Chapter 3

D-continuous maps in Topological Spaces

3.1 Introduction

Several authors [7, 14, 30, 41, 59, 67] working in the field of general topology have shown much interest in the concepts of generalizations of continuous maps. A weak form of continuous map called $g$-continuous map was introduced by Balachandran [7]. In this chapter D-continuous maps are defined and their relations with various generalized continuous maps and few properties are discussed. Strongly D-continuous maps, perfectly D-continuous maps, D-compact and D-connected spaces are defined and developed.
3.2 D-continuous functions

In this section the concept of $D$-continuous functions in topological spaces are introduced and their relations with various generalized continuous maps are discussed.

**Definition 3.2.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $D$-continuous if $f^{-1}(F)$ is $D$-closed in $(X, \tau)$ for every closed set $F$ of $(Y, \sigma)$.

**Example 3.2.2.** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is $D$-continuous. It is observed the closed set $F = \{b\}$, $f^{-1}(F) = \{b\}$ is $D$-closed.

**Theorem 3.2.3.** Every continuous is $D$-continuous.

**Proof.** Let $f : (X, \tau) \to (Y, \sigma)$ be continuous. Let $F$ be closed in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(F)$ is closed in $(Y, \sigma)$. By theorem 2.2.2, $f^{-1}(F)$ is $D$-closed in $(X, \tau)$. Hence $f$ is $D$-continuous. \qed

**Remark 3.2.4.** The converse of the above theorem need not be true as seen from the following example:

**Example 3.2.5.** Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = b = f(c)$. Then $f$ is $D$-continuous but not continuous. It is observed the closed set $F = \{b\}$, $f^{-1}(F) = \{b, c\}$ is $D$-closed but not closed.
**Proposition 3.2.6.** Every contra continuous and pre-continuous is D-continuous

**Proof.** Let \( f : (X, \tau) \to (Y, \sigma) \) be contra continuous and pre-continuous. Let \( F \) be closed in \((Y, \sigma)\). Then \( f^{-1}(F) \) is pre-closed and open in \((X, \tau)\). Hence by theorem 2.2.3, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( f \) is D-continuous. \(\square\)

**Remark 3.2.7.** The converse of the above proposition need not be true as seen from the following example:

**Example 3.2.8.** Let \( X = \{a, b, c\} = Y, \tau = \{\emptyset, \{b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(b) = c \) and \( f(c) = b \). Then \( f \) is D-continuous but neither contra continuous nor pre-continuous. It is observed the closed set \( F = \{c\} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{a, b\} \) is D-closed but it is neither pre-closed nor open in \((X, \tau)\).

**Proposition 3.2.9.** Every D-continuous is gp-continuous.

**Proof.** By theorem 2.2.6, every D-closed set is gp-closed, the proof follows. \(\square\)

**Remark 3.2.10.** The converse of the above proposition need not be true as seen from the following example:

**Example 3.2.11.** Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = a; f(b) = c \) and \( f(c) = b \). Then \( f \) is gp-continuous but not D-continuous. It is
observed the closed set \( F = \{c\} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{b\} \) is \( gpr \)-closed but not D-closed in \((X, \tau)\).

**Proposition 3.2.12.** Every D-continuous is \( gpr \)-continuous.

**Proof.** By theorem 2.2.9, every D-closed set is \( gpr \)-closed, the proof follows. \(\square\)

**Remark 3.2.13.** The converse of the above proposition need not be true as seen from the following example:

**Example 3.2.14.** By Example 3.2.11, \( f \) is \( gpr \)-continuous but not D-continuous. It is observed the closed set \( F = \{c\} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{b\} \) is \( gpr \)-closed but not D-closed in \((X, \tau)\).

