CHAPTER VII

CHARACTERIZATIONS OF MINIMALLY NONOUTERPLANAR GRAPHS

The object of this Chapter is to establish characterizations of nonseparable minimally nonouterplanar graphs in terms of forbidden subgraphs (Kuratowski-like), of dual graphs (Whitney-like) and of base of cycles (MacLane-like).
7.1. INTRODUCTION

We need the following definitions

Two graphs $G$ and $G^*$ are duals to each other if there is a bijection between their sets of edges. In [1], Coxeter reported that Cayley and Tait have studied dual graphs in which the exterior region is not considered to be a vertex of the dual; this will be called the weak dual and is denoted by $G^w$. Thus $G^w = G^* - v$, where $v$ is a vertex which corresponds to the exterior region of $G$. Of course, a planar graph may have different weak duals depending on how it is embedded in the plane. Let $E$ denote the set of edges of a graph $G$ and $L$ a base of cycles of $G$. The length of a base $L = \{C_i\}$ of cycles of a graph $G$ is $|L| = \sum_{C_i \in L} C_i$. The intersection graph of $L$ over $E$ is called a cycle graph of $G$ with respect to $L$. This concept was introduced in [7].

A quasi-wheel is a graph which consists of a cycle $C$ together with a vertex $v$ on $C$ such that $v$ is adjacent with at least 3 vertices of $C$. Any graph homeomorphic from $K_{2,3}$ is called a theta-graph, and the two vertices $u_1$ and $u_2$ of degree 3 are called primary vertices. A vertex adjacent
with $u_1$ is called secondary to $u_i$. If $G$ has a vertex which is secondary to both $u_1$, then $G$ is called a basic theta-graph.

A graph $G$ is said to be $k$-minimally nonouterplanar if $i(G) = k$, $k \geq 1$. An 1-minimally nonouterplanar graph is called minimally nonouterplanar (see $\square_3$).

The following are characterizations of minimally nonouterplanar graphs.

**THEOREM A** $\square_3$. A graph $G$ is minimally nonouterplanar if and only if one block of $G$ is minimally nonouterplanar and each of its remaining blocks is outerplanar.

**THEOREM B** $\square_6$. A connected graph $G$ is minimally nonouterplanar if and only if the following hold:

1. All but exactly one block $B$ of $G$ is outerplanar.

2. Block $B$ contains a spanning subgraph $H$ which is a quasi-wheel or a basic theta-graph.

3. Any elements of $E(B) - E(H)$ join vertices $v_i, v_j$ of some path $P = v_1, v_2, \ldots, v_n$. 


where \( v_1 \) and \( v_n \) have degree 3 in \( H \).
and all intermediate vertices have
degree 2 in \( H \). Furthermore, if \( v_i v_j \)
are adjacent with \( i < j \) then \( v_k v_m \) are
not adjacent for \( 1 < k > j \) and \( m > j \).

Characterizations of planar graphs are
given by Kuratowski \( \{2,5\} \), Whitney \( \{9\} \) and MacLane \( \{5\} \)
in different forms. The purpose of this Chapter
is to give Kuratowski-like, Whitney-like and MacLane-like
characterizations of nonseparable minimally nonouterplanar
graphs.

The following is noted for use in the
proof of one of our results.

**Theorem C \( \{2,8\} \).** A graph \( G \) is outerplanar if and
only if it has a weak dual \( G^w \) which is a forest.

We can simplify Theorem C for nonseparable
graphs in the following way.

**Corollary C.1.** A nonseparable graph \( G \) is outerplanar
if and only if its weak dual \( G^w \) is a tree.

7.2. MINIMALLY NONOUTERPLANAR GRAPHS

We now present a characterization of minimally
nonouterplanar graphs in terms of forbidden subgraphs.
THEOREM 7.1. A nonseparable graph G is minimally nonouterplanar if and only if it fails to contain a subgraph homeomorphic to one of the graphs of Fig. 7.1.

PROOF. Suppose G is a nonseparable minimally nonouterplanar graph. Then clearly G has no subgraph homeomorphic to any one of the graphs of Fig. 7.1.

Conversely, suppose G contains no subgraph homeomorphic to any one of the graphs of Fig. 7.1, but assume G is not minimally nonouterplanar. Then G is either outerplanar or k-minimally nonouterplanar (k ≥ 2). If G is outerplanar, then there is nothing to prove. If G is k-minimally nonouterplanar (k ≥ 2), then it must contain more than four vertices. Embed G in the plane so that a maximum number of vertices lie on the exterior cycle C. Since i(G) ≥ 2, there are at least two vertices which lie in the interior of G. Let v_1 and v_2 be the vertices interior to C which are respectively adjacent to vertices u_1 and u_2 on C. G is a block and deg v_i ≥ 2, i = 1, 2. Then

(1) there is a path P from v_1 to v_2,

or

(2) there are two paths P_i, i = 1, 2, from v_1 to some vertex w_i on C

or

(3) there are two paths P_1(v_1 - w_2) and
Fig. 7.1

$G_1$, $G_2$, $G_3$, $G_4$, $G_5$, $G_6$, $G_7$, $G_8$
\[ P_2(v_2 - w_2) \] where \( w_1, w_2 \) are on \( C \).

