In this Chapter, we obtain a necessary and sufficient condition for the semitotal-block graph of a connected graph to be eulerian. Also we present a characterization of graphs whose semitotal-block graphs are hamiltonian. Further we establish a criterion for the semitotal-block graph of a graph to be planar.
1.1 INTRODUCTION

We need the following definitions.

Unless otherwise stated, a graph $G$ is a finite nonempty set $V(G)$ together with a set $E(G)$ of two-element subsets of distinct elements of $V(G)$. Each element of $V(G)$ is referred to as a vertex and $V(G)$ itself as the vertex set of $G$. The members of the edge set $E(G)$ are called edges. The vertices and edges of a graph are called its elements. The edge $e = (u,v)$ is said to join the vertices $u$ and $v$. If $e = (u,v)$ is an edge of a graph $G$, then $u$ and $v$ are adjacent vertices while $e$ and $u$ are incident, as are $e$ and $v$. If $e$ and $e'$ are distinct edges of $G$ incident with a common vertex, then $e$ and $e'$ are adjacent edges. Henceforth, we denote an edge by $uv$ or $vu$ rather than $(u,v)$.

Two graphs $G_1$ and $G_2$ are isomorphic if there exists a one-to-one mapping from $V(G_1)$ onto $V(G_2)$ such that two vertices of $G_1$ are adjacent if and only if the corresponding vertices of $G_2$ are adjacent. The equality of two graphs means that they are isomorphic.

If $v$ is a vertex of $G$, then the number of edges of $G$ incident with $v$ is the degree of $v$ and is often denoted $\deg G_v$. Let $\Delta(G)$ denote the maximum degree among the
vertices of $G$.

A sequence $v_1, v_2, ..., v_n$ of distinct vertices of a graph $G$ in which every two consecutive vertices are adjacent is called a $v_1 \rightarrow v_n$ path of $G$. If there exists a $u \rightarrow v$ path for every two distinct vertices $u, v$ of a graph $G$, then $G$ is connected. Two $u \rightarrow v$ paths of $G$ are called disjoint if the paths have no vertices in common, other than $u$ and $v$. If, in the above sequence, $n \geq 3$ and $v_1 v_n$ is also an edge of $G$, then the sequence $v_1, v_2, ..., v_n, v_1$ is a cycle of $G$. The length of a path or cycle is the number of edges in it.

Let $G$ be a graph; if $V'$ and $E'$ are subsets of $V(G)$ and $E(G)$, respectively, which together form a graph $H$, then $H$ is a subgraph of $G$.

A component of $G$ is a maximal connected subgraph. The removal of a vertex $v$ from a graph $G$ results in that graph $G - v$ which is the maximal subgraph of $G$ not containing $v$. The removal of an edge $e$ from a graph $G$ results in that graph $G - e$ which is the maximal subgraph of $G$ not containing $e$.

A cutvertex of a connected graph $G$ is a vertex $v$ such that $G - v$ is disconnected. A bridge of a connected graph $G$ is an edge whose removal disconnects $G$. 
A graph $G$ is called a block if it has more than one vertex, is connected and has no cutvertices. A block of a graph $G$ is maximal subgraph of $G$ which is itself a block. If $B = \{u_1, u_2, ..., u_r; r \geq 2\}$ is a block of $G$, then we say that vertex $u_1$ and block $B$ are incident with each other, as are $u_2$ and $B$ and so on. If two distinct blocks $B_1$ and $B_2$ are incident with a common cutvertex, then they are adjacent blocks. The degree of a vertex $v$ in $B$ is the number of edges of $B$ incident with $v$ and it is denoted by $\deg_B v$. The vertices and blocks of a graph are called its members.

A graph is planar if it can be drawn on the plane in such a way that no two of its edges intersect. A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the exterior region.

A graph $G$ is called eulerian if it has a closed path which contains every edge of $G$ exactly once and contains every vertex of $G$. Such a path is referred to as an eulerian path.

A cycle of a graph is called a hamiltonian cycle if it contains every vertex of $G$; a hamiltonian graph is one which contains a hamiltonian cycle.
A connected graph with no cycles is referred to as a tree. Any graph without cycles is a forest.

The trivial graph is consisting of exactly one vertex.

The line graph $L(G)$ of a graph $G$ is the graph whose vertices can be put in one-to-one correspondence with the edges of $G$ in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent.

By $L^2(G)$ we mean $L(L(G))$.

In general, $L^n(G) = L(L^{n-1}(G))$ for $n \geq 1$

where $L^1(G) = G$ and $L^0(G) = G$.

