with the property that the variance of the estimate bounded above can be obtained for any other process provided one knows the sufficient statistic and an unbiased estimate of \( \theta \) by choosing \( T \) properly.