CHAPTER VI

FORBIDDEN SUBGRAPHS

FOR

MINIMALLY NONOUTERPLANAR LINE GRAPHS

The purpose of this Chapter is to present a characterization of graphs with minimally nonouterplanar line graphs in terms of forbidden subgraphs.
6.1. INTRODUCTION

The inner vertex number \( i(G) \) of a planar graph \( G \), introduced in \( \text{[2]} \), is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of \( G \) in the plane. Obviously \( G \) is outerplanar if and only if \( i(G) = 0 \). A graph \( G \) is said to be minimally nonouterplanar if \( i(G) = 1 \).

The following will be useful to prove our result.

**THEOREM A** \( \text{[1]} \). A graph has a planar line graph if and only if it has no subgraph homeomorphic to \( K_5, K_{3,3}, P_4 + K_1 \) or \( K_2 + \overline{K}_5 \).

**THEOREM B** \( \text{[2]} \). The line graph \( L(G) \) of a finite connected graph \( G \) is minimally nonouterplanar if and only if \( G \) satisfies the following conditions:

1. \( \deg v \leq 4 \) for every vertex \( v \) of \( G \)
2. \( G \) has exactly one vertex \( v \) of degree 4, \( v \) lies on at least three blocks of \( G \) in which one block has an endvertex of \( G \) and if \( \deg v_1 = 3 \) for any other vertex \( v_1 \) of \( G \), then \( v_1 \) is a cutvertex.

or

1. \( \deg v \leq 3 \) for every vertex \( v \) of \( G \),
G has exactly two noncutvertices of degree 3 and these are adjacent.

6.2. A RESULT

We now prove

THEOREM. Let G be a connected planar graph. Then it has a minimally nonouterplanar line graph if and only if it has no subgraph homeomorphic to any one of the graphs of Fig. 6.1.

PROOF. Let G be a connected planar graph with a minimally nonouterplanar line graph. We now show that all graphs homeomorphic to any one of the graphs of Fig. 6.1, have not minimally nonouterplanar line graphs. It follows from Theorem B, since graphs homeomorphic to $G_1$ have $\Delta(G_1) > 4$, graphs homeomorphic to $G_2$ or $G_5$ have two or more vertices of degree four, graphs homeomorphic to $G_4$ or $G_5$ have a vertex of degree 4 which lies on two blocks, graphs homeomorphic to $G_6$ or $G_7$ have a vertex of degree 4 which lies on 3 or 4 blocks and each block containing $v$ has no end vertex of $G$, graphs homeomorphic to $G_8$ or $G_9$ have a vertex of degree 4 and vertices of degree 3 which are noncutvertices, graphs homeomorphic to $G_{10}$, $G_{11}$ or $G_{12}$ have more than two noncutvertices of degree three and graphs homeomorphic to $G_{13}$ have exactly two noncutvertices of degree three which are not adjacent.
Fig. 6.1
Conversely, suppose that $G$ contains no subgraph homeomorphic to any one of the graphs of Fig. 6-1. Assume that $\Delta(G) \geq 5$. Then $G$ contains a subgraph homeomorphic to $G^*$, a contradiction. Hence $\Delta(G) \leq 4$.

Let $v$ be a vertex of $G$ and $\deg v = 4$. We prove that $v$ is a cutvertex. If not, let $a$, $b$, $c$ and $d$ be the vertices of $G$ adjacent to $v$, then there exists a path $a - b$ containing $c$ and $d$ not containing $v$, or there exist paths $a - b$, $a - c$ and $a - d$, each of which does not contain $v$. By Theorem A, $G$ contains a subgraph homeomorphic to $G_4$ or $G_5$ which is a contradiction. Thus $v$ is a cutvertex and every vertex of degree 4 is a cutvertex.

Assume that every cutvertex of degree 4 lies on two blocks of $G$. Let $v$ be a cutvertex of degree 4 and it lies on two blocks. We consider three cases:

CASE 1. If there exist two paths between $a$, $b$ and $c$, $d$ not containing $v$, then $G$ has a subgraph homeomorphic to $G_4$.