**Proposition 3.2.15.** Every D-continuous is \( gsp \)-continuous.

**Proof.** By theorem 2.2.15, every D-closed set is \( gsp \)-closed, the proof follows. \(\square\)

**Remark 3.2.16.** The converse of the above proposition need not be true as seen from the following example.

**Example 3.2.17.** By example 3.2.14, \( f \) is \( gsp \)-continuous but not D-continuous. It is observed the closed set \( F = \{c\} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{b\} \) is \( gsp \)-closed but not D-closed in \((X, \tau)\).

**Proposition 3.2.18.** Every D-continuous is \( \pi gp \)-continuous.

**Proof.** By theorem 2.2.12, every D-closed is \( \pi gp \)-closed, the proof follows. \(\square\)
Remark 3.2.19. The converse of the above proposition need not be true as seen from the following example:

Example 3.2.20. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then $f$ is $\pi gp$-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is $\pi gp$-closed but not D-closed in $(X, \tau)$.

Remark 3.2.21. D-continuous and pre-continuous are independent. It is shown by the following example:

Example 3.2.22. By example 3.2.20, $f$ is pre-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is pre-closed but not D-closed in $(X, \tau)$.

Example 3.2.23. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = b = f(c)$. Then $f$ is D-continuous but not pre-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b, c\}$ is D-closed but not pre-closed.

Remark 3.2.24. D-continuous is independent of semi-continuous and semi-pre-continuous. It is shown by the following example:

Example 3.2.25. By example 3.2.23, $f$ is D-continuous but neither semi-continuous nor semi-pre-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b, c\}$ is D-closed but neither semi-closed nor semi-pre-closed.
Example 3.2.26. Let $X = Y = \{a, b, c\}$, $\tau = \emptyset, \{a\}, \{a, b\}, X$ and $\sigma = \emptyset, \{a, c\}, Y$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is semi-continuous and semi-pre-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b\}$ semi-closed and semi-pre-closed but not D-closed.

Remark 3.2.27. D-continuous and pre-semi-continuous are independent. It is shown by the following example:

Example 3.2.28. Let $X = Y = \{a, b, c\}$, $\tau = \emptyset, \{c\}, X$ and $\sigma = \emptyset, \{b, c\}, Y$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(c) = a$ and $f(b) = b$. Then $f$ is D-continuous but not pre-semi-continuous. It is observed the closed set $F = \{a\}$, $f^{-1}(F) = \{a, c\}$ is D-closed but not pre-semi-closed.

Example 3.2.29. By example 3.2.20, $f$ is pre-semi-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is pre-semi-closed but not D-closed in $(X, \tau)$.

Remark 3.2.30. D-continuous and pg-continuous are independent. It is shown by the following examples:

Example 3.2.31. By example 3.2.23, $f$ is D-continuous but not pg-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b, c\}$ is D-closed but not pg-closed in $(X, \tau)$.

Example 3.2.32. By example 3.2.20, $f$ is pg-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is pg-closed but not D-closed in $(X, \tau)$.
Remark 3.2.33. D-continuous and $g^*p$-continuous are independent. It is shown by the following example:

Example 3.2.34. By example 3.2.28, $f$ is D-continuous but not $g^*p$-continuous. It is observed the closed set $F = \{a\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{a, c\}$ is D-closed but not $g^*p$-closed.

Example 3.2.35. By example 3.2.20, $f$ is $g^*p$-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is $g^*p$-closed but not D-closed.

Remark 3.2.36. D-continuous and g-continuous are independent. It is shown by the following example:

Example 3.2.37. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is D-continuous but not g-continuous. It is observed the closed set $F = \{a\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{a\}$ is D-closed but not g-closed in $(X, \tau)$.

Example 3.2.38. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is g-continuous but not D-continuous. It is observed the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b\}$ is g-closed but not D-closed in $(X, \tau)$.

Proposition 3.2.39. Every D-continuous is $\rho$-continuous.

