2. ESTIMATION OF RELIABILITY IN SHOCK MODELS
WITH CONTINUOUS IID DAMAGES

2.1 Introduction.

In this chapter we consider the problem of estimation of reliability in shock models with continuous, i.i.d damages. Esary, Marshall and Proschan (1973) have studied some models for the life distribution of a device subjected to a sequence of shocks occurring randomly in time as events in a Poisson process. If the device has the probability $\bar{F}_n$ of surviving $n(=0,1,2,...)$ shocks then the reliability $R(t_0)$ of the device at the mission time $t_0$, that is, the probability that it operates beyond time $t_0$ is given by

\begin{equation}
R(t_0) = \sum_{n=0}^{\infty} \bar{F}_n e^{-\beta_1 t_0} \frac{(\beta_1 t_0)^n}{n!},
\end{equation}

for some $\beta_1 > 0$. The above authors have shown among other things, that some properties such as increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE), decreasing failure rate (DFR), decreasing failure rate average (DFRA), new worse than used (NWU) etc. of the shock survival probabilities $\bar{F}_n$, natural in reliability are
reflected as properties of the life distribution, \( F(t) = 1 - R(t) \). They have considered the following three models for shock survival probabilities \( F_n \).

1. Cumulative damage fixed threshold model.
2. Cumulative damage random threshold model.

In cumulative damage fixed threshold model it is assumed that each shock causes a random damage, that damages caused by successive shocks are i.i.d with a common distribution \( F \), and that failure occurs when the accumulated damage exceeds a specified threshold \( y \). Cumulative damage random threshold model is similar to fixed threshold model except that here the threshold \( y \) is assumed to be a random variable with some distribution \( G \) such that \( G(0) = 0 \). In maximum shock threshold model, it is assumed that shocks do not affect the device unless the amount of damage due to a shock exceeds a fixed threshold \( y \). If the threshold is exceeded the device fails; otherwise it remains as good as new.

Under the cumulative damage fixed threshold model assuming that damages are i.i.d continuous random variables, having negative exponential distribution UMVU estimator of the component reliability is obtained in Section 2.2 and ML estimator is obtained in Section 2.3. Particular case when one of the parameters is known is considered and UMVU estimator of the reliability is obtained in Section 2.3.
We have obtained in Section 2.4 the UMVU estimator of 
\[ P(T > T') \] where \( T \) is the strength of a device which is subjected to a stress \( T' \). The device fails whenever \( T < T' \) and there is no failure when \( T > T' \). Further from the life test data on \( r(>m) \) components UMVU estimators of the reliability of \( k \)-out-of-\( m \) system and standby redundant system are obtained in Section 2.5. In the case of cumulative damage random threshold model if the distribution \( G \) of random threshold \( y \) is exponential, the life distribution \( F(t) = 1 - R(t) \) given by (2.1) reduces to exponential distribution. Again in maximum shock threshold model it is shown in Theorem 6.1 of Esary et al (1973) that the life distribution is exponential if the damages are i.i.d with some distribution function \( F \) such that \( F(0) = 0 \). The UMVU estimation of reliability in exponential case has been considered among others by Laurent (1963), Pugh (1963), Basu (1964), Rutemiller (1966), Holla (1967) and Sathe and Varde (1969). 

2.2 UMVU Estimator of the component Reliability

Under cumulative damage fixed threshold model we suppose that the damages \( X_i \)'s are i.i.d as negative exponential with density

\[ f(x ; \beta_2) = \beta_2 e^{-\beta_2 x}, \quad \beta_2 > 0, \quad x > 0. \]
Then the general expression of the reliability function (2.1) at mission time \( t_0 \) reduces to

\[
R(t_0) = \sum_{n=1}^{\infty} \left[ e^{-\beta_1 t_0} \frac{\beta_1 t_0}{(n-1)!} \int_0^{t_0} e^{-\beta_2 s} \frac{s^{n-2}}{(n-2)!} ds \right],
\]

where \( s \) is the total damage due to \((n-1)\) shocks.

