5. RELIABILITY ESTIMATION IN SHOCK MODEL WITH MARKOV DEPENDENT DAMAGES

5.1. Introduction.

In this chapter we consider a model which is similar to the one proposed by Barlow and Proschan (1967, p 140). As in the case of shock models of Chapter 2, it is assumed that a device is subjected to a sequence of shocks which occur in a Poisson process with intensity parameter $\beta$. Each shock causes a random damage to the device. Unlike we assumed that damages due to shocks are i.i.d random variables in Chapters 2 and 3 here we assume that the damages are Markov dependent as explained below. Initially the device is new and subsequently due to damages caused by shocks it deteriorates. At time $t$ the device is assumed to be in any one of the $(m+1)$ states $0, 1, \ldots, m$. These states $0, 1, \ldots, m$ represent the increasing order of deterioration, the brand new state being designated by $0$ and the inoperative state by $m$. If the device is in state $i (\leq m)$ at a particular time due to a shock it remains in the same state with probability $q_i$, moves to the next state $(i+1)$ with probability $p_i$ or enters the absorbing state $m$ with probability $r_i$, where $p_i + q_i + r_i = 1$. For this model reliability function is found and using RELS Theorem UMVU estimator of this reliability is obtained in Section 5.2. Special cases with (i) $r_j = 0, j = 0, 1, \ldots, m-1$ and (ii) $r_j = 0, p_j = p, j = 0, 1, \ldots, m-1$ are considered in
Section 5.3. In the case of discrete time the corresponding estimator is obtained in Section 5.4 assuming that shocks are coming in regular intervals of time instead of Poisson process and retaining all other assumptions regarding damages. We obtain the discrete analogue of the previous model considered by Saksena (1976). For this model UMVU estimator of reliability is obtained in Section 5.4. Section 5.5 gives the ML estimator of reliability.

5.2 UMVU Estimator of reliability-continuous time.

As in the shock model let \( t_n \) be the time instant when the \( n \)-th shock occurs and \( X(t_n) \) be the state of the device at \( t_n \). If the device is in state \( 1 \), at time \( t_n \), at time \( t_{n+1} \) it will be in state \( 1, 1+1 \) or \( n \) with probabilities \( q_1, p_1 \) or \( r_1 \) respectively. Thus the transition probabilities are given by

\[
P[ X(t_{n+1}) = j \mid X(t_n) = 1 ]
\]

\[
= q_1, \text{ if } j = 1 \\
= p_1, \text{ if } j = 1+1 \\
= r_1, \text{ if } j = n \\
= 0 \text{ otherwise.}
\]

In this case the reliability of the device at time \( t_0 \) is

\[
R(t_0) = \sum_{n=0}^{\infty} e^{-\beta t_0} \frac{(\beta t_0)^n}{n!} \bar{F}_n ,
\]

\[(5.2)\]
with $P_n$ denoting the probability of surviving $n$ shocks. 
In the case of the present model the $R(t_0)$ can be written as

$$R(t_0) = \sum_{n=0}^{\infty} \left[ \frac{e^{-\beta t_0} (\beta t_0)^n}{n!} \right] \text{Prob( the device is in a transient state )]}
$$

$$= \sum_{n=0}^{\infty} \left[ \frac{e^{-\beta t_0} (\beta t_0)^n}{n!} \sum_{s=0}^{n-1} \text{Prob( the device is in state s after n-th shock )} \right].
$$

In view of (5.1) it can be written as

$$R(t_0) = \sum_{n=0}^{\infty} \left[ \frac{e^{-\beta t_0} (\beta t_0)^n}{n!} \sum_{s=0}^{n-1} \prod_{i=0}^{s-1} p_i \right] \left( q_0^{n_0} q_1^{n_1} \ldots q_s^{n_s} \right),
$$

where $\sum_{s}$ stands for the summation over $n_{s}$'s $s = 0, 1, \ldots, s$ such that $\sum_{i=0}^{s} n_{s} = n-s$.