Proof. By theorem 2.2.42, every D-closed set is $\rho$-closed, the proof follows. $\square$

Remark 3.2.40. The converse of the above theorem need not be true as seen from the following example:
Example 3.2.41. Let \( X = Y = \{a, b, c\} \), \( \tau = \{\emptyset, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). Then the identity function \( f : (X, \tau) \to (Y, \sigma) \) is \( \rho \)-continuous but not D-continuous. It is observed the closed set \( F = \{b\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{b\} \) is \( \rho \)-closed but not D-closed.

Remark 3.2.42. We have the following relationship between D-continuous and other related generalized continuous. \( A \to B(A \nrightarrow B) \) represents \( A \) implies \( B \) but not conversely (\( A \) and \( B \) are independent of each other).

3.3 Characterization of D-continuous functions

In this section the characterization of D-continuous functions in the sense of definition 3.2.1 is obtained.

Theorem 3.3.1. A function \( f : (X, \tau) \to (Y, \sigma) \) is D-continuous if and only if \( f^{-1}(U) \) is D-open in \( (X, \tau) \) for every open set \( U \) in \( (Y, \sigma) \).
Proof. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be D-continuous and \( U \) be an open set in \( (Y, \sigma) \). Then \( f^{-1}(U^c) \) is D-closed in \( (X, \tau) \). But \( f^{-1}(U^c) = (f^{-1}(U))^c \) and so \( f^{-1}(U) \) is D-open in \( (X, \tau) \). Conversely, let \( U \) be an open set in \( (Y, \sigma) \). Then \( U^c \) is a closed set in \( (Y, \sigma) \). Since \( f^{-1}(U) \) is D-open in \( (X, \tau) \), \( (f^{-1}(U))^c \) is D-closed in \( (X, \tau) \). Therefore \( f^{-1}(U^c) = (f^{-1}(U))^c \) is D-closed in \( (X, \tau) \).

\[\square\]

Remark 3.3.2. The composition of two D-continuous functions need not be D-continuous. It is shown by the following example:

Example 3.3.3. Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{a, b\}, X\} \), \( \sigma = \{\emptyset, \{a\}, X\} \) and \( \eta = \{\emptyset, \{a, c\}, X\} \). Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = b, f(b) = a, f(c) = c \) and define \( g : (Y, \sigma) \rightarrow (Z, \eta) \) by \( g(x) = x \). Then \( f \) and \( g \) are D-continuous but \( g \circ f \) is not D-continuous. Since \( \{b\} \) is closed in \( (Z, \eta) \), \( (g \circ f)^{-1}(\{b\}) = f^{-1}(g^{-1}(\{b\})) = f^{-1}(\{b\}) = \{a\} \) is not D-closed in \( (X, \tau) \).

Definition 3.3.4. 1. A space \( (X, \tau) \) is said to be D-\( T_s \)-space if every D-closed set is closed.

2. A space \( (X, \tau) \) is said to be D-\( T_{\frac{1}{2}} \)-space if every D-closed set is pre-closed.

Theorem 3.3.5. Let \( (X, \tau) \) and \( (Z, \eta) \) be topological spaces and \( (Y, \sigma) \) be D-\( T_s \)-space. Then the composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) of D-continuous (resp. continuous) function \( f : (X, \tau) \rightarrow (Y, \sigma) \) and the D-continuous function \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is D-continuous (resp. continuous).
Proof. Let $G$ be any closed set of $(Z, \eta)$. Then by assumption $g^{-1}(G)$ is closed in $(Y, \sigma)$. Since $f$ is D-continuous (resp. continuous), $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is D-closed (resp. closed) in $(X, \tau)$. Thus $g \circ f$ is D-continuous (resp. continuous). \qed

**Theorem 3.3.6.** Let $(X, \tau)$ and $(Z, \eta)$ be topological spaces and $(Y, \sigma)$ be $T\frac{1}{2}$-space (resp $Tw$-space, $T\tilde{g}$-space, $gsT\frac{1}{2}$-space). Then the composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ of D-continuous function $f : (X, \tau) \rightarrow (Y, \sigma)$ and the g-continuous (resp $\omega$-continuous, $\tilde{g}$-continuous, $#gs$-continuous) function $g : (Y, \sigma) \rightarrow (Z, \eta)$ is D-continuous.

