CHAPTER VIII

ON LINE GRAPHS WITH CROSSING NUMBER 1

In this Chapter, we deduce a necessary and sufficient condition for graphs with line graphs having crossing number 1. Also we obtain characterizations of graphs whose repeated line graphs have crossing number 1. Further, we prove that the line graph of any nonplanar graph has crossing number greater than 1.
8.1. INTRODUCTION

Harary and Hill\[2\] introduced the crossing number $c(G)$ of a graph $G$, which is the least number of intersection of pair of lines in any embedding of $G$ in the plane. Obviously $G$ is planar if and only if $c(G) = 0$. A graph $G$ has crossing number 1 if $c(G) = 1$. Clearly $G$ has crossing number 1 if and only if $G$ is nonplanar and it has an edge $e$ such that $G-e$ is planar. According to the classical theorem of Kuratowski\[3\], a graph is nonplanar if and only if it contains a subgraph homeomorphic with the complete graph $K_5$ or the complete bigraph $K_{3,3}$.

The following will be useful in our results.

**THEOREM A**\[4\]. Let $G$ be a planar graph. A necessary and sufficient condition for $G$ to have a planar line graph is that

1. $\Delta(G) \leq 4$

and
2. if $\deg v = 4$, then $v$ is a cutvertex of $G$.

8.2. LINE GRAPHS WITH CROSSING NUMBER 1

We now give the main theorem of this Chapter.
**THEOREM 8.1.** The line graph of a planar graph $G$ has crossing number 1 if and only if (1) or (2) holds:

1. $\Delta(G) = 4$ and there is a unique noncutvertex of degree 4.

2. $\Delta(G) = 5$, every vertex of degree 4 or 5 is a cutvertex, there is a unique vertex of degree 5 and it does not have degree 4 in any block.

**PROOF.** First assume (1) holds. Then $L(G)$ has crossing number at least 1. Form $G'$ from $G$ by making the transformation in Figure 8.1. Then $L(G')$ is planar and must contain the configuration in Figure 8.2(a). This can be transformed to give a drawing of $L(G)$ with only one crossing.

![Fig. 8.1](image1)

![Fig. 8.2](image2)
Now assume (2) holds. The edges at the vertex v of degree 5 can be split into sets of size 2 and 3 so that no edges in different sets are in the same block. Transform G to G' as in Figure 8.3. Then L(G') is again planar and L(G) can be drawn with one crossing as indicated in Figure 8.4.

Fig. 8.3

Fig. 8.4
For the converse, assume $L(G)$ has crossing number 1. Clearly $\Delta(G) = 4$ or 5.

**CASE 1.** Assume $\Delta(G) = 4$ and that $G$ has two noncut vertices $v$ and $v'$ of degree 4. It is easily seen that they must lie in the same block of $G$. By inserting a vertex of degree 2 if necessary, we may assume they are not adjacent. In a drawing of $L(G)$ with one crossing, the vertices from the four edges at $v$ or $v'$ must form a complete 4-graph of either Type 1 or Type 2 in Figure 8.5, and not both can be of Type 1.

![Type 1](image1)

![Type 2](image2)

Fig.8.5
Suppose both are of Type 2; see Figure 8.6(a).
Then in $G$ edges $e$ and $e'$ are on a cycle which contains only one other edge at each of $v$ and $v'$. In $L(G)$ such a cycle must give two crossings. Similarly if $v$ gives rise to Type 1 and $v'$ to Type 2, one deduces that $L(G)$ has at least two crossings (Figure 8.6(b)).
CASE 2. Assume $\Delta(G) = 5$. If $v$ has degree 5 and is not a cutvertex, then the removal of any edge at $v$ must leave a graph whose line graph is non-planar (Theorem A). It follows that $L(G)$ would have two crossings. Now suppose $v$ has degree 5 and has four edges in one block. It will be sufficient to show that the graph $H$, formed by adding to this block $B$ an end edge $e$ at $v$, has crossing number at least two. In $L(H)$ the four edges other than $e$ form a complete 4-graph and $L(B)$ has a crossing. Hence, if $e$ is to be added without a new crossing, we must have Figure 8.7.
But there must be a cycle containing a and b but not c or d. Hence L(H) must have at least two crossings. Finally, one needs to show that there are no noncut vertices u of degree 4, but this is easily done (say, by removing some edge at v in a block other than the one with such a vertex u), completing the proof of the theorem.

The following theorems are characterizations of graphs whose repeated line graphs have crossing number 1.

**Theorem 8.2.** The second line graph $L^2(G)$ of a connected planar graph G has crossing number 1 if and only if G satisfies the following conditions.

1. $\deg v \leq 4$ for every vertex v of G
2. $\deg u + \deg v \leq 7$ for every edge $(u,v)$ of G, G has exactly one edge $(u,v)$ such that $\deg u + \deg v = 7$ and $(u,v)$ is a bridge of G and one of the vertices u and v lies on four blocks of G

or

1'. $\deg u + \deg v \leq 6$ for every edge $(u,v)$ of G, G has exactly one edge $(u,v)$ such that $\deg u + \deg v = 6$ and it is not a bridge of G.