Expressing the incomplete gamma integral in (2.3) as sum of Poisson probabilities, changing the order of summations and then using once again the relationship between Poisson and gamma distributions, the above expression (2.3) can be alternatively written as

\[
R(t_0) = \sum_{n=1}^{\infty} \left[ e^{-\beta_2 t_0} \frac{(\beta_2 v)^{n-1} e^{-\beta_1 t_0} \frac{\beta_1 t_0}{(n-1)!}}{(n-1)!} \right].
\]

To obtain the UMVU estimator of \( R(t_0) \), a sample of \( r \) brand new identical devices each having the life distribution \( F(t) = 1 - R(t) \) given by (2.4) are simultaneously placed on a life testing experiment. Each of these devices is subjected to a sequence of shocks which occur in an independent Poisson process of rate \( \beta_1 \). Let \( X_{ij} (i = 1, 2, \ldots, r; j = 1, 2, \ldots) \) denote the random damage due to \( j \)-th shock to the \( i \)-th device. Then \( X_{ij} \)'s are i.i.d random variables with a common exponential density (2.2). The damages accumulate additively and the \( i \)-th device
survives the k-th shock if the accumulated damage,

\[ X_{11} + X_{12} + \ldots + X_{1k} \]

does not exceed the fixed threshold \( y \). Suppose the life test is continued until all the \( r \) devices fail. Let \( N_1 \)-th shock be the fatal shock for the \( i \)-th device \( i = 1, 2, \ldots, r \), that is the \( i \)-th device survives \( N_1-1 \) shocks and fails due to \( N_1 \)-th shock. We assume that the damage \( X_{11,N_1} \) due to fatal shock is not observable but is known to exceed \( y - \sum_{j=1}^{N_1-1} X_{1j} \). Let \( T_{11}, T_{12}, \ldots, T_{1,N_1} \) be the time epochs at which the shocks occur to the \( i \)-th device. Then the interarrival times

\[ T_{11}, (T_{12} - T_{11}), \ldots, (T_{1,N_1} - T_{1,N_1-1}) \]

are i.i.d. random variables with exponential density \( \beta_1 \exp(-\beta_1 t) \). Now the joint density of the random variables

(2.5) \( (N_1, T_{11}, T_{12}, \ldots, T_{1,N_1}, X_{11}, X_{12}, \ldots, X_{1,N_1-1}) \)

for the \( i \)-th device is

(2.6) \[ \beta_1 \beta_2 e^{-\beta_1 t_1 - \beta_2 y} n_1^{-\beta_1 t_1} n_1^{-1} - \beta_2 y \]

with \( 0 < t_{11} < t_{12} < \ldots < t_{1,N_1} < \infty \);
\[ x_{1j} \geq 0 \text{ for } j = 1,2,\ldots, n_1-1 \]
\[ \sum_{j=1}^{n_1-1} x_{1j} < y \text{ and } n_1 = 1,2,\ldots, \]

and \( n_1, t_1, x_1 \) denoting the values taken by the random variables \( N_1, T_1, X_1 \) respectively. Thus the joint density of the variables

\[(N_1, T_{11}, \ldots, T_{1r}, N_{11}, X_{11}, X_{12}, \ldots, X_{1r}, N_{1r-1})\]

for \( i = 1,2,\ldots, r \) corresponding to all the \( r \) devices is.

\[ (2.7) \quad \beta_1^n e^{-\beta_1 t} \beta_2^r e^{-r\beta_2 y}, \]

with \( \sum_{i=1}^{r} n_i = n, \sum_{i=1}^{r} t_i, n_i = t \) and the ranges of \( t_1, x_1 \) and \( n_1 \) as given in (2.6) for each \( i = 1,2,\ldots, r \). The joint pdf (2.7) can be rewritten as

\[ (2.8) \quad \exp[ n \log(\beta_1\beta_2) - \beta_1 t + r \log\beta_1 - r\beta_2 y ] . \]

Obviously, this density belongs to two parameter exponential family. So by Theorem 1 of Lehmann (1959, p 132), \( (T,N) \) with \( T = \sum_{i=1}^{r} T_{1i}, N_1 \) and \( N = \sum_{i=1}^{r} N_i \) is complete sufficient statistic for the family of densities (2.8).
Now we obtain the UMVU estimator of reliability $R(t_0)$ given by (2.3) by applying Rao-Blackwell-Lehmann-Scheffe (RBL-S) Theorem and proceeding in the lines of Basu (1964). For brevity, let $T_l = T_{1,l}, l = 1, 2, \ldots, r$. Then for the first device we have

\[(2.9) \quad f(n_1, t_1) = f_1(n_1) f_2(t_1 | n_1).\]