CASE 2. If there exists a path $a - b$ containing either $c$ or $d$, then $G$ has a subgraph homeomorphic to $G_5$.

CASE 3. If there exists a path $a - b$ not containing $c$ and $d$, since $v$ lies on two blocks, there is a path
either c - a, c - b, d - a or d - b. Without loss of generality, we assume that there is a path c - a. Let u be the first vertex of intersection (starting from c) of a path c - a with a - b. Depending on the location of u on a - b, we consider two sub cases:

**SUBCASE 3.1.** If u = a or b, then in either case G has a subgraph homeomorphic to $G_5$.

**SUBCASE 3.2.** u $\neq$ a and b. Without loss of generality we can assume that u lies between a and b, then G has a subgraph homeomorphic to $G_5$ or $G_{15}$.

In each case, G has a subgraph homeomorphic to $G_4$, $G_5$, or $G_{15}$, a contradiction. Hence v lies on either 3 or 4 blocks of G.

Suppose there are two or more cutvertices of degree four, each of which lies on either 3 or 4 blocks. Let $v_1$ and $v_2$ be the two cutvertices of degree 4 in which $v_1$ and $v_2$ are connected by a path P and let $a_i$, $i = 1, 2, 3$ and $b_j$, $j = 1, 2, 3$, be the vertices adjacent to $v_1$ and $v_2$ respectively. We consider three cases:

**CASE 1.** Assume $v_1$ and $v_2$ both lie on 3 blocks.

We now consider four subcases:
SUBCASE 1.1. If there exists a path between a vertex of \( a_1 \) and a vertex of \( b_j \), then \( G \) has a subgraph homeomorphic to \( G_3 \) (see Fig. 6.2(b)).

SUBCASE 1.2. If one of \( a_1 \) and one of \( b_j \) are joined by a path passing through \( u_1 \) and \( w_1 \) (not passing through \( v_1 \) and \( v_2 \)), where \( u_1 \) and \( w_1 \) are the vertices of \( P \), then \( G \) has a subgraph homeomorphic to \( G_2 \) (see Fig. 6.2(c)).

SUBCASE 1.3. If there is a path between any two vertices of \( a_1 \) not containing \( v_1 \) and also a path between any two vertices of \( b_j \) not containing \( v_2 \), then a subgraph of \( G \) is homeomorphic to \( G_2 \) (see Fig. 6.2(d)).

SUBCASE 1.4. If there is a path between any two vertices of \( a_1 \) not containing \( v \) and also a path from one of \( b_j \) to a vertex of \( P \) not containing \( v_2 \), then \( G \) has a subgraph homeomorphic to \( G_2 \) or \( G_6 \) (see Fig. 6.2(e)).

CASE 2. Assume either \( v_1 \) or \( v_2 \) lies on \( 4 \) blocks, say \( v_2 \).

We consider two subcases:

SUBCASE 2.1. If there exists a path between any two vertices of \( a_1 \), then \( G \) has a subgraph homeomorphic to \( G_2 \) (see Fig. 6.2(f)).

SUBCASE 2.2. If one of \( a_1 \) is joined by a path with
Fig. 6.2
a vertex of \( P \), then \( G \) has a subgraph homeomorphic to \( G_2 \) (see Fig. 6.2(g)).

**CASE 3.** Assume \( v_1 \) and \( v_2 \) both lie on 4 blocks. Then \( G \) contains a subgraph homeomorphic to \( G_2 \) (see Fig. 6.2(a)).

We have exhausted all the cases and we arrive at the conclusion that \( G \) has exactly one cutvertex of degree 4 which lies on either 3 or 4 blocks.

Suppose \( \deg v = 4 \) and \( v \) lies on either 3 or 4 blocks of \( G \) in which each of these blocks has no end vertex of \( G \).

We consider two cases.

**CASE 1.** If \( v \) lies on 3 blocks of \( G \), then one of these blocks contains a cycle and each of the remaining blocks is an edge. Then clearly \( G \) has a subgraph homeomorphic to \( G_6 \).