In order to obtain an estimator of the reliability $R(t_0)$ (5.4), let the life testing experiment be conducted on $k$ new identical devices. Let the experiment be continued till all the $k$ devices fail. Let the $N_{1j}$-th shock be the fatal shock for the $j$-th device, $T_{1j}$ be its life length and $N_{1j}$ be the number of shocks received by it in the $j$-th state. Then
the joint density of the random observations \((T_i, X_{ij}, N_{ij})\) for the \(i\)-th device can be written as

\[
\begin{align*}
(5.5) \quad & \beta_{ij} e^{-\beta t_{ij}} m - 2 \quad n_{ij} x_{ij}^{1-x_{ij}} x_{io} x_{il} \cdots x_{i,j-1} \\
& \pi \left( \sum_{j=0}^{n_{1,m-1}} x_{io} x_{il} \cdots x_{i,m-2} \right),
\end{align*}
\]

where \(x_{ij} = 1\), if \(i\)-th device moves from state \(j\) to state \(j + 1\),

\(= 0\), if \(i\)-th device moves from state \(j\) to \(m\).

\(j = 0, 1, \ldots, m - 2; \quad i = 1, 2, \ldots, k, \) with \(x_{1,-1} = 1\).

Now for the \(k\) devices the joint density of

\[
(5.6) \quad (T_i, X_{ij}, N_{ij}, i = 1, 2, \ldots, k; \quad j = 0, 1, \ldots, m - 2)
\]

can be written as

\[
(5.7) \quad \begin{align*}
& \sum_{i=1}^{k} n_{i} e^{-\beta t_{i}} m - 2 \quad \pi \left[ \sum_{j=0}^{n_{i,m-1}} x_{io} x_{il} \cdots x_{i,j-1} n_{ij} \right] \\
& + \sum_{i=1}^{k} x_{io} x_{il} \cdots x_{i,j} \sum_{j=0}^{n_{i,m-1}} \sum_{r=1}^{n_{i,-1}} x_{io} x_{il} \cdots x_{i,j-1} (1-x_{ij}) \quad r_{ij}, \\
& + \sum_{i=1}^{k} x_{io} x_{il} \cdots x_{i,m-2} n_{i,m-1} \sum_{p=1}^{n_{i,m-1}} x_{io} x_{il} \cdots x_{i,m-2}. 
\end{align*}
\]
with \( x_1, -1 = 1 \). Let \( t = \sum_{l=1}^{k} t_l \) be the total time on test and \( n = \sum_{l=1}^{k} n_l \) be the total number of shocks received by all the \( k \) devices together. Then the joint density (5.7) reduces to

\[
\begin{align*}
\beta^n e^{-\theta t} \frac{\Gamma(m-1)}{\Gamma(j)} \sum_{j=0}^{k} x_{10} x_{11} \cdots x_{1, j-1} n_{1j} \\
\sum_{j=1}^{k} x_{10} x_{11} \cdots x_{1j} \sum_{j=1}^{k} x_{10} x_{11} \cdots x_{1, j-1}(1-x_{1j})
\end{align*}
\]

with \( x_{1, m-1} = 1 \) for \( i = 1, 2, \ldots, k \) and \( r_{m-1} = 1 \).

Now let,

\[
\begin{align*}
X_{10} X_{11} \cdots X_{1, j-1} N_{1j} &= U_{1j} \\
X_{10} X_{11} \cdots X_{1j} &= V_{1j} \\
\sum_{l=1}^{k} U_{1j} &= U_{j} \text{ and } \sum_{l=1}^{k} V_{1j} &= V_{j}
\end{align*}
\]

Then \( V_{1j} \) is the number of shocks received by the \( 1 \)-th device in the \( j \)-th state and \( U_{j} \) is the total number of shocks received by all the \( k \) devices in the same state \( j \). It may be noted that \( V_{1j} = 1 \) if the \( 1 \)-th device moves from state \( j \) to state \( j+1 \) and \( V_{1j} = 0 \), if the \( 1 \)-th device
moves from state \( j \) to state \( m \). Thus \( V_{.j} \) stands for the number of devices that have not failed in the \( j \)-th state.