Since the damages are assumed to be i.i.d negative exponential random variables, $f_1(n_1)$ is the same as the probability that $n_1-1$ events occur in time $y$ in a Poisson process of rate $\beta_2$. And the conditional distribution of $T_l$ given $N_l = n_1$, is the same as the distribution of the waiting time for $n_1$ events to occur in Poisson process of rate $\beta_1$. Thus

\[(2.10) \quad f(n_1, t_1) = e^{-\beta_2 y (\beta_2 y)^{n_1-1}} (n_1-1)! \cdot e^{-\beta_1 t_1 (\beta_1 t_1)^{n_1-1}} (n_1-1)! .\]

Now let

\[(2.11) \quad \varphi_{T_1}(t_0) = 1, \text{ if } T_1 > t_0 \]

\[= 0, \text{ otherwise,}\]

where $t_0$ is the mission time. Then it can be easily seen that $\varphi_{T_1}(t_0)$ is an unbiased estimator of $R(t_0)$. Actually,

\[(2.12) \quad E[\varphi_{T_1}(t_0)] = \sum_{n_1=1}^{\infty} \left[ e^{-\beta_2 y (\beta_2 y)^{n_1-1}} (\beta_2 y)^{n_1} \frac{1}{(n_1-1)!} \cdot e^{-\beta_1 t_1 (\beta_1 t_1)^{n_1-1}} (\beta_1 t_1)^{n_1} \frac{1}{(n_1-1)!} \cdot e^{-\beta_1 t_1 n_1 (\beta_1 t_1)^{n_1-1}} t_1 \right] \]
which is \( R(t_0) \), given by (2.4). Since \((T,N)\) is complete sufficient statistic, in virtue of RBLT theorem the UMVU estimator of \( R(t_0) \) is given by

\[
(2.13) \quad R(t_0) = E[ \varphi_{T^1}(t_0) \mid t, n ]
\]

To find this conditional expectation, we find the conditional density of \( T_1 \) for a given \((t,n)\). For this purpose consider the last \((r-1)\) devices and let

\[
N' = \sum_{i=2}^{r} N_i \quad \text{and} \quad T' = \sum_{i=2}^{r} T_i.
\]

Following the argument used in obtaining the joint density (2.10) and noting that \( n' \) is the sum of \((r-1)\) i.i.d Poisson variates, the density of \( N' \) can be obtained as

\[
(2.14) \quad f(n') = e^{-\beta_2 y} \frac{[(r-1)\beta_2 y]^{n'-r+1}}{(n'-r+1)!}.
\]

Also the conditional distribution of \( T' \) given \( N' = n' \) is the same as the distribution of the waiting time for \( n'-1 \) events to occur in a Poisson process of rate \( \beta_1 \). Thus the density of \((N',T')\) is

\[
(2.15) \quad f(n',t') = e^{-\beta_2 y} \frac{[(r-1)\beta_2 y]^{n'-r+1}}{(n'-r+1)!} \cdot e^{-\beta_1 t'} \beta_1^{n'(t')} \frac{n'-1}{(n'-r+1)!}.
\]

Since \((T_1, N_1)\) and \((T', N')\) are independent random vectors, the joint density of \((T_1, N_1, T', N')\), in virtue of (2.10)
and (2.15), is given by

\[ (2.16) f(n_1, t_1, n, t') = \left( \beta_1 y \right)^{n_1-1} e^{-\beta_2 y} e^{-\beta_1 t_1} \frac{\beta_1^{n_1} n_1!}{(n_1-1)! (n-1)!} \]

Letting \[ N = N' + N_1 \] and \[ T = T' + T_1 \] and then summing over all possible values of \[ N_1 \] we get the probability density of \( (T_1, T, N) \) as

\[ (2.17) f(t_1, n, t) = \sum_{n_1=1}^{n} \left[ \frac{(r-1)}{(n-r+1)!} \left( \beta_2 y \right)^{n_1} \frac{\beta_1^{n_1} n_1!}{(n_1-1)! (n-r+1)!} \right] e^{(r-1)\beta_2 y} - \beta_1 t_1 \frac{n_1^{n_1} (t-t_1)}{(n-n_1+r+1)!} \]

with \( n = r, r+1, \ldots \); \( 0 < t_1 < t < \infty \).