Proof. Let $G$ be any closed set of $(Z, \eta)$. Then $g^{-1}(G)$ is g-closed (resp $\omega$-closed, $\tilde{g}$-closed, $#gs$-closed) in $(Y, \sigma)$ and by assumption, $g^{-1}(G)$ is closed in $(Y, \sigma)$. Since $f$ is D-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is D-closed in $(X, \tau)$. Thus $g \circ f$ is D-continuous. \qed

**Theorem 3.3.7.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be D-continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be continuous. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is D-continuous.

Proof. Let $G$ be any closed set of $(Z, \eta)$. Then $g^{-1}(G)$ is closed in $(Y, \sigma)$. Since $f$ is D-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is D-closed in $(X, \tau)$. Thus $g \circ f$ is D-continuous. \qed

**Theorem 3.3.8.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be contra continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be contra continuous. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is D-continuous.
Proof. Let $G$ be any closed set of $(Z, \eta)$. Since $g$ is contra continuous, then $g^{-1}(G)$ is open in $(Y, \sigma)$. Since $f$ is contra continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is closed in $(X, \tau)$. Then by theorem 2.2.2, $(g \circ f)^{-1}(G)$ is D-closed in $(X, \tau)$. Hence $g \circ f$ is D-continuous.  

Definition 3.3.9. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called D-irresolute if $f^{-1}(F)$ is D-closed (resp.D-open) in $X$ for every D-closed (resp.D-open) subset $F$ of $Y$.

Example 3.3.10. Let $X = \{a, b, c\} = Y$ with topologies $\tau = \{\emptyset, \{b\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, X\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = c$; $f(b) = a$; $f(c) = b$. Then the function $f$ is D-irresolute.

Theorem 3.3.11. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be D-irresolute and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be D-continuous. Then their composition $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is D-continuous.

Proof. Let $G$ be any closed set of $(Z, \eta)$. Since $g$ is D-continuous, $g^{-1}(G)$ is D-closed in $(Y, \sigma)$. Since $f$ is D-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is D-closed in $(X, \tau)$. Thus $g \circ f$ is D-continuous.  

Theorem 3.3.12. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be D-continuous then $f$ is continuous if $(X, \tau)$ is D-$T_s$.

Proof. Let $G$ be any closed set of $(Y, \sigma)$. Since $f$ is D-continuous and by assumption $f^{-1}(G)$ is closed in $(X, \tau)$, $f$ is continuous.  

Definition 3.3.13. 1. Let $x$ be a point of $(X, \tau)$ and $V$ be a subset of $X$. Then $V$ is called a D-neighborhood of $x$ in $(X, \tau)$ if there exists a D-open set $U$ of $(X, \tau)$ such that $x \in U \subseteq V$. 

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2. The intersection of all D-closed sets containing a set $A$ in a topological space $X$ is called a D-closure of $A$ and is denoted by $D\text{-}cl(A)$.

**Theorem 3.3.14.** Let $A$ be a subset of $(X, \tau)$. Then $x \in D\text{-}cl(A)$ if and only if for any D-neighborhood $N_x$ of $x$ in $(X, \tau)$ such that $A \cap N_x \neq \emptyset$.

**Proof.** Necessity: Assume that $x \in D\text{-}cl(A)$. Suppose that there exists a D-neighborhood $N_x$ of $x$ such that $A \cap N_x = \emptyset$. Since $N_x$ is a D-neighborhood of $x$ in $(X, \tau)$, by definition 3.3.13, there exists a D-open set $V_x$ such that $x \in V_x \subseteq N_x$. Therefore, we have $A \cap V_x = \emptyset$ and so $A \subseteq (V_x)^c$. Since $(V_x)^c$ is a D-closed set containing $A$, we have $D\text{-}cl(A) \subseteq (V_x)^c$ and therefore $x \notin D\text{-}cl(A)$, which is a contradiction.

