INTRODUCTION

This thesis is in the area of Graph Theory in Mathematics. It consists of eight Chapters. Chapters I to III deal with traversability and planarity of some graph valued functions such as semitotal-block graph, total-block graph and semientire graph, Chapters IV to VII with forbidden subgraphs and Chapter VIII with line graphs having crossing number 1.

The graphs considered are finite, undirected without loops or multiple edges. The definitions and notations not given in this thesis may be found in [1,7].

We begin with the graph valued functions which have received the most attention. The operation of forming the graph valued functions of a graph is probably the most interesting operation by which one graph is obtained from another. The line graph, total graph, block-cutvertex graph, entire graph, dual graph etc., are the examples of graph valued functions which have been defined on a collection of graphs. Moreover, graphs have been associated with entities which are not themselves graphs. The line graph \( L(G) \) of a graph \( G \) has the edge set of \( G \) as its vertex set and two vertices of \( L(G) \) are adjacent whenever the corresponding edges in \( G \)
have a vertex in common. This concept is natural that it has been originated by Whitney in 1932 and also it has been independently discovered by many authors. There are several variations of this concept, each of which gives rise to a graph valued function.

For example the line graph of a graph $G$ can be considered as determined by its complete subgraphs of order two. If $G$ is a tree, then each such subgraph is a clique. This suggests the clique graph $K(G)$ of a graph $G$. If the subgraphs are considered as blocks then this again suggests the following two more graph valued functions which were introduced in $\chi_9$. The semitotal-block graph $T_B(G)$ of a graph $G$ is the graph whose set of vertices is the union of the set of vertices and blocks of $G$ and in which two vertices are adjacent if and only if the corresponding vertices of $G$ are adjacent or the corresponding members of $G$ are incident (if one member is a vertex and the other a block). The total-block graph $T_T(G)$ of a graph $G$ is the graph whose set of vertices is the union of the set of vertices and blocks of $G$ and in which two vertices are adjacent if and only if the corresponding members of $G$ are adjacent (if both members are vertices or both are blocks) or incident (if one member is a vertex and the other a block). Some basic properties of $T_B(G)$ and $T_T(G)$ have been studied in $\chi_9$. 
In Chapter I, we prove that if a nontrivial connected graph $G$ is eulerian, then the semitotal-block graph $T_b(G)$ of $G$ is not eulerian. Next, we establish the main result of this Chapter as follows:

The semitotal-block graph $T_b(G)$ of a nontrivial connected graph $G$ is eulerian if and only if $G$ satisfies the following condition:

1. If a vertex $v$ of $G$ lies on blocks $B_1, B_2, \ldots, B_r$ of $G$, then $\deg_{B_1} v, \deg_{B_2} v, \ldots, \deg_{B_r} v$ are all odd.

Further, a necessary and sufficient condition for a graph whose semitotal-block graph is hamiltonian is presented in the following theorem:

Let $G$ be a nontrivial connected graph. Then the semitotal-block graph of $G$ is hamiltonian if and only if $G$ is a block containing a hamiltonian path.

Furthermore, we establish the following two characterization theorems:

The semitotal-block graph $T_b(G)$ of a graph $G$ is planar if and only if $G$ is outerplanar.

The semitotal-block graph $T_b(G)$ of a graph $G$ is outerplanar if and only if each component of $G$ is a tree.
In Chapter II, before proceeding with a necessary and sufficient condition for a nontrivial connected graph $G$ to have an eulerian total-block graph of $G$, we prove that if a nontrivial connected graph $G$ is eulerian, then $T_B(G)$ is not eulerian.

Let $G$ be a nontrivial connected graph. The total-block graph $T_B(G)$ of $G$ is eulerian if and only if $G$ satisfies the following condition:

(1) if a vertex $v$ of $G$ lies on blocks $B_1, B_2, ..., B_r$ of $G$, then $\deg_{B_1} v, \deg_{B_2} v, ..., \deg_{B_r} v$ and $r$ are all either even or odd.

Next, we obtain that if a nontrivial connected graph $G$ has a hamiltonian path, then $T_B(G)$ is hamiltonian.

We close this Chapter with the following observations characterizing graphs which have planar total-block graphs or outerplanar total-block graphs.

The total-block graph $T_B(G)$ of a graph $G$ is planar if and only if $G$ is outerplanar and every cutvertex of $G$ lies on at most three blocks.