By the arguments similar to those used in obtaining the density of \( (N', T') \), the joint density of \( (N, T) \) can be obtained as

\[ (2.18) f(n, t) = e^{-\beta_2 y} \left( \beta_2 y \right)^{n-r} e^{-\beta_1 t} \frac{\beta_1^{n-1} n!}{(n-r)! (n-1)!} \]

Dividing the density (2.17) by the density (2.18), the conditional density of \( T_1 \) for a given \( (T, N) \) can be obtained as
\begin{align*}
(2.19) \ f(t_1 \mid n, t) &= \sum_{n_1=1}^{n-r+1} \left[ \binom{n-r}{n_1-1} \left( \frac{1}{r} \right)^{n_1-1} \left( 1 - \frac{1}{r} \right)^{n-n_1-1-r+1} 
\frac{1}{\beta(n_1, n-n_1)} \left( \frac{t_1}{t} \right)^{n_1-1} \left( 1 - \frac{t_1}{t} \right)^{n-n_1-1} \right].
\end{align*}

Finally the UMVU estimator of \( R(t_o) \) is obtained by integrating this conditional density over the interval \([t_o, t]\) and is given by

\begin{align*}
(2.20) \ \overline{R}(t_o) &= \sum_{n_1=1}^{n-r+1} \left[ \binom{n-r}{n_1-1} \left( \frac{1}{r} \right)^{n_1-1} \left( 1 - \frac{1}{r} \right)^{n-n_1-1-r-1} 
\frac{1}{\beta(n_1, n-n_1)} \int_{t_o/t}^{1} u^{n_1-1} (1-u)^{n-n_1-1} \, du \right], \ t_o < t,
\end{align*}

= 0, \text{ for } t_o > t.

Particular case 1: \( \beta_1 \) is known.

Using the relationship between gamma and Poisson distribution, the reliability function (2.3) can be rewritten as

\begin{align*}
(2.21) \ R(t_o) &= \sum_{n_1=1}^{\infty} \left[ e^{-\beta_1 t_o} \left( \frac{\beta_1 t_o}{n_1-1} \right)^{n_1-1} \sum_{j=n_1-1}^{\infty} e^{-\beta_2 y} \left( \frac{\beta_2 y}{j} \right)^{j} \right] 
\end{align*}

When \( \beta_1 \) is known, in order to estimate \( R(t_o) \) we have to estimate only

Now from the joint density (2.8) of the observations on $r$ devices, the total number of shocks received $N$, is complete sufficient for the family. By applying RBLS Theorem we get the UMVU estimator of (2.22) as

$$\sum_{j=n_1-1}^{\infty} e^{-\beta_2 y} \frac{\left(\beta_2 y\right)^j}{j!}.$$  

Thus the UMVU estimator of $R(t_0)$ given by (2.21), is

$$\sum_{j=n_1-1}^{n-r} \left(\frac{n-r}{j}\right) \left(1 - \frac{1}{r}\right)^{n-j}.$$  

Remark 1. When $\beta_2$ is known, the family of densities given by (2.8) is not complete and so only ML estimator is obtained in Section 2.3.

2.3 ML Estimator.

Using the same life test data obtained on a sample of $r$ components in Section 2.2 ML estimator of reliability is obtained in this section. From the joint pdf (2.7) of the observations, the ML estimators of the parameters are found to be
\( \hat{\beta}_1 = \frac{n}{t}, \hat{\beta}_2 = \frac{n-r}{r+y} \), where \( n = \sum_{i=1}^{r} n_i \)

is the total number of shocks occurred and \( t = \sum_{i=1}^{r} t_i \) is the total time on test. By the invariance property of ML estimators the ML estimator of the reliability is given by

\[
(2.26) \hat{R}(t_0) = \sum_{n_1=1}^{\infty} \left[ e^{-\hat{\beta}_1 t_0} \frac{(\hat{\beta}_1 t_0)^{n_1-1}}{(n_1-1)!} \sum_{s=0}^{\infty} e^{-\hat{\beta}_2 s} \frac{(\hat{\beta}_2 s)^{n_1-2}}{(n_1-2)!} \right] \hat{\beta}_2 ds
\]

with \( \hat{\beta}_1 \) and \( \hat{\beta}_2 \) given by (2.25).

Particular case 2:

When \( \hat{\beta}_1 \) is known, the ML estimator of reliability (2.21) is,

\[
(2.27) \hat{R}(t_0) = \sum_{n_1=1}^{\infty} \left[ e^{-\hat{\beta}_1 t_0} \frac{(\hat{\beta}_1 t_0)^{n_1-1}}{(n_1-1)!} \sum_{j=0}^{\infty} e^{-\hat{\beta}_2 j} \frac{(\hat{\beta}_2 j)^{n_1-1}}{j!} \right] \hat{\beta}_2^n
\]

with \( \hat{\beta}_2 \) given by (2.25). Similarly when \( \hat{\beta}_2 \) is known the ML estimator of reliability is obtained by substituting

\( \hat{\beta}_1 = n/t \) for \( \hat{\beta}_1 \) in (2.21).