Sufficiency: Assume that for each D-neighborhood $N_x$ of $x$ in $(X, \tau)$ such that $A \cap N_x \neq \emptyset$. Suppose $x \notin D\text{-}cl(A)$. Then there exists a D-closed set $F$ of $(X, \tau)$ such that $A \subseteq F$ and $x \notin F$. Thus $x \in F^c$ is D-open in $(X, \tau)$. But $A \cap F^c = \emptyset$, which is a contradiction.  

**Theorem 3.3.15.** Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:

1. The function $f$ is D-continuous

2. The inverse of each open set in $(Y, \sigma)$ is D-open in $(X, \tau)$

3. The inverse of each closed set in $(Y, \sigma)$ is D-closed in $(X, \tau)$
4. For each \( x \) in \( (X, \tau) \) the inverse of every neighborhood of \( f(x) \) is a D-neighborhood of \( x \).

5. For each \( x \) in \( (X, \tau) \) and each neighborhood \( N \) of \( f(x) \), there is a D-neighborhood \( W \) of \( x \) such that \( f(W) \subseteq N \).

6. For each subset \( A \) of \( (X, \tau) \), \( f(D-cl(A)) \subseteq cl(f(A)) \).

7. For each subset \( B \) of \( (Y, \sigma) \), \( D-cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \).

**Proof.** 1 \( \iff \) 2 This follows from theorem 3.3.1.

2 \( \iff \) 3 The proof is clear from the result \( f^{-1}(A^c) = (f^{-1}(A))^c \).

2 \( \iff \) 4 Let \( x \in X \) and let \( N \) be a neighborhood of \( f(x) \). Then there exists an open set \( V \) in \( (Y, \sigma) \) such that \( f(x) \in V \subseteq N \). Consequently \( f^{-1}(V) \) is D-open in \( (X, \tau) \) and \( x \in f^{-1}(V) \subseteq f^{-1}(N) \). Thus \( f^{-1}(N) \) is a D-neighborhood of \( x \).

4 \( \iff \) 5 Let \( x \in X \) and let \( N \) be a neighborhood of \( f(x) \). Then by assumption, \( W = f^{-1}(N) \) is a D-neighborhood of \( x \) and \( f(W) = f(f^{-1}(N)) \subseteq N \).

5 \( \iff \) 6 Suppose that (5) holds. Let \( y \in f(D-cl(A)) \) and let \( N \) be any neighborhood of \( y \). Then there exists \( x \in X \) and a D-neighborhood \( W \) of \( x \) such that \( f(x) = y, x \in W \). Hence \( x \in D-cl(A) \) and \( f(W) \subseteq N \). By theorem 3.3.14, \( W \cap A \neq \emptyset \) and hence \( f(A) \cap N \neq \emptyset \). Hence \( y = f(x) \in cl(f(A)) \). Therefore \( f(D-cl(A)) \subseteq cl(f(A)) \). Conversely, suppose that (6) holds. Let \( x \in X \) and \( N \) be any neighborhood of \( f(x) \). Let \( A = f^{-1}(N^c) \). Since \( f(D-cl(A)) \subseteq cl(f(A)) \subseteq N^c \), \( D-cl(A) \subseteq A \). Hence \( D-cl(A) = A \). Since \( x \notin D-cl(A) \), there exists a D-
neighborhood \( W \) of \( x \) such that \( W \cap A = \emptyset \). Hence \( f(W) \subseteq f(A^c) \subseteq N \).  