The total graph $T(G)$ of a graph $G$ is the graph whose vertices can be put in one-to-one correspondence with the set of vertices and edges of $G$ in such a way that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of $G$ are adjacent (if both elements are vertices or both are edges) or incident (if one element is a vertex and the other an edge). The line graph originated with Whitney while the total graph originated with Behzad.

The semitotal-block graph $T_b(G)$ of a graph $G$
is the graph whose set of vertices is the union of the set of vertices and blocks of $G$ and in which two vertices are adjacent if and only if the corresponding vertices of $G$ are adjacent or the corresponding members of $G$ are incident. This concept was introduced in $[6]$. In Figure 1.1, a graph $G$ and its semitotal-block graph $T_b(G)$ are shown.

By $T^2_b(G)$ we mean $T_b(T_b(G))$. In general,

$$T^n_b(G) = T_b(T_b^{n-1}(G))$$

for $n \geq 2$ where $T^1_b(G) = T_b(G)$.

In $[4]$, graphs whose line graphs are eulerian or hamiltonian are investigated and characterizations of these graphs are given. Furthermore, necessary and sufficient conditions are presented for iterated line graphs to be eulerian or hamiltonian. Characterizations of graphs having hamiltonian total graphs and of graphs whose total graphs are eulerian are presented in $[3]$. Characterizations of planar line graphs and outerplanar line graphs were given in $[7]$ and $[5]$ respectively. Also characterizations of planar total graphs and outerplanar total graphs were given in $[2]$ and $[5]$ respectively. The purpose of the first chapter is to establish some similar results for semitotal-block graphs.

The following result is of use.
THEOREM A. A graph $G$ is the semitotal-block graph of some graph $H$ if and only if

1. every block $B_i$ of $G$ has a noncutvertex $v_i$ such that $G$ and $G - v_i$ have the same number of blocks

2. the degree of $v_i$ is $n_i - 1$ where $n_i$ is the number of vertices in $B_i$.

The following observations are of use.

REMARK 1.1. For $v$ a vertex of a graph $G$

$$\deg_{T_b(G)} v = \deg_G v + r$$

where $r$ is the number of blocks containing $v$.

REMARK 1.2. Let $B$ be a block of a graph $G$, then

$$\deg_{T_b(G)} B = n$$

where $n$ is the number of vertices in $B$.

1.2. EULERIAN SEMITOTAL-BLOCK GRAPHS

THEOREM 1.1. Let $G$ be a nontrivial graph. If $G$ is eulerian, then $T_b(G)$ is not eulerian.

PROOF. It is known that every nontrivial connected graph has at least two vertices which are not cutvertices and since $G$ is eulerian, it has a noncutvertex $v$ of even
degree. It follows by remark 1.1 that the degree of v in $T^e(G)$ is odd. Hence $T^e(G)$ is not eulerian.

The following Lemma follows immediately from the first theorem of Graph Theory and it is useful to prove out next theorem.

**Lemma.** In any block B of a graph G, there is an even number of vertices $v_i$ such that the degree of each vertex $v_i$ in B is odd.

We now present the main result of this section.

**Theorem 1.2.** Let G be a nontrivial connected graph. Then $T^e(G)$ is eulerian if and only if G satisfies the following condition:

1) if a vertex v of G lies on blocks $B_1, B_2, ..., B_r$ of G, then $\deg_{B_1} v, \deg_{B_2} v, ..., \deg_{B_r} v$ are all odd.

**Proof.** Suppose $T^e_b(G)$ is eulerian. Let B be a block of G and v a vertex of B. Assume B has an even number of vertices. Let u be the corresponding vertex of B in $T^e_b(G)$. Remark 1.2 implies that the degree of u in $T^e_b(G)$ is odd; a contradiction. This proves that every block of G contains an even number of vertices.
Suppose \( v \) is not a cutvertex of \( G \), and \( \deg_G v \) is even. Then by Remark 1.1, the degree of \( v \) in \( T_b(G) \) is odd, which is a contradiction. This proves that every noncutvertex of \( G \) is of odd degree.

Suppose \( v \) is a cutvertex of \( G \) and it lies on blocks \( B_1, B_2, ..., B_r, r \geq 2 \) of \( G \).

Assume \( r \) is even. We consider the following possibilities.

(I) \( \deg_{B_i} v \) is odd for \( 1 \leq i \leq 2n-1 \leq r \)
and \( \deg_{B_i} v \) is even for \( 2n \leq i \leq r \).