Remark 2: Maximum shock threshold model.

Certain devices made up of brittle materials such as glass do not accumulate damages. In such cases it is
appropriate to assume that the device is as good as new so long as damage due to a shock does not exceed a fixed threshold \( y \) and it fails once the damage exceeds \( y \). Such a model is known as maximum shock threshold model. Suppose that damages are i.i.d random variables having exponential distribution with parameter \( \beta_2 \). If the shocks are occurring in Poisson process with rate \( \beta_1 \), the life distribution of the device is given by

\[
(2.28) \quad f_T(t) = \sum_{n=1}^{\infty} \left[ (1-e^{-\beta_2 y}) \frac{\beta_2 y^{n-1}}{(n-1)!} e^{-\beta_2 y} \frac{\beta_1 t^{n-1}}{(n-1)!} \right].
\]

This can be written as

\[
(2.29) \quad f_T(t) = \beta_3 e^{-\beta_3 t}, \quad \text{where} \quad \beta_3 = \beta_1 e^{-\beta_2 y}.
\]

So using the life test data on \( r \) components, UMVU estimator of the reliability can be easily obtained as

\[
(2.30) \quad \bar{R}(t_0) = (1 - t_0/t)^{r-1}, \quad \text{for} \quad t_0 < t
\]

\[
= 0, \quad \text{for} \quad t_0 > t.
\]

Remark 3: Cumulative damage random threshold model.

If the threshold \( y \) is assumed to have exponential distribution, life distribution is exponential as is shown in Theorem 4.10 of Esary et al (1973) and UMVU estimator can be obtained as in the case of the above model.
2.4 UMVU Estimator of \( P(T > T') \)

Let the random variables \( T \) and \( T' \) denote the life lengths of two components with probability distribution functions \( F \) and \( G \) respectively. Then

\[
(2.31) \quad P(T > T') = \int [1 - F(t)] dG(t)
\]

is the probability that component with life time \( T \) operates for a longer period than the other. Also this probability gives the reliability of a device in stress-strength model where \( T \) is strength of the device which is subjected to a random stress \( T' \). The component fails whenever \( T < T' \) and there is no failure when \( T > T' \). The problem of estimation of this probability in the context of reliability has been considered by many authors under the assumption that \( F \) and \( G \) are two different univariate distributions. Among them Tong (1974) and Bartoszewicz (1977) have assumed the \( F \) and \( G \) to be exponential distribution functions whereas Church and Harris (1970), Downtone (1973) and Woodward and Kelley (1977) have assumed them to be normal distributions. The above single component version of Tong (1974) has been extended to multi-component \( k \)-out-of-\( m \) system of stress-strength model by Bhattacharyya and Johnson (1974). Distribution free confidence bound for reliability is given by Owen et al (1964), Govindarajulu (1968) and others. Assuming that \( F \) and \( G \) are life distributions of two devices, for the case of cumulative
damage fixed threshold model, the UMVU estimator of \( P(T > T') \) is obtained in this section.

Suppose that a component with fixed threshold \( y_1 \) receives shocks in Poisson process with rate \( \beta_1 \) and the shocks cause random damages which are i.i.d as exponential with parameter \( \alpha_1 \). In virtue of (2.10) the probability density function of time to failure, \( T \), of this component is given by

\[
(2.32) \quad f_T(t; \beta_1, \alpha_1) = \sum_{n=1}^{\infty} \frac{e^{-\alpha_1 y_1} [(\alpha_1 y_1)^{n-1}]}{(n-1)!} \left( e^{-\beta_1 t} \beta_1^n \right)^{n-1} \frac{1}{(n-1)!}
\]

with \( \alpha_1, \beta_1 > 0 \). Suppose that \( r_1 \) such new identical components with life distribution (2.32) are put on life test till all of them fail. From (2.7) the joint density of the random variables

\[
(2.33) \quad \left( N_1, T_{11}, T_{12}, \ldots, T_{1r_1}, X_{11}, X_{12}, \ldots, X_{1r_1} \right)
\]

for \( i = 1, 2, \ldots, r_1 \) corresponding to all the \( r_1 \) components can be written as

\[
(2.34) \quad \beta_1^n e^{-\beta_1 t} \alpha_1^{r_1} e^{-r_1 \alpha_1 y_1} ,
\]