**Proof.** Suppose $L^2(G)$ has crossing number 1. To prove
the necessity, assume $G$ has a vertex $v$ of degree $\geq 5$. Then $L(G)$ contains a subgraph $H$ with five vertices and in addition $H$ is a complete graph isomorphic to $K_5$. It is easy to see that $L(G)$ does not satisfy the condition (1) of Theorem 8.1. Hence $L^2(G)$ has crossing number greater than 1. Thus $\deg v \leq 4$ for every vertex $v$ of $G$.

By (1), $\deg u + \deg v \leq 8$ for every edge $(u,v)$ of $G$. Assume $\deg u + \deg v = 8$ for some edge $(u,v)$ of $G$. Then $L(G)$ has a vertex of degree 6. Clearly $L(G)$ does not satisfy the condition (2) of Theorem 8.1, which is a contradiction. This proves that $\deg u + \deg v \leq 7$ for every edge $(u,v)$ of $G$.

Suppose that edge $(u,v)$ where $\deg u + \deg v = 7$, is not a bridge of $G$. Then it lies on a cycle of $G$. In $L(G)$, its degree is 5 and is not a cutvertex, a contradiction to condition (2) of Theorem 8.1.

We now prove that $G$ has exactly one bridge $(u,v)$ such that $\deg u + \deg v = 7$. Suppose $G$ has at least two such bridges. Then $L(G)$ has at least two cutvertices of degree five. By the condition (2) of Theorem 8.1, $L^2(G)$ has crossing number greater than 1, a contradiction. Thus $(u,v)$ is exactly one bridge of $G$ such that $\deg u + \deg v = 7$. In view of (1) assume
deg u = 4. Then deg v = 3. The proof of condition (2) will be completed by showing that u lies on four blocks of G. Suppose u does not lie on four blocks. Then two different possible cases arise.

**CASE 1.** Suppose u lies on three blocks. Then \( L(G) \) contains at least two noncutvertices of degree four and a cutvertex of degree five. By Theorem 8.1, \( L^2(G) \) has crossing number greater than 1.

**CASE 2.** Suppose u lies on two blocks. Then \( L(G) \) contains at least two noncutvertices of degree four, a cutvertex of degree five and a noncutvertex of degree five. Theorem 8.1 implies that \( L^2(G) \) has crossing number greater than 1.

Thus we have proved u lies on four blocks.

Suppose next that \( \deg u + \deg v \leq 6 \) for every edge \((u,v)\) of G. Assume \( \deg u + \deg v = 6 \) for some edge \((u,v)\) of G. Then we prove that \((u,v)\) is not a bridge of G. Suppose \((u,v)\) is a bridge of G. Then \( L(G) \) has a cutvertex of degree 4. By Theorem A, \( L^2(G) \) is planar, which is a contradiction. Suppose G has at least two nonbridges \((u_1, v_1)\) such that \( \deg u_1 + \deg v_1 = 6 \). Then \( L(G) \) contains at least two noncutvertices of degree four, a contradiction to condition (1) of Theorem 8.1.
This proves (2').

Conversely, assume that $G$ satisfies the conditions (1) and either (2) or (2'). By (2) every vertex of $L(G)$ has degree at most five. By (1) and (2), $L(G)$ has exactly one vertex of degree five and it lies on two blocks in which none of them is $K_2$. Clearly $L(G)$ satisfies the conditions of Theorem 8.1. Hence $L^2(G)$ has crossing number 1. Also by (1) and (2'), $L(G)$ contains exactly one noncutvertex of degree four. It follows again by Theorem 8.1 that $L^2(G)$ has crossing number 1.

**Theorem 8.3.** The third line graph $L^3(G)$ of a connected planar graph $G$ has crossing number 1 if and only if $G$ satisfies the following conditions.

1. $\deg v \leq 3$ for every vertex $v$ of $G$

and

2. $G$ has exactly one vertex $v$ of degree three and $v$ lies on at least two blocks of $G$ in which exactly one block has an endvertex of $G$.

**Proof.** Suppose $L^3(G)$ has crossing number 1. If $G$ has a vertex $v$ of degree $\geq 4$, then $K_4$ is a subgraph of $L(G)$. Obviously $L(G)$ does not satisfy the condition (2') of Theorem 8.2, a contradiction. Thus $\deg v \leq 3$ for every
vertex $v$ of $G$.

Suppose $G$ has a vertex of degree 3 which is not a cutvertex. Then $L(G)$ has at least two nonbridges $(u_1, v_1)$ such that $\deg u_1 + \deg v_1 \geq 6$. $L(G)$ does not satisfy the condition $(2')$ of Theorem 8.2, which is a contradiction. Hence $G$ has a cutvertex of degree 3.