**CASE 1.** Suppose there is a path \( P \) from \( v_1 \) to \( v_2 \). Then there are two subcases to consider.

**SUBCASE 1.1.** The vertices \( u_1 \) and \( u_2 \) are consecutive on \( C \). In this case, some vertex of \( P \) must have degree at least 3; otherwise, the path could be transferred outside of \( C \) to produce a planar embedding of \( G \) having a longer exterior cycle. Thus, there is a path from a vertex \( x \) of \( P \) to a vertex \( u_2 \) of \( C \) not containing any other vertex of \( P \) and \( u_2 \) is adjacent with one of \( u_1 \) and \( u_2 \). The edges of \( C \) and the three paths from \( x \) to \( C \) induce a subgraph homeomorphic from \( G^1 \) (see Fig. 7.2). Furthermore if \( u_2 \) is also adjacent with \( u_1 \), then there exist two more paths from \( v_1 \) (or \( v_2 \)) to \( u_2 \) and from \( x \) to \( u_2 \) (or \( u_2 \)). Otherwise \( P \) could be transferred outside of \( C \) to produce a minimally nonouterplanar embedding of \( G \).

The edges of \( C \), the three paths from \( x \) to \( C \), the two paths from \( v_1 \) to \( C \) and the path from \( v_1 \) to \( x \) induce a subgraph homeomorphic from \( G_2 \) (see Fig. 7.3).

**SUBCASE 1.2.** The vertices \( u_1 \) and \( u_2 \) are not consecutive on \( C \). There are two subcases to consider.

In the first subcase, if the length of a
path from \( u_1 \) to \( u_2 \) on \( C \) is 2, then some vertex of \( P \) must have degree at least 3; otherwise, the path \( P \) can be transferred outside of \( C \) to produce a planar embedding of \( G \) with exactly one inner vertex. Hence, there exists a path from a vertex \( x \) of \( P \) to a vertex \( u_j \) (adjacent to one of \( u_1 \) say \( u_2 \)) of \( C \) not containing any other vertex of \( P \). The edges of \( C \) and the three paths from \( x \) to \( C \) induce a subgraph homeomorphic from \( G_j \) (see Fig. 7.4).

In the second subcase, the length of each path from \( u_1 \) to \( u_2 \) on \( C \) is at least three. Otherwise the path \( P \) can be transferred outside of \( C \) to produce a minimally nonouterplanar embedding of \( G \). Clearly, the edges of \( C \) and those of the path through \( v_1, v_2 \) from \( u_1 \) to \( u_2 \) induce a subgraph homeomorphic from \( G_4 \) (see Fig. 7.5).

**CASE 2.** Suppose there are two paths \( P_i, i = 1, 2 \) from \( v_i \) to some vertex \( w_i \) on \( C \). We consider two subcases.

**SUBCASE 2.1.** The vertices \( u_i \) and \( w_i, i = 1, 2 \) are consecutive on \( C \).

In this case, some vertex of \( P_i, i = 1, 2 \) different from \( w_i \), must have degree at least 3; otherwise
the paths \( P_i, i = 1, 2 \) can be transferred outside of \( C \) to produce an outerplanar embedding of \( G \). Hence, there is a path from a vertex \( x \) of \( P_1 \) to a vertex \( y \) of \( P_2 \). Then the edges of \( C \) and the paths from the vertices \( x, y \) to \( C \) induce a subgraph homeomorphic from \( G_5 \) (see Fig. 7.6).

**Subcase 2.2.** The vertices \( u_i \) and \( w_i, i = 1, 2 \) are not consecutive on \( C \). Clearly, the edges of \( C \) and those of the paths through \( v_i, i = 1, 2 \) from \( u_i \) to \( w_i \) induce a subgraph homeomorphic from \( G_6 \) (see Fig. 7.7). Further if \( u_1 = u_2 \), then the edges of \( C \) and those of the paths through \( v_i, i = 1, 2 \) from \( u_i \) to \( w_i \) induce a subgraph homeomorphic from \( G_7 \) (see Fig. 7.8). Furthermore if we put \( w_1 = w_2 \), then the edges of \( C \) and those of the paths through \( v_i, i = 1, 2 \) from \( u_i \) to \( w_i \) induce a subgraph homeomorphic from \( G_8 \) (see Fig. 7.9).

**Case 3.** Suppose there are two paths \( P_1(v_1 - w_2) \) and \( P_2(v_2 - w_1) \) where \( w_1, w_2 \) are on \( C \). Then the edges of \( C \) and those of the paths \( u_1 - w_2 \) and \( u_2 - w_1 \) through \( v_1 \) and \( v_2 \) respectively induce a subgraph homeomorphic from \( G_9 \) (see Fig. 7.10).