with \( \sum_{i=1}^{r_1} N_1 = n \) and \( \sum_{i=1}^{r_1} \sum_{j=1}^{N_1} t_{ij} = t \). Similarly suppose that another component with a fixed threshold \( y_2 \) receives shocks in Poisson process with rate \( \beta_2 \) and the random damages due to these shocks are i.i.d as exponential
with parameter $\alpha_2$. Again from (2.10) the density of the life length $T'$ of this component is given by

$$f_T(t' ; \beta_2, \alpha_2) = \sum_{n=1}^{\infty} \frac{(-\alpha_2 t')^n}{(n-1)!} e^{-\beta_2 t'} \frac{(\alpha_2 y_2)^{n-1}}{(n-1)!},$$

with $\alpha_2, \beta_2 > 0$. If such $r_2$ new components with life density (2.35) are put on life test, the joint density of the random variables

$$(N'_1, T'_{11}, T'_{12}, \ldots, T'_{1N'_1}, x'_1, \ldots, x'_1, N'_1 - 1)$$

for $i = 1, 2, \ldots, r_2$, can be written as

$$\beta_2 e^{-\beta_2 t'} \alpha_2^{r_2 - r_2} e^{-r_2 \alpha_2 y_2},$$

with $\sum_{i=1}^{r_2} n'_i = n'$ and $\sum_{i=1}^{r_2} t'_{1i} = t'$.

Now we define

$$\varphi_{T'_1}(t'_1) = \begin{cases} 1 & \text{if } T'_1 > t'_1 \\ 0 & \text{otherwise}, \end{cases}$$

where $T'_1$ is the life length of the first component with density (2.32) and $T'_1$ is the life length of the first component with density (2.35). Clearly $\varphi_{T'_1}(t'_1)$ is an unbiased estimator of $P(T > T')$. Since $(T'_1, N'_1, T'_1, N'_1)$ is complete sufficient, by RBLS Theorem the UMVU estimator
of $P(T > T')$ is given by

$$
\bar{P}(T > T') = \mathbb{E}[ \varphi_{T_1}(t_1') | n., t., n', t'] 
= P[ (T_1 > T_1') | n., t., n', t'] 
= P(T_1 > t_1' | t., n.) f(t_1' | t', n') dt_1'
$$

Denoting the conditional pdfs of $T_1$ and $T_1'$ by $f_1$ and $f_2$ respectively we have

$$(2.38) \quad P(T > T') = \int_0^{t_1'} \int_0^{t'} f_1(t_1', t., n.) f_2(t_1' | n', t') dt_1 dt_1'.
$$

These conditional pdfs are given by (2.19) substituting these conditional densities in (2.38) and considering only the terms involving $t_1$ and $t_1'$, we have

$$
(2.39) \quad \sum_{j=0}^{t_1'} \left( \frac{t_1-n_1-1}{t} \right) \left( 1-\frac{t_1}{t'} \right) \frac{\beta(n_1+n_1', n.-n_1)}{n_1+n_1'} (dt_1'/t') (dt_1/t'), \quad t_1 < t_1'.
$$

On interchanging the order of integration and simplifying (2.39) reduces to

$$(2.40) \quad \sum_{j=0}^{t_1'} \left[ (-1)^j \frac{(n_1-n_1'-1)_j}{j!} \frac{n_1'+j}{\beta(n_1+n_1', n.-n_1)} \right].
$$

Thus finally the UMVU estimator of $P(T > T')$ is obtained as
(2.41) \( \bar{F}(T > T') = \sum_{n_1} \sum_{n_1'} \left[ q_1 q_2 q(n_1, n_1') \right] \)

where \( q(n_1, n_1') \) stands for the expression (2.40) and

\[
(2.42) \quad Q_1 = \binom{n_r - r_1}{n_1 - 1} \left( \frac{1}{r_1} \right)^{n_1 - 1} \left( 1 - \frac{1}{r_1} \right)^{n_r - n_1 - r_1 + 1} / \beta(n_1, n_r - n_1) \quad \text{and} \\
Q_2 = \binom{n_r - r_2}{n_1' - 1} \left( \frac{1}{r_2} \right)^{n_1' - 1} \left( 1 - \frac{1}{r_2} \right)^{n_r - n_1' - r_2 + 1} / \beta(n_1', n_r - n_1').
\]

2.5 UMVU Estimator of system reliability

(1) \( k \)-out-of-\( m \) system.