The total-block graph $T_B(G)$ of a graph $G$ is outerplanar if and only if each component of $G$ is a path.
The entire graph \( e(G) \) of a plane graph \( G \) is the graph with vertex set the vertices, edges, and regions of \( G \). Two vertices of \( e(G) \) are adjacent if and only if they are adjacent elements of \( G \). The vertex-semientire graph \( e_v(G) \) of a plane graph \( G \) is the graph with vertex set the vertices and regions of \( G \). Two vertices of \( e_v(G) \) are adjacent if and only if they are adjacent elements of \( G \). The edge-semientire graph \( e_e(G) \) of a plane graph \( G \) is the graph with vertex set the edges and regions of \( G \). Two vertices of \( e_e(G) \) are adjacent if and only if they are adjacent elements of \( G \). J. Mitchem \cite{13} has obtained hamiltonian and eulerian properties of entire graphs. In Chapter III, we study some basic properties of semientire graphs. Among the basic properties we show that for any plane graph \( G \), the vertex (or edge)-semientire graph has at most one cutvertex.

We next, investigate a relation between the vertex-semientire graph and edge-semientire graph of a plane graph in the following theorem:

For a graph \( G \), \( e_v(G) = e_e(G) \) if and only if \( G \) is a cycle.

We now give eulerian and hamiltonian properties of semientire graphs.
Let $G$ be a plane graph. A necessary and sufficient condition for a vertex (or edge)-semientire graph to be eulerian is that each of the following holds:

1. Each vertex (or edge) of $G$ is adjacent to even number of elements.
2. Each region of $G$ has an even number of elements adjacent to it.

If the vertex-semientire graph $e_v(G)$ of a plane graph $G$ is hamiltonian, then $G$ is connected and no region of $G$ has a boundary which contains two bridges with a common end vertex.

If the edge-semientire graph $e_e(G)$ of a plane graph $G$ is hamiltonian, then $G$ is connected and no region of $G$ has a boundary which contains one of the following:

1. Three paths $P_i: x, e_i, a_i, e'_i, b_i, i = 1, 2$ and 3 with a common endvertex $x$,
2. Two paths $P_i: x, e_i, a_i, e'_i, b_i, i = 1$ and 2 with a common endvertex $x$ and third path $P_3: x_3, e_3, a_3, e'_3, b_3$ at other endvertex $x_3$,
3. Each path $P_i: x_i, e_i, a_i, e'_i, b_i$ at each endvertex $x_i, i = 1, 2$ and 3.

If $G$ is a hamiltonian plane graph, then $e_v(G)$
(or \( e(G) \)) is also Hamiltonian.

We conclude this chapter with the discussion of planarity (outerplanarity) of semientire graphs. We give a necessary and sufficient condition for \( e_v(G) \) to be planar. Further we prove that a necessary and sufficient condition for \( e_e(G) \) to be planar is that \( G \) is a tree and every vertex of \( G \) has degree at most three. Furthermore we prove that the vertex (or edge)-semientire graph of a connected graph \( G \) is outerplanar if and only if \( G \) is a path.

Normally a characterization of graphs having a given property by means of "forbidding" a certain family of subgraphs has a great interest due to its practical applications. In \( \mathcal{J} \), Greenwell, Hemminger and Klerlein reported that a question concerning the possible characterization of a certain class of graphs in terms of forbidden subgraphs was raised at Michigan Conference in March 1972. In \( \mathcal{J} \), they found a simple criterion to determine when a given class of graphs can be characterized. Let \( H \) be a class of graphs. Then \( H \) can be characterized in terms of forbidden subgraphs if and only if there is a class of graphs \( F \) such that \( G \) is in \( H \) if and only if no subgraph of \( G \) is in \( F \). They have taken a more general setting. Let \( L \) be a partially
ordered class under $\leq$. If $G \in L$, then $G$ a structure
and if $G, H \in L$, and $H \leq G$, then $H$ a substructure of $G$.
Let $H$ be a subclass of $L$. Then $H$ can be characterized
in terms of forbidden substructures if there exists
a subclass $F$ of $L$ such that $G \in H$ if and only if no
substructure of $G$ is in $F$.

The next four Chapters deal with forbidden
subgraphs.

In 1930, Kuratowski \cite{12} characterized the
planar graphs in terms of forbidden subgraphs. In the
year 1963, Sedláček \cite{15} found a result which is a
characterization of graphs with planar line graphs by
giving some restrictions on the given graphs. With the
help of this result, Greenwell and Hemminger in \cite{6}
characterized graphs with planar line graphs in terms
of forbidden subgraphs. In \cite{14}, O. Ore proposed the
problems concerning some extensions of the Sedláček
type result. In \cite{11}, the complete solutions to Ore's
problems are given. With the help of these solutions
in Chapter IV, we obtain alternative solutions to Ore's
problems in terms of forbidden subgraphs.

Chartrand and Harary \cite{2} introduced the idea
of outerplanar graphs and also they found a characterization
of outerplanar graphs. Chartrand, Geller and Hedtkefi
in \(\Delta(G) \leq 3\) and if \(\deg v = 3\) for a vertex \(v\) of \(G\), then \(v\) is a cutvertex. In Chapter V, we characterize those graphs whose repeated line graphs are outerplanar in the following theorems.