Assume $G$ has at least two cutvertices of degree three. It is easy to see that $L(G)$ does not satisfy $(2')$ of Theorem 8.2. Thus $G$ has exactly one cutvertex of degree three. We now prove that the cutvertex $v$ of degree three lies on either two or three blocks of $G$ in which one block has an end vertex of $G$.

We consider two cases:

CASE 1. Suppose $v$ lies on two blocks of $G$. Then one of the blocks is a cycle of length $\geq 3$ and other is $K_2$. Assume the block $K_2$ containing $v$ has no end vertex of $G$. Then $L(G)$ has three nonbridges $(u_1, v_1)$ such that $\deg u_1 + \deg v_1 = 6$. Clearly $L(G)$ does not satisfy the condition $(2')$ of Theorem 8.2. Hence $L^3(G)$ has crossing number exceeding 1, which is a contradiction.

CASE 2. Suppose $v$ lies on three blocks of $G$. Then each of these blocks is $K_2$. We consider 3 subcases.

SUBCASE 2.1. Assume each block $K_2$ containing $v$ has no
end vertex of $G$. Then $L(G)$ contains three nonbridges $(u_1, v_1)$ such that $\deg u_1 + \deg v_1 = 6$. Condition $(2')$ of Theorem 8.2 implies that $L^3(G)$ has crossing number greater than 1, a contradiction.

**SUBCASE 2.2.** Assume one of the blocks $K_2$ containing $v$ has no end vertex of $G$. Then $\Delta(L(G)) \leq 3$. $L(G)$ does not satisfy the condition (1) of Theorem 8.2. Thus $L^3(G)$ has crossing number greater than 1, a contradiction.

**SUBCASE 2.3.** Assume each block $K_2$ containing $v$ has an end vertex of $G$. Then $L(G)$ is $K_3$. Clearly $L(G)$ does not satisfy (1) of Theorem 8.2. Hence $L^3(G)$ has crossing number greater than 1, which is a contradiction.

Now we conclude that exactly one block $K_2$ containing $v$ has an end vertex of $G$.

Conversely, suppose $G$ satisfies the conditions (1) and (2). Then $L(G)$ is either a cycle $C$ together with a vertex which is adjacent to two adjacent vertices of $C$ or a path of length $\geq 3$ together with a vertex which is adjacent to two adjacent non-endvertices of path. Then $L(G)$ satisfies (1) and (2) of Theorem 8.2. Thus $L^3(G)$ has crossing number 1. This completes the proof of the theorem.

**Theorem 8.4.** The fourth line graph $L^4(G)$ of a connected
planar graph $G$ has crossing number 1 if and only if $G$ is a path of length two together with two endedges adjacent to one endvertex.

**Proof.** Suppose $L^4(G)$ has crossing number 1. Then $L(G)$ satisfies the conditions of Theorem 8.3. It is known that any cutvertex of line graph lies on exactly two blocks. Thus a cutvertex of $L(G)$ lies on two blocks. Then by Theorem 8.3, $L(G)$ is a path of length one together with a triangle adjacent to one endvertex. This implies that $G$ is a path of length two together with two endedges adjacent to one endvertex.

The converse part is immediate, we omit the proof.

**Theorem 8.5.** For any graph $G$, $L^n(G)$ has crossing number greater than 1 for $n \geq 5$.

**Proof.** For any graph $G$, $L(G)$ does not contain an induced subgraph $K_{4,3}$. (Harary (77, p.74, Theorem 8.4)). Hence in view of Theorem 8.4, $L^5(G)$ has crossing number greater than 1. It is easy to see that $L^n(G)$ has crossing number greater than 1 for $n \geq 5$.

Finally we prove that for any nonplanar graph $G$, $c(L(G)) > 1$. 
**THEOREM 8.6.** The line graph of any nonplanar graph has crossing number greater than 1.

**PROOF.** Let $G$ be a nonplanar graph. It is well known that every nonplanar graph contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$. Then $G$ contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$. Suppose $G$ contains a subgraph homeomorphic to $K_5$. Then the removal of any edge of $K_5$ must leave a graph whose line graph is nonplanar (Theorem A). It follows that the line graph of $G$ would have at least two crossings. Suppose next, $G$ has a subgraph homeomorphic to $K_{3,3}$. It is sufficient to show that the line graph of $K_{3,3}$ has crossing number two. Let $e = uv$ be an edge of $K_{3,3}$ and let it be adjacent to $e_1, e_2$ at $u$ and $e_3, e_4$ at $v$ (see Figure 8.8(a)). Then $L(K_{3,3} - e)$ is planar and $L(K_{3,3})$ can be drawn with two crossings as indicated in Figure 8.8(b). This completes the proof of the theorem.
Fig. 8.8
# REFERENCES

1. F. Harary,  

2. F. Harary and A. Hill,  

3. K. Kuratowski,  

4. J. Sedláček,  