A system with \( m \) components which operates if and only if at least \( k \) of them operate is known as \( k \)-out-of-\( m \) system. If the components are independent and identical and have reliability \( R(t_o) \) at mission time \( t_o \), then the reliability of the \( k \)-out-of-\( m \) system \( R_{k,m}(t_o) \) at time \( t_o \) is given by

\[
(2.43) \quad R_{k,m}(t_o) = \sum_{j=k}^{m} \binom{m}{j} R^j(t_o) (1 - R(t_o))^{m-j}.
\]

On expansion it can be written as

\[
(2.44) \quad R_{k,m}(t_o) = \sum_{j=k}^{m} \binom{m}{j} (-1)^{j} \binom{m}{j} R^{j+1}(t_o).
\]
Series and parallel systems become particular cases of this system when \( k = m \) and \( k = 1 \) respectively. Several authors like Ruttimiller (1966), Wani and Kabe (1971) Bhattacharyya and Johnson (1974), Basu and Mawaziny (1978) have obtained UMVU estimators of system reliabilities. They have used component life test data to estimate the system reliability. Also several different systems may be considered at one time and the same data used to draw inferences about the system reliabilities. System testing is important at the design stage and it contrasts markedly with the situation where a completed system must be constructed and tested. In this section assuming that the components are identical and independent and have life distribution

\[
(2.45) \quad f_{T_1}(t_1) = \sum_{n_1=1}^{\infty} \left[ e^{-\beta_2 Y} \frac{(\beta_2 Y)^{n_1-1}}{(n_1-1)!} e^{-\beta_1 t_1} \frac{\beta_1 t_1^{n_1-1}}{(n_1-1)!} \right],
\]

[see, (2.10)] we obtain the UMVU estimator of the system reliability (2.44).

Consider a life test on \( r(>m) \) new and identical components with life distribution (2.45). Let \((N_1, T_1), 1 = 1, 2, \ldots, r\) be the number of shocks received and time operated by the \( i \)-th component respectively. Also let

\[
\sum_{i=1}^{r} T_i = T \quad \text{be the total time on test and} \quad \sum_{i=1}^{r} N_i = N \quad \text{be the total number of shocks received by the} \quad r \quad \text{components.}
\]

Let us denote the UMVU estimator of \( R^S(t_0) \) by \( \bar{R}(s, t_0, n, t) \).
where \( i+j = s \), \( t_0 \) is the mission time and \((N,T)\) is complete sufficient for the family. Then

\[
(2.46) \quad \mathcal{R}(s,t_0,n,t) = P[ (T_1 > t_0, T_2 > t_0, \ldots, T_s > t_0) | n, t ].
\]

This conditional probability is given by

\[
(2.47) \quad \mathcal{R}(s,t_0,n,t) = \int_A f(t_1, t_2, \ldots, t_s | n, t) \, dt_1 \ldots dt_s,
\]

where

\[
A = \{ (t_1, t_2, \ldots, t_r) : t_0 = \sum_{i=1}^{r} t_i = t \}.
\]

To obtain the conditional density \( f(t_1, t_2, \ldots, t_s | n, t) \) consider the joint densities of \((N_1, T_1), i = 1, 2, \ldots, s, \) and \((N', T')\) where

\[
N' = \sum_{1=s+1}^{r} N_1 \quad \text{and} \quad T' = \sum_{1=s+1}^{r} T_1.
\]

Finding the product of these joint densities and then making the transformation

\[
N' = N - \sum_{1=s+1}^{S} N_1, \quad T' = T - \sum_{1=s+1}^{S} T_1,
\]

we get the joint density of \((N_1, T_1, 1 = 1, 2, \ldots, s; N, T)\). Dividing this by the joint density (2.18) of \((N, T)\) we get the conditional density of \((N_1, T_1, 1 = 1, 2, \ldots, s)\) for a given value of \((N, T)\) as