**CASE 2.** If \( v \) lies on 4 blocks of \( G \), then each of these blocks is an edge of \( G \). Then \( G \) contains a subgraph homeomorphic to \( G_7 \).

In each case we have a contradiction. Thus if \( \deg v = 4 \), then \( v \) lies on either 3 or 4 blocks of \( G \) in which at least one block has an end vertex of \( G \).
Further suppose $\deg v_1 = 3$ for any other vertex $v_1 \neq v$ of $G$ and $v_1$ is not a cutvertex. The proof of the condition (2) of Theorem B will be completed if we disprove the above statement by deducing a contradiction.

We consider two cases:

**CASE 1.** $v$ is adjacent to $v_1$.

Let $v_2$ and $v_3$ be the vertices of $G$ adjacent to $v_1$. Since $v_1$ is not a cutvertex, there exist two shortest paths $Z_1 (v - v_3)$ and $Z_2 (v - v_2)$ which do not contain $v_1$. The minimality of the paths assures that either $v_2 \notin Z_1$ or $v_3 \notin Z_2$. Let $x$ be the first vertex of intersection (starting from $v_2$) of $Z_2$ with $Z_1$. If $x = v_2$ or $v_3$, then in either case $G$ has a subgraph homeomorphic to $G_8$ (see Fig. 6.3(b1) or 6.3(b2)). If $x = v$, then $G$ contains a subgraph homeomorphic to $G_4$ or $G_5$ (see Fig. 6.3(c)). If $x \neq v_2, v_3$ and $v$, then $G$ has a subgraph that is homeomorphic to $G_8$ (see Fig. 6.3(a)).

**CASE 2.** $v$ is not adjacent to $v_1$.

Since $G$ is connected, there exists a path $Z_1 (v - v_1)$. Let $v_2$ and $v_3$ be the vertices of $G$ adjacent to $v_1$ and let $v_4$ be a vertex on $Z_1$ which divides $Z_1$.
Fig. 6.3
into two subpaths $Z'_1 (v - v_4)$ and $Z''_1 (v_4 - v_1)$. Since $v_1$ is not a cutvertex, there are two paths $Z_2 (v_4 - v_2)$ and $Z_3 (v_4 - v_3)$ which do not contain $v_1$. There are four subcases to consider depending on whether or not $v$ is on $Z_2$ and $Z_3$.

**Subcase 2.1.** $v$ is on $Z_2$ and $Z_3$.

Let $x$ be the last but one vertex of $Z_2$ which also belongs to $Z_3$. If $x = v_2$ or $v_3$, then in either case we have a subgraph of $G$ that is homeomorphic to $G_8$ (see Fig. 6.3(e1) or 6.3(e2)). If $x = v$, then $G$ has a subgraph homeomorphic to $G_{11}, G_5$ or $G_{13}$ (see Fig. 6.3(f)). If $x \neq v_2, v_3$ and $v$, then $G$ contains a subgraph homeomorphic to $G_8$ (see Fig. 6.3(d)).

**Subcase 2.2.** $v$ lies on $Z_2$ or $Z_3$. In either case we have a subgraph homeomorphic to $G_8$ (see Fig. 6.3(g1) or 6.3(g2)).

**Subcase 2.3.** $v$ does not lie on $Z_2$ and $Z_3$.

Again we consider two subcases of subcase 2.3.

**Subcase 2.3.1.** $v$ lies on three blocks.

(a) There exists a path (other than $Z'_1$) $v - y$ where $y \in Z'_1$. 
(I) Assume $Z_2$ and $Z_3$ are not disjoint. Let $x$ be the last but one vertex of $Z_2$ which also belongs to $Z_3$. If $x = v_2$ or $v_3$, then in either case $G$ has a subgraph homeomorphic to $G_8$ or $G_{11}$ (see Fig.6.3(i) or 63(i)). If $x = v_4$, then $G$ contains a subgraph homeomorphic to $G_9$, $G_8$ or $G_{13}$ (see Fig.6.3(j)). If $x \neq v_2$, $v_3$ and $v_4$, then $G$ has a subgraph homeomorphic to $G_8$, $G_{11}$ or $G_{13}$ (see Fig.6.3(h)).

If $y = v_1$, then $G$ has a subgraph homeomorphic to $G_8$ (see Fig.6.3(k)). If $y \neq v_1$, then $G$ contains a subgraph homeomorphic to $G_8$ (see Fig.6.3(l)).

(II) Assume $Z_2$ and $Z_3$ are disjoint. If $y = v_1$, then $G$ contains a subgraph homeomorphic to $G_8$ or $G_{13}$ (see Fig.6.3(m)). If $y \neq v_1$, then $G$ has a subgraph homeomorphic to $G_7$, $G_8$ or $G_{13}$ (see Fig.6.3(n)).

(b) There exists a path (other than $Z_1$) $v - y$ where $y \in Z_1$ or there exists a path between pair of adjacent vertices of $v$.

(I) Assume $Z_2$ and $Z_3$ are not disjoint.
Let $x$ be the last but one vertex of $Z_2$ which also belongs to $Z_3$. If $x = v_2$ or $v_3$, then in either case $G$ contains a subgraph homeomorphic to $G_9$ (see Fig. 6.3(p_1) or 6.3(p_2). If $x \neq v_2$, $v_3$, then $G$ has a subgraph homeomorphic to $G_9$ or $G_{13}$ (see Fig. 6.3(e)).

(II) Assume $Z_2$ and $Z_3$ are disjoint. Then $x = v_4$. In this case $G$ has a subgraph homeomorphic to $G_5$ (see Fig. 6.3(q)).

**SUBCASE 2.3.2.** $v$ lies on four blocks. Then there is no path between pairs of vertices adjacent to $v$ or no $v - y$ path exists, where $y \notin Z'$.

This subcase is analogous to subcase 2.3.1.(b).

We omit the proof.

We have completed all cases. In each case, we found that $G$ contains a subgraph homeomorphic to one of the forbidden subgraphs of Fig. 6.1. Hence $v_4$ is a cutvertex.

Lastly, we prove condition (2') of Theorem B. If $\deg v \neq 4$ then $\Delta(G) \leq 3$. Suppose $G$ contains more than two noncutvertices of degree 3. We consider
the following three cases.

CASE 1. If $G$ is nonouterplanar, then $G$ has a nonouterplanar block $H$ with more than 3 vertices. If $H$ is drawn in the plane, then the maximum number of vertices lie on the exterior cycle $C$. Since $H$ is nonouterplanar, there exists at least one vertex which lies in the interior of $C$. Let $v$ be the vertex interior to $C$ and adjacent to 2 vertices of $C$. Degree of $v$ must be three. Otherwise $H$ contains two noncut vertices of degree three. Hence there is a path from $v$ to some other vertex of $C$. Thus a subgraph of $H$ is homeomorphic to $G_{10}$.

CASE 2. If $G$ has at least two diagonal edges, then there are 2 subcases to consider depending on whether the 2 diagonal edges exist in one cycle or in two different edge disjoint cycles.

SUBCASE 2.1. If two diagonal edges exist in one cycle of $G$, then $G$ has a subgraph homeomorphic to $G_{10}$ or $G_{11}$.

SUBCASE 2.2. If two diagonal edges exist in two different edge disjoint cycles of $G$, then $G$ has a subgraph homeomorphic to $G_{12}$. 
CASE 5. If $G$ is nonouterplanar with at least two diagonal edges, then $G$ contains a subgraph homeomorphic to $G_{10}$.

In each case we have a contradiction. Hence $G$ has exactly two noncutvertices of degree 3.

Suppose $G$ has exactly two nonadjacent noncutvertices of degree three. Then there exist 3 disjoint paths between these two noncutvertices of degree 3. Clearly $G$ contains a subgraph homeomorphic to $G_{13}$, a contradiction. Thus $G$ has exactly two adjacent noncutvertices of degree 3. Thus Theorem B implies that $G$ has a minimally nonouterplanar line graph.
REFERENCES