This completes the proof of the theorem.

In the following theorem we present a characterization of minimally nonouterplanar graphs.
Let \( C_2 \) be a cycle with two vertices and two multiple edges.

**Theorem 7.2.** A nonseparable plane graph \( G \) is minimally nonouterplanar if and only if its weak dual has exactly one cycle \( C_n, n \geq 2 \).

**Proof.** Suppose \( G \) is minimally nonouterplanar. Assume \( G^w \) has no cycles. Then by Theorem \( C \), \( G \) is outerplanar, a contradiction.

Suppose \( G^w \) contains at least two cycles \( C_n, n \geq 2 \). We consider three cases.

**Case 1.** Suppose there are two cycles in \( G^w \) with a cutvertex. Then there must be a sequence of distinct interior regions \( R_0, R_1, \ldots, R_n, R_n', R_{n-1}', \ldots, R_1', R_0' \), such that

1. \( R_0 = R_n = R_0' \),
2. successive regions in the sequence have a common boundary edges

and

3. \( R_n \) is bounded by at least four edges.

Clearly \( G \) must have at least two inner vertices in this embedding of \( G \), a contradiction.

**Case 2.** Suppose there are at least two cycles in \( G^w \) with a common boundary as a path of length at least
one. In $G$, there exist at least two interior regions with a common boundary as a path of length at least three in any embedding of $G$. Then it is easy to see that $G$ contains at least two inner vertices, which is a contradiction.

**CASE 3.** Suppose there are at least two disjoint cycles in $G^W$. Then $G$ contains at least two inner vertices in any embedding of $G$. Then it is not minimally nonouterplanar, a contradiction.

Conversely, suppose $G$ is not minimally nonouterplanar. Then there are two cases to consider.

**CASE 1.** $G$ is outerplanar. Then by Corollary C.1, $G$ is a tree.

**CASE 2.** $G$ is $k$-minimally nonouterplanar, for $k \geq 2$. Then there exist at least two vertices $u_1$ and $u_2$ interior to $G$. Assume $u_i$, $i = 1, 2$ has degree 2. If $u_1$ and $u_2$ are adjacent, then weak dual of $G$ contains at least one pair of triple multiple edges. If $u_1$ and $u_2$ are not adjacent, then $G^W$ has a pair of cycles $C_2$. If $\deg u_1 = 2$ and $\deg u_2 > 2$, then for this embedding of $G$, $G^W$ will contain $C_2$ and $n$-cycle, $(n > 2)$. If each of $u_1$ and $u_2$ has degree $> 2$, then weak dual of $G$ will contain a pair of $n$-cycles, $n \geq 3$. 
In each case, $G^w$ does not contain exactly one cycle $C_n$, $n \geq 2$. Thus if $G$ is not minimally nonouterplanar, no weak dual of $G$ contains exactly one cycle $C_n$, $n \geq 2$.

This completes the proof of the theorem.

In the next theorem we establish a characterization of minimally nonouterplanar graphs in terms of base of cycles of a graph.

**Theorem 7.3.** A nonseparable graph $G$ is minimally nonouterplanar if and only if $G$ has a base of cycles with respect to which the cycle graph of $G$ is unicyclic and one pair of cycles have one or two common edges and remaining every pair of cycles have at most one common edge.

**Proof.** Suppose $G$ is a nonseparable minimally nonouterplanar graph. Then the cycle graph with respect to its minimally nonouterplanar base is isomorphic to the simple graph of $G^w - v$ which is unicyclic where $v$ corresponds to the exterior region of the embedding. Since the cycle graph is unicyclic, it contains exactly one cycle of length $n \geq 2$. If $n = 2$, then one pair of cycles of $G$ has exactly two
common edges. If $n \geq 3$, then every pair of cycles of $G$ has at most one common edge.

Conversely, suppose a graph $G$ has a base of cycles with respect to which the cycle graph is unicyclic. Then one edge of $G$ belongs to no more than three cycles of the base and remaining every edge of $G$ belongs to no more than two cycles of the base which form the planar base of $G$ according to Mclane's characterization of planar graph. Since one pair of cycles of the base have one or two common edges and remaining every pair of cycles of the base have at most one common edge, the cycle graph of $G$ with respect to the minimally nonouterplanar base is isomorphic to $G^* - v$. Thus, $G$ is minimally nonouterplanar.

It is known in [8] that every planar base $L$ of a $p$-vertex and $q$-edge graph $G$ satisfies $|L| \geq 2q - p$.

One can easily prove the following characterization of minimally nonouterplanar graphs.

**Theorem 7.4.** A nonseparable planar graph $G$ is minimally nonouterplanar if and only if the minimal planar base of cycles is of length $2q - p + 1$. 
REFERENCES


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