\[
(2.48) \quad f(n_1, t_1, 1 = 1, 2, \ldots, s | n, t) = \frac{(n-r)!}{s!} \left( \frac{n-1}{\prod_{1=1}^{n} (n_1-1)!} \right)^2 \frac{(n-r-\sum_{1=1}^{s} n_1+s)!}{(n-\sum_{1=1}^{s} n_1+s)!}.
\]
The conditional density of \((T_1, T_2, \ldots, T_s)\) given \((N, T)\) is obtained by summing with respect to \(n_1, n_2, \ldots, n_s\) over the range \(n_1 \geq 1, s = 1, 2, \ldots, s\) and \(\sum_{i=1}^{s} n_i = n - (r-s)\). Integrating this conditional density with respect to \(t_1, t_2, \ldots, t_s\) over the region \(t_0 < t_1 < t-s t_0, 1 = 1, 2, \ldots, s\) and \(\sum_{i=1}^{s} t_i = t\), we get the UMVU estimator of \(R_s(t_0)\). Thus for the \(k\)-out-of-\(m\) system the UMVU estimator of reliability is

\[
(2.49) \quad \bar{R}_{k,m}(t_0) = \sum_{k=1}^{m} \sum_{l=0}^{m-l} (-1)^l \binom{m}{l} \binom{m-l}{j} R(s, t_0, n, t)
\]

Special case: Putting \(k = m\) and \(k = 1\) in \((2.49)\) we get the UMVU estimator of reliability for series and parallel systems respectively. In the case of series system the estimator is

\[
(2.50) \quad \bar{R}_{m,m}(t_0) = \bar{R}(m, t_0, n, t).
\]
A system in which one component is operating and remaining (s-1) components are in standby is called a standby redundant system. Whenever the operating component fails a standby component is switched on. If T_1, T_2, ..., T_s are the life lengths of s components then the reliability of this system at time t_0 is given by

$$R_s(t_0) = P[T_1 + T_2 + ... + T_s > t_0].$$

If the components are identical and have life distribution with density given by (2.45), the life distribution of the redundant system is obtained by the arguments similar to those used in obtaining (2.15). From this life distribution the reliability of the standby redundant system can be obtained as

$$R_s(t_0) = \sum_{n=s}^{\infty} \frac{(-s \beta_2) \beta_2^n}{(n-s)!} \int_{t_0}^{\infty} e^{-\beta_1 t} \beta_1^r t^{n-1} \beta_1 dt.$$  

To obtain the UMVU estimator of $R_s(t_0)$, a life test is conducted on r (> s) new components and the observations made are $(N_1, T_1, i = 1,2, ..., r)$. Here $T_1$ is the life length and $N_1$ is the total number of shocks received by the i-th component. The joint distribution of $(N_1, T_1, i = 1,2, ..., r)$ belongs to exponential family and $(N, T)$ with $N = \sum_{i=1}^{r} N_i, T = \sum_{i=1}^{r} T_i$ is complete sufficient for the family. If
(2.53) \[ Z = T_1 + T_2 + \ldots + T_s \],

by virtue of RBLS Theorem, the UMVU estimator of (2.52) is given by

(2.54) \[ \hat{R}_s(t_0) = \int_{t_0}^{\infty} f(z | n, t) \, dz . \]

Let \( N' = \sum_{i=1}^{s} N' \), \( N'_{r-s} = \sum_{i=s+1}^{r} N' \) and \( T'_{r-s} = \sum_{i=s+1}^{r} T' \).

Then from the joint density of \((Z, N'_s, N'_{r-s}, T'_{r-s})\) the conditional density of \((Z, N'_s)\) for a given \((N, T)\) is obtained as

(2.55) \[ f(z, n'_s | n, t) = \binom{n-r}{r} \binom{s}{r} (1 - \frac{r}{n})^{n'_{r-s}} \left( 1 - \frac{r}{n} \right)^{n-n'_{r-s}+r} \frac{1}{t} \]

\[ \left( \frac{s}{t} \right)^{n'_{r-1}} (1 - \frac{s}{t})^{n-n'_{r-1}} / \beta(n'_s, n-n'_s) . \]

Summing this density over \( n'_s \) from \( s \) to \( n-r \) we get the conditional density \( f(z | n, t) \). Integrating it with respect to \( z \) over the range \( t_0 < z < t \), we obtain the UMVU estimator of reliability \( R_s(t_0) \) as under

(2.56) \[ \hat{R}_s(t_0) = \sum_{n'_s=s}^{n-r+s} \left[ \binom{n-r}{r} \binom{s}{r} (1 - \frac{r}{n})^{n'_{r-s}} \left( 1 - \frac{r}{n} \right)^{n-n'_{r-s}+r} \right] \left[ \frac{1}{t} \int_{t_0}^{t} u^{n'_{r-1}} (1-u)^{n-n'_{r-1}} \, du / \beta(n'_s, n-n'_s) \right], \]

if \( t_0 < t \)

= 0 if \( t_0 > t \).