The second line graph \(L^2(G)\) of a graph \(G\) is outerplanar if and only if

1. \(\deg v \leq 3\) for every vertex \(v\) of \(G\),
2. \(\deg u + \deg v \leq 5\) for every edge \((u,v)\) of \(G\), and
3. if \(\deg u + \deg v = 5\), then \((u,v)\) is a bridge of \(G\).

The \(n\)th line graph \(L^n(G)\), \(n \geq 3\), of a graph \(G\) is outerplanar if and only if

1. \(\deg v \leq 3\) for every vertex \(v\) of \(G\), and
2. if \(\deg v = 3\) for some vertex \(v\) of \(G\), then the component of \(G\) containing \(v\) is \(K_{1,3}\).

Further, we use the above result of Chartrand, Geller and Hedetniemi to show that \(G\) has an outerplanar line graph if and only if \(G\) has no subgraph homeomorphic to \(K_{1,4}\) or \(K_4\)-\(x\).
Furthermore by using above two theorems we characterize outerplanarity of $L^2(G)$ and $L^n(G)$, $n \geq 3$ in terms of forbidden subgraphs in the following way.

The second line graph $L^2(G)$ of a graph $G$ is outerplanar if and only if $G$ fails to contain a subgraph homeomorphic to $K_{1,4}$, $K_{3,3} - C_4$ or $K_3K_2$, where $K_{3,3} - C_4$ is a spanning subgraph of $K_{3,3}$ obtained by removal of edges of $C_4$.

The $n$th line graph $L^n(G)$, $n \geq 3$ of a graph $G$ is outerplanar if and only if $G$ fails to contain a subgraph homeomorphic to $K_{1,4}$, $K_{1,3}K_2$ or $K_3K_2$, where $K_{1,3}K_2$ is the graph $K_{1,3}$ together with an edge at some endvertex.

Kulli introduced the concept of minimally nonouterplanar graphs and he found a result which is a characterization of graphs with minimally nonouterplanar line graphs. In Chapter VI, we use this result to give a characterization of graphs with minimally nonouterplanar line graphs in terms of forbidden subgraphs.

The dual $G^*$ of a planar graph $G$ is a well-known concept (see Whitney or Harary, P.113). We will also consider an earlier formulation studied by Cayley and Tait, as reported in Coxeter, in which the exterior region is not considered to be a vertex of the dual; this will be called the weak dual and it is
denoted by $G^\diamondsuit$. Thus $G^\diamondsuit$ is obtained from $G^*$ by deleting one vertex which corresponds to the exterior region of $G$. In Chapter VII, we present characterizations of minimally nonouterplanar graphs in terms

1. of forbidden subgraphs (Kuratowski-like),
2. of dual graphs (Whitney-like)
and 3. of base of cycles (MacLane-like).

The crossing number $c(G)$ of $G$ which is the least number of intersections of pairs of edges in any embedding of $G$ in the plane. This concept was introduced by Harary and Hill in \cite{8}. In the last Chapter, we characterize planar graphs whose line graphs have crossing number 1 in the following theorem:

The line graph of a connected planar graph $G$ has crossing number 1 if and only if (1) or (2) holds:

1. $\Delta(G) = 4$ and there is a unique noncutvertex of degree 4.
2. $\Delta(G) = 5$, every vertex of degree 4 or 5 is a cutvertex, there is a unique vertex of degree 5 and it does not have degree 4 in any block.

We obtain characterizations of repeated line
graphs with crossing number 1 in the next three theorems.

The second line graph $L^2(G)$ of a connected planar graph $G$ has crossing number 1 if and only if

1. $\deg v \leq 4$ for every vertex $v$ of $G$

and

2. $\deg u + \deg v \leq 7$ for every edge $(u,v)$ of $G$, $G$ has exactly one edge $(u,v)$ such that $\deg u + \deg v = 7$ and $(u,v)$ is a bridge of $G$ and one of $u,v$ lies on four blocks of $G$.

or

2'. $\deg u + \deg v \leq 6$ for every edge $(u,v)$ of $G$, $G$ has exactly one edge $(u,v)$ such that $\deg u + \deg v = 6$ and it is not a bridge of $G$.

The third line graph $L^3(G)$ of a connected planar graph $G$ has crossing number 1 if and only if

1. $\deg v \leq 3$ for every vertex $v$ of $G$

and

2. $G$ has exactly one vertex $v$ of degree 3 and $v$ lies on at least 2 blocks of $G$ in which one block has an end vertex of $G$.

The $n^{\text{th}}$ line graph $L^n(G)$, $n \geq 4$ of a finite connected graph $G$ has crossing number 1 if and only if $G$
is a path of length two together with two end edges adjacent to one endvertex.

Finally, we prove that the line graph of any nonplanar graph has crossing number greater than 1.
REFERENCES