6 \( \iff \) 7 Suppose that (6) holds. Let \( B \) be any subset of \((Y, \sigma)\). Then replacing \( A \) by \( f^{-1}(B) \) in (6), we obtain \( f(D-cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B) \). That is \( D-cl(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \). Conversely, suppose (7) holds. Let \( B = f(A) \), where \( A \) is a subset of \((X, \tau)\). Then \( D-cl(A) \subseteq D-cl(f^{-1}(B)) \subseteq f^{-1}(cl(f(A))) \) and so \( f(D-cl(A)) \subseteq cl(f(A)) \).  

\[ \blacksquare \]

**Proposition 3.3.16.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \omega \)-irresolute and \( M \)-preclosed function then \( f(A) \) is \( D \)-closed in \((Y, \sigma)\) for every \( D \)-closed set \( A \) of \((X, \tau)\).

**Proof.** Let \( U \) be any \( \omega \)-open set of \((Y, \sigma)\) such that \( f(A) \subseteq U \). Then \( A \subseteq f^{-1}(U) \). Since \( f \) is \( \omega \)-irresolute then \( f^{-1}(U) \) is \( \omega \)-open. Since \( A \) is \( D \)-closed in \((X, \tau)\), we have \( pcl(A) \subseteq int(f^{-1}(U)) \). Hence \( f(pcl(A)) \subseteq int(U) \). Since \( f \) is \( M \)-preclosed, \( f(pcl(A)) \) is pre-closed in \((Y, \sigma)\). Now \( pcl(f(A)) \subseteq pcl(f(pcl(A))) = f(pcl(A)) \subseteq int(U) \). Hence \( f(A) \) is \( D \)-closed in \((Y, \sigma)\).  

\[ \blacksquare \]

**Theorem 3.3.17.** If the bijective function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is pre-irresolute and \( \omega^* \)-open then \( f \) is \( D \)-irresolute.

**Proof.** Let \( A \) be \( D \)-closed in \((Y, \sigma)\) and let \( U \) be any \( \omega \)-open set in \((X, \tau)\) such that \( f^{-1}(A) \subseteq U \). Then \( A \subseteq f(U) \). Since \( f \) is \( \omega^* \)-open, \( f(U) \) is \( \omega \)-open in \((Y, \sigma)\). Since \( A \) is \( D \)-closed in \((Y, \sigma)\), we have \( pcl(A) \subseteq int(f(U)) \). Thus \( f^{-1}(pcl(A)) \subseteq f^{-1}(int(f(U))) \subseteq int(f^{-1}(f(U))) = int(U) \). Since \( f \) is pre-irresolute, we have \( f^{-1}(pcl(A)) \subseteq int(U) \).
is pre-closed in \((X, \tau)\). Now, 
\[ pcl(f^{-1}(A)) \subseteq pcl(f^{-1}(pcl(A))) = f^{-1}(pcl(A)) \subseteq int(U). \]
Hence \(f^{-1}(A)\) is D-closed in \((X, \tau)\) and so \(f\) is D-irresolute.

\[ \square \]

**Theorem 3.3.18.** 1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( gp\)-continuous and contra continuous then \( f \) is D-continuous.

2. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( gpr\)-continuous and RC-continuous then \( f \) is D-continuous.

**Proof.** 1. Let \( F \) be any closed set of \((Y, \sigma)\). Since \( f \) is \( gp\)-continuous and contra continuous, \( f^{-1}(F) \) is \( gp\)-closed and open in \((X, \tau)\). By theorem 2.3.16, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( f \) is D-continuous.

2. Let \( F \) be any closed set of \((Y, \sigma)\). Since \( f \) is \( gpr\)-continuous and RC-continuous, \( f^{-1}(F) \) is \( gpr\)-closed and regular open. By theorem 2.3.14, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( f \) is D-continuous.

\[ \square \]

**Theorem 3.3.19.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is D-irresolute and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is D-irresolute then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is D-irresolute.

**Proof.** Let \( G \) be any D-closed set of \((Z, \eta)\). Since \( g \) is D-irresolute, \( g^{-1}(G) \) is D-closed in \((Y, \sigma)\). Since \( f \) is D-irresolute, \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \) is D-closed in \((X, \tau)\). Thus \( g \circ f \) is D-irresolute.  

\[ \square \]
The following are regarding the restriction of a D-continuous function:

**Theorem 3.3.20.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a D-continuous function and \( H \) be a open D-closed subset of \( X \). Assume that \( \text{DC}(X, \tau) \) (the class of all D-closed sets of \( (X, \tau) \)) is D-closed under finite intersections. Then the restriction \( f|H : (H, \tau_H) \to (Y, \sigma) \) is D-continuous.