Now the joint density of the random vector

\[
(5.9) \quad (N, T, U_{.j}, V_{.j}, j = 0, 1, \ldots, m-1)
\]

can be written as

\[
(5.10) \quad \beta^n e^{-\beta t} \prod_{j=0}^{m-1} (q_{.j}^{u_{.j}} p_{.j}^{p_{.j}} r_{.j}^{r_{.j}})^{J_{.j}^{J_{.j}} v_{.j}^{v_{.j}}}
\]

with \( v_{.m-2} = k \) and \( v_{.m-1} = v_{.m-2} \). It can be easily seen that

\[
n = \sum_{i=1}^{k} n_i = \sum_{i=1}^{k} \sum_{j=0}^{m-1} (u_{i,j} + v_{i,j-1}) = \sum_{j=0}^{m-1} (u_{.j} + v_{.j-1}).
\]

The joint density (5.10) can be rewritten as

\[
(5.11) \quad \exp[-\beta t + n \log \beta + \sum_{j=0}^{m-1} u_{.j} \log q_{.j} + \sum_{j=0}^{m-1} v_{.j} \log p_{.j}]
\]

\[
+ \sum_{j=0}^{m-1} (v_{.j-1} - v_{.j}) \log r_{.j}
\]

This belongs to the exponential family and the statistics

\[
(5.12) \quad [T, (u_{.0}, u_{.1}, \ldots, u_{.m-1}), (v_{.0}, v_{.1}, \ldots, v_{.m-2})]
\]

are complete sufficient for the family of densities given by (5.11). The joint density of the statistics

\[
(5.13) \quad (T_{10}, U_{10}, U_{11}, \ldots, U_{1,m-1}, V_{10}, V_{11}, \ldots, V_{1,m-2})
\]
corresponding to the first device is

\[(5.14) \ f(t_1, u_1, v_1, j = 0, 1, \ldots, m-1) = \frac{\beta^m t_1^n}{n!} \prod_{j=0}^{m-1} \left( q_j u_j p_j v_j r_j v_1, j-1, v_1 \right), \]

with \( v_{1,-1} = 1, v_{1,m-2} = v_{1,m-1} \) and \( u_1 = v_{1,j-1} \).

The joint density (5.14) follows by noting that for a fixed value \( n_j \) of \( N_j, T_j \) is the sum of \( n_j \) i.i.d random variables each having exponential distribution with parameter \( \beta, U_{1j} \) is the number of ineffective shocks received by the first device at the \( j \)-th state before it moves from that state and \( V_{1s} \) is the Bernoulli random variable under the condition that the device has moved from the \( s \)-th state.

By the k-fold convolution of density (5.14), the joint density of the complete sufficient statistics (5.12) can be obtained as

\[(5.15) \ f(t, u, v, j = 0, 1, \ldots, m-1) = e^{-\beta t} \frac{\beta^n t^{n-1}}{(n-1)!} \]

\[= \frac{\beta^n t^{n-1}}{(n-1)!} \prod_{j=0}^{m-1} \left( q_j u_j p_j v_j r_j v_1, j-1, v_1 \right), \]

with \( v_{1,-1} = k \). This density follows by noting that sum of i.i.d Bernoulli random variables follows binomial distribution, sum of i.i.d geometric random
variables follows negative binomial distribution and sum of the independent gamma variates with the same scale parameter is gamma distribution. For \((k-1)\) devices let

\[
N' = \sum_{i=2}^{k} N_i, \quad T' = \sum_{i=2}^{k} T_i, \quad U'_{i,j} = \sum_{i=2}^{k} U_{i,j} \text{ and } V'_{j} = \sum_{i=2}^{k} V_{i,j}.
\]