(II) \( \deg_{B_i} v \) is odd for \( 1 \leq i \leq 2n \leq r \)
and \( \deg_{B_i} v \) is even for \( 2n+1 \leq i \leq r \).

Assume \( r \) is odd. We now consider the following two possibilities.

(III) \( \deg_{B_i} v \) is even for \( 1 \leq i \leq 2n-1 \leq r \)
and \( \deg_{B_i} v \) is odd for \( 2n \leq i \leq r \).

(IV) \( \deg_{B_i} v \) is even for \( 1 \leq i \leq 2n \leq r \)
and \( \deg_{B_i} v \) is odd for \( 2n+1 \leq i \leq r \).

Assume a cutvertex \( v \) satisfies either (I) or (III). Then by Remark 1.1, \( \deg_{T_b(G)} v \) is odd, which is a contradiction. Thus it proves that any cutvertex of
G does not satisfy either (I) or (III).

Assume now v satisfies either (II) or (IV).

Then there exist at least two blocks $B_j$ and $B_k$ of $G$ containing $v$ such that $\deg_{B_j} v$ and $\deg_{B_k} v$ are even, $1 \leq j < k \leq r$. Since every block of $G$ contains an even number of vertices, it follows from Lemma that the block $B_j(B_k)$ contains an even number of vertices such that the degree of each of which in $B_j(B_k)$ is even.

Without loss of generality, assume that $B_j$ contains exactly two vertices, say $v$ and $u$, such that $\deg_{B_j} v$ and $\deg_{B_j} u$ are even. It is clear that $u$ is a cutvertex for otherwise by Remark 1.1, $\deg_{T_B}(u)$ is odd, a contradiction.

Then the same reasoning as in above $u$ does not satisfy (I) or (III). If $u$ satisfies (II) or (IV), then there exist at least two blocks $B_j$ and $B_m$ of $G$ containing $u$ such that $\deg_{B_j} u$ and $\deg_{B_m} u$ are even. It follows from the same reasoning made in the above that the block $B_m$ contains an even number of vertices such that the degree of each of which in $B_m$ is even. This procedure must continue without end. This is nonsense since the graph is finite. This proves that any cutvertex of $G$ does not satisfy either (II) or (IV).

Thus we conclude that $\deg_{B_i} v$ is odd for each $1, 1 \leq i \leq r$. 
We now prove the converse.

By Remark 1.1, if \( v \) is a vertex of \( G \), then
\[
\deg_{T_b(G)} v = \deg_G v + r
\]
where \( r \) is the number of blocks containing \( v \). Suppose all vertices of a block of \( G \) are of odd degree. Then clearly

(1) if \( \deg_G v \) is odd, then \( r \) is odd
and (2) if \( \deg_G v \) is even, then \( r \) is even.

Hence \( \deg_{T_b(G)} v \) is even. Also it is clear that the number of vertices of each block of \( G \) is even. Thus the vertex in \( T_b(G) \) which corresponds to a block of \( G \) has even degree. Hence \( T_b(G) \) is eulerian.

**THEOREM 1.3.** Let \( G \) be a nontrivial graph. If \( T_b(G) \) is eulerian, then \( G \) is not eulerian.

**PROOF.** Suppose \( G \) is eulerian, but \( T_b(G) \) is eulerian. Then every vertex of \( G \) is of even degree and according to Theorem 1.2, all vertices of a block of \( G \) are of odd degree. Thus every vertex of a block of \( G \) is a cutvertex of even degree. That is, each vertex of \( G \) is a cutvertex, which is impossible.

### 1.3 HAMILTONIAN SEMITOTAL-BLOCK GRAPHS

The following observation was made in [6]
and it is useful in our next theorem.

**REMARK 1.3.** If \( v \) is a cutvertex of a graph \( G \) then it is also a cutvertex of \( T_b(G) \), conversely.

A necessary and sufficient condition for a graph whose semitotal-block graph is hamiltonian is presented in the following theorem.

**THEOREM 1.4.** Let \( G \) be a nontrivial connected graph. Then \( T_b(G) \) is hamiltonian if and only if \( G \) is a block containing a hamiltonian path.

**PROOF.** Let \( G \) be a nontrivial connected graph and let \( T_b(G) \) be hamiltonian. Assume \( G \) is not a block. Then \( G \) has at least one cutvertex. Hence by Remark 1.3, \( T_b(G) \) has at least one cutvertex, which contradicts the fact that \( T_b(G) \) is hamiltonian.

Suppose now \( G \) is a block and has no hamiltonian path. Then \( G \) has a longest path \( P \) of length at most \( p-2 \), where \( p \) is the number of vertices of \( G \). Let \( v_1 \) and \( v_j \) be the endvertices of \( P \). Let \( w \) be a vertex of \( T_b(G) \) which corresponds to \( G \). Then the graph \( T_b(G) \) is a graph \( G \) together with vertex \( w \) and all edges \( wv_1, v_1 \in V(G) \). But a path \( P \) together with edges \( wv_1 \) and \( v_j \) form a cycle of length at most \( p \) in \( T_b(G) \). However, this implies that \( T_b(G) \) is not hamiltonian, a contradiction.
Conversely, suppose $G$ is a block containing a hamiltonian path

$$v_1 v_2 \cdots v_n, \quad n \geq 2.$$

Then $T_b(G)$ has a vertex $v$ corresponding to $G$ and $vv_i$, $i = 1, 2, \ldots, n$, are edges in $T_b(G)$. Thus

$$v_1 v_2 \cdots v_n v v_1$$

is a hamiltonian cycle of $T_b(G)$.

The following corollary follows from Theorem 1.4 immediately.

**Corollary 1.4.1.** If $G$ is hamiltonian, then $T_b(G)$ is also hamiltonian.

The converse of this corollary is not true always.

For example, the semitotal-block graph of $K_{2,3}$ is hamiltonian; but $K_{2,3}$ is not hamiltonian.

1.4. PLANARITY OF SEMITOTAL BLOCK GRAPHS

A criterion for the semitotal-block graph of a graph to be planar is given below.

**Theorem 1.5.** The semitotal-block graph $T_b(G)$ of a graph $G$ is planar if and only if $G$ is outerplanar.
PROOF. Suppose $T_b(G)$ is planar. Then each block of $T_b(G)$ is planar. Let $b$ be a block of $T_b(G)$. Then by Theorem A, $b$ has a noncutvertex $v$ such that $b-v$ is a block of $G$. Now it is sufficient to prove that each block of $G$ is outerplanar. Suppose one is not outerplanar. Then it has a subgraph homeomorphic to $K_4$ or $K_{2,3}$. Clearly the corresponding block of $T_b(G)$ has a subgraph homeomorphic to $K_5$ or $K_{3,3}^*$ which is nonplanar. Hence $T_b(G)$ is nonplanar, a contradiction.

Conversely suppose $G$ is outerplanar. Then each block of $G$ is outerplanar. Thus each can be drawn on the plane so that all its vertices lie on the exterior region. Let $B$ be a block of $G$ and let $v$ be the corresponding vertex of $B$ in $T_b(G)$. The block $B$ and all edges incident with $v$ form a block of $T_b(G)$. Then clearly this block is planar. Thus each block of $T_b(G)$ is planar and hence $T_b(G)$ is planar.

The next result is a characterization of graphs whose semitotal-block graphs are outerplanar.

**Theorem 1.6.** The semitotal-block graph $T_b(G)$ of a graph $G$ is outerplanar if and only if each component of $G$ is a tree.

**Proof.** For the direct part of the theorem, we assume
that G has a cycle C whose vertices are \( v_1, v_2, \ldots, v_n \), \( n \geq 3 \).

Then G has a block B containing C. Let u be the vertex of \( T_b(G) \) which corresponds to B. The cycle C in \( T_b(G) \) together with the edges \( uv_1, uv_2, \ldots, uv_n \) produce a wheel with \( n + 1 \) vertices as a subgraph. Then \( T_b(G) \) has a subgraph homeomorphic to \( K_4 \) or \( K_{2,3} \) and hence \( T_b(G) \) is not outerplanar, a contradiction.

Conversely, suppose each component of G is a tree. It implies that each block of G is a tree edge. Thus each block of \( T_b(G) \) is \( K_3 \) and hence outerplanar. This proves that \( T_b(G) \) is outerplanar, completing the proof.

From Theorems 1.5 and 1.6, we have

**THEOREM 1.7.** The second semitotal-block graph \( T_{b}^{2}(G) \) of a graph G is planar if and only if each component of G is a tree.

**THEOREM 1.8.** For any nontrivial connected graph G, \( T_{b}^{n}(G) \), \( n \geq 3 \), is nonplanar.

**PROOF.** For any nontrivial connected graph G, \( T_{b}(G) \) is not a tree. In view of Theorem 1.7, \( T_{b}^{3}(G) \) is nonplanar. Then clearly \( T_{b}^{n}(G) \) is nonplanar for \( n \geq 3 \).
REFERENCES