**Proof.** Let \( F \) be a closed subset of \( Y \). By hypothesis and assumption, 
\[ f^{-1}(F) \cap H = H_1 \text{ (say)} \] is D-closed in \( X \). Since \( (f|H)^{-1}(F) = H_1 \), it is sufficient to show that \( H_1 \) is D-closed in \( H \). Let \( G_1 \) be an \( \omega \)-open set in \( H \) such that \( H_1 \subseteq G_1 \). Then by hypothesis and by Lemma 1.1.19(2), \( G_1 \) is \( \omega \)-open in \( X \). Since \( H_1 \) is D-closed in \( X \), \( \text{pcl}_X(H_1) \subseteq \text{int}(G_1) \). Since \( H \) is open and by lemma 1.1.15, \( \text{pcl}_H(H_1) = \text{pcl}_X(H_1) \cap H \subseteq \text{int}(G_1) \cap H = \text{int}(G_1) \cap \text{int}(H) = \text{int}(G_1 \cap H) \subseteq \text{int}(G_1) \). Hence \( H_1 = (f|H)^{-1}(F) \) is D-closed in \( H \). Thus \( f|H \) is D-continuous. \( \square \)

**Theorem 3.3.21.** Let \( A \) and \( Y \) be subsets of \( (X, \tau) \) such that \( A \subseteq Y \subseteq X \). Let \( A \) be \( \omega \)-closed and regular closed in \( (X, \tau) \). If \( A \) is D-closed in \( (Y, \sigma) \) and \( Y \) is open and D-closed in \( (X, \tau) \) then \( A \) is D-closed in \( (X, \tau) \).

**Proof.** Let \( U \) be an \( \omega \)-open set of \( (X, \tau) \) such that \( A \subseteq U \). Since \( Y \) is open in \( (X, \tau) \) and \( A \) is D-closed in \( (Y, \sigma) \), we have \( \text{pcl}_Y(A) \subseteq \text{int}_Y(U \cap Y) \). Thus \( \text{pcl}(A) \cap Y \subseteq \text{pcl}_Y(A) \subseteq \text{int}_Y(U \cap Y) = \text{int}(U \cap Y) \). By lemma 1.1.19(1), \( (\text{pcl}(A))^c \) is \( \omega \)-open in \( (X, \tau) \). Hence \( \text{int}(U \cap Y) \cup (\text{pcl}(A))^c \) is \( \omega \)-open in \( (X, \tau) \) and it contains \( Y \). Since \( Y \) is D-closed...
in \((X, \tau)\), we have \(\text{pcl}(A) \subseteq \text{pcl}(Y) \subseteq \text{int}[\text{int}(U \cap Y) \cup (\text{pcl}(A))^c] \subseteq \text{int}(U) \cup (\text{pcl}(A))^c\). Thus \(\text{pcl}(A) \subseteq \text{int}(U)\). Hence \(A\) is D-closed in \((X, \tau)\). □

Theorem 3.3.22. Let \(X = G \cup H\) be a topological space with topology \(\tau\) and \(Y\) be a topological space with topology \(\sigma\). Let \(f : (G, \tau_G) \to (Y, \sigma)\) and \(g : (H, \tau_H) \to (Y, \sigma)\) be D-continuous functions such that \(f(x) = g(x)\) for every \(x \in G \cap H\). Assume that \(D[E] \subseteq D_p[E]\), for any \(E \subseteq X\). Suppose that both \(G\) and \(H\) are open and D-closed in \((X, \tau)\). Then their combination \(f \Delta g : (X, \tau) \to (Y, \sigma)\