Then by the \((k-1)\)-fold convolution of the density (5.14) by similar argument the joint density of

\[
(5.16) \quad (T', U'_{i,j}, V'_{j}, \quad j = 0, 1, \ldots, m-1)
\]

can be written as

\[
(5.17) \quad f(t',u'_{i,j},v'_{j}, \quad j = 0, 1, \ldots, m-1) = e^{-\beta t'} \mu_1^{n'} (t')^{n'-1}/(n'-1)! \cdot \\
\prod_{j=0}^{m-1} \left[ \frac{u'_{i,j} + v'_{j}}{v'_{i,j}} \right]^{u'_{i,j} - 1} \left( v'_{i,j} \right)^{v'_{j} - 1} \prod_{j=0}^{m-1} \left( q_{j}^{u'_{i,j}} r_{j}^{v'_{j}} \right)^{e_{j}}.
\]

Since vectors (5.13) and (5.16) are independent their joint pdf is obtained by the product of their densities (5.14) and (5.17). In this joint density, transforming \((T_i, U'_j, V'_j, j = 0, \ldots, m-1)\) to \((T, U'_j, V'_j, j = 0, \ldots, m-1)\) by the relations

\[
T' = T - T_1, \quad U'_{i,j} = U_{i,j} - U_{i} \quad V'_{j} = V_{j} - V_{1,j}, \quad j = 0, 1, \ldots, m-1,
\]

we obtain the joint density of

\[
( T_1, U_{i,j}, V_{1,j}, T, U'_j, V'_j, \quad j = 0, 1, \ldots, m-1 )
\]
The conditional pdf of \((T_p, U_p, V_p, \theta = 0, 1, \ldots, m-1)\) \hspace{1cm} given \((T, U, V, \theta = 0, 1, \ldots, n-1)\) is obtained by dividing (5.18) by (5.15). By summing this conditional density over \(U_p\) from 0 to \(U, \theta = 0, 1, \ldots, m-1\) and \(V_p\) from 0 to \(V, \theta = 0, 1, \ldots, n-1\), we obtain the conditional density of \(T_p\) given \((T, U, V, \theta = 0, 1, \ldots, m-1)\)

as

\[
(5.19) f( t_1 | t, u, j, v, j, j = 0,1, \ldots, m-1 ) = \sum \sum \left[ 1/\beta(n_1, n-n_1) \right] (t_1/t)^{n_1-1} (1-t_1/t)^{n-n_1-1} (1/t)
\]

\[
m-1 \pi \left[ ( u_j - u_{1j} + v, j-1 - v_1, j-1 ) ( v, j-1 - v_1, j-1 ) \right] / \]

\[
\sum_{2} \sum_{3} \text{and } \sum_{2} \text{ and } \sum_{3} \text{ stand for summations over } \xi_{1j} \text{ and }
\]

...
\( \{ v_{1j} \}, \ j = 0,1, \ldots, m-1 \) respectively. Following the argument given in Chapter 2 we get the UMVU estimator of \( R(t_0) \) by integrating the above conditional pdf over the interval \([t_0, t]\).

5.3 Special cases.

Case (1) \( r_j = 0 \), \( j = 0,1, \ldots, m-1 \).

Here we consider a particular case of the model given by Section 5.2, assuming that \( r_j = 0 \), that is \( p_j + q_j = 1 \) for \( j = 0,1, \ldots, m-1 \). In this case the reliability function of the device remains the same as that given by (5.4).

The conditional density of \( T_1 \) given \( (T_0, U_1, U_2, \ldots, U_m) \) is obtained from (5.19) by letting \( v_{1j} = 1 \) and noting that \( v_{1j} = k \) for \( j = 0,1, \ldots, m-1 \), as

\[
(5.20) \quad f(t_{\frac{1}{m}}, v_{10}, v_{11}, \ldots, v_{1m-1})
\]

\[
= \sum_{j=0}^{n-1} \left[ \frac{(u_{1j} - u_{1j} + k-2)}{u_{1j} + k-1} \right] /
\]

\[
(t_{\frac{1}{m}}/t)^{n_{1j}-1} (1 - t_{\frac{1}{m}}/t)^{n_{1j}-1} (1/t) / \beta(n_{1j}, n-n_{1j})
\]

where \( \sum_{j=0}^{n_{1j}} \) stands for summation over \( u_{1j} \) such that \( 0 \leq u_{1j} < u_{\frac{1}{m}}, \)

\( j = 0,1, \ldots, m-1, \sum_{j=0}^{m-1} u_{1j} = n_1 \) and \( \sum_{l=1}^{k} n_{1l} = n \). Integrating this density over \([t_0, t]\) we get the UMVU estimator of \( R(t_0) \) as
(5.21) \( \tilde{R}(t_o) = \sum_{j=0}^{\infty} \left[ \frac{\beta_{t_o}^j}{j!} \frac{\beta_{t_o}^{n-j-1}}{(n-j)!} \int_{t_o/t}^1 \frac{z^{n_j-1} (1-z)^{n-n_j-1}}{\beta(n_1,n-n_1)} \, dz \right] \).

Case (ii) \( p_j = p \), and \( r_j = 0, j = 0,1,\ldots,n-1 \).

In this case the reliability function (5.4) reduces to

\[
(5.22) \tilde{R}(t_o) = \sum_{n=0}^{\infty} e^{-\beta_{t_o}} \frac{(\beta_{t_o})^n}{n!} \sum_{s=0}^{n-1} p^s (1-p)^{n-s} \sum_{l \leq s} (1),
\]

where \( \sum_{l \leq s} (1) \) is as defined in (5.4). Then

\[
(5.23) \tilde{R}(t_o) = \sum_{n=0}^{\infty} e^{-\beta_{t_o}} \frac{(\beta_{t_o})^n}{n!} \sum_{s=0}^{n-1} \binom{n}{s} p^s q^{n-s},
\]

since \( \sum_{l \leq s} (1) \) is the number of ways of distributing \( (n-s) \) balls among \((s+1)\) cells. Changing the order of summation, we have

\[
(5.24) \tilde{R}(t_o) = \sum_{s=0}^{n-1} \frac{1}{s!} \beta_{t_o}^s (\beta_{t_o} p)^s / s!.
\]

Using the relation between Poisson and gamma distribution the above expression can be written as

\[
(5.25) \tilde{R}(t_o) = \int_{t_o}^{\infty} e^{- \beta_p s} (\beta_p)^s s^{m-1} ds / (m-1)!,
\]
which is the reliability in the case of gamma life distribution. The UMVU estimator of (5.25) is given by Basu (1964) as follows

\[ R(t_o) = \frac{Ynp}{Yp} \left( \frac{1}{(nx)^{p-1}} \sum_{j=0}^{n-1} \frac{(nx)^j}{(n-l)^{p+j}} \right) t^{p-1} \left( 1 - \frac{t}{nx} \right)^{n-1}. \]

5.4 Estimator of reliability in Markov model with discrete time

As in the model given in Section 5.2 we assume that the device can be in any one of the (m+1) states 0, 1, \ldots, m, with 0 denoting brand new state, m denoting the inoperative state and all other states denoting the increasing order of deterioration. Let \( X(n), n = 0, 1, \ldots \) denote the state of the device due to \( n \)-th job. Then one step transition probabilities are given by

\[ P[ X(n+1) = j \mid X(n) = i ] = q_j, \quad \text{if } j = i \]
\[ = p_j, \quad \text{if } j = i+1 \]
\[ = r_j, \quad \text{if } j = m \]
\[ = 0, \quad \text{for other values of } j. \]

The reliability of the device after \( n \)-th job is given by

\[ R(n) = \sum_{s=0}^{m-1} \text{Prob (the device is in state } s \text{ after } n \text{-th job)}. \]

With the transition probabilities (5.27) the above expression for the reliability reduces to

\[ = \sum_{s=0}^{m-1} \left[ p_0 p_1 \cdots p_{s-1} \sum_{j=0}^{n} q_j \right] q_1 q_2 \cdots q_s.\]
where \( \sum_{l} \) is as defined in (5.4) with \( p_{-l} = 1 \). This model has been considered by Saksena (1978) and he has obtained different replacement policies. But here we find the UMVU estimator of reliability function (5.28).

As earlier \( k \) new devices are put on test and observed till all of them fail. The number of jobs performed at each stage are observed and the joint density is given by

\[
(5.29) \exp \left[ \sum_{j=0}^{m-1} u_{j} \log q_{j} + \sum_{j=0}^{m-1} v_{j} \log p_{j} + \sum_{j=0}^{m-1} (v_{j} - v_{j-1}) \log r_{j} \right],
\]

where \( u_{j}, v_{j} \) are as defined in 5.2. The complete sufficient statistics for the family can be easily found to be

\[
(5.30) (U_{o}, U_{1}, \ldots, U_{m-1}, V_{0}, \ldots, V_{m-2}).
\]

The conditional density of \((U_{jd}, V_{jd})\) \( j = 0,1, \ldots, m-1 \) for a given \((U_{jd}, V_{jd})\) \( j = 0,1, \ldots, m-1 \) can be obtained as

\[
(5.31) f[ (u_{jd}, v_{jd}, j = 0,1, \ldots,m-1) | (u_{jd}, v_{jd}, j = 0, \ldots,m-1) ]
\]

\[
= \prod_{j=0}^{m-1} \left( \frac{u_{j} v_{j}^{(m-j-1)} v_{j+1}^{(j-1)}}{v_{j} v_{j+1}} \right) \phi_{j}(u_{jd}, v_{jd}, j = 0, \ldots,m-1).
\]
Summing this density with respect to $u_{l_j}, v_{l_j}$ over the range $0 \leq u_{l_j} \leq u_{j}, 0 \leq v_{l_j} \leq v_{j}$ and $\sum (u_{l_j} + v_{l_j}) \geq n$ we obtain the UMVU estimator of $R(n)$.

5.5 ML Estimator of the reliability.

The reliability function (5.4) involves $2m$ parameters $(\rho, p_0, p_1, \ldots, p_{m-2}, q_0, \ldots, q_{m-1})$. From the likelihood function (5.10) the ML estimators of these parameters are found to be

$$\hat{\beta} = n/t$$
$$\hat{q}_j = u_{.j} v_{.j} / (u_{.j} - u_{.j} v_{.j} - v_{.j-1} v_{.j} + v_{.j}^2),$$
$$j = 0, 1, \ldots, m-1$$
$$\hat{p}_j = u_{.j}^2 / (u_{.j} - u_{.j} v_{.j} - v_{.j-1} v_{.j} + v_{.j}^2),$$
$$j = 0, 1, \ldots, m-2.$$  

The ML estimator of the reliability (5.4) is given by

$$R(t_0) = \sum_{n=0}^{\infty} \left[ e^{\hat{\beta}t_0} \left( \frac{\hat{\beta}t_0}{n!} \right)^n \sum_{s=0}^{m-1} (\hat{p}_0 \hat{p}_1 \cdots \hat{p}_{s-1} \sum_{\mathcal{L}} q_{0}^{n_0} \cdots q_{s}^{n_s} ) \right]$$

with $\hat{\beta}, \hat{p}_j, \hat{q}_j, j = 0, 1, \ldots$ given by (5.32). Using the estimators $\hat{q}_j$ and $\hat{p}_j, j = 0, 1, \ldots$ given in (5.32) ML estimator of the reliability (5.28) can be obtained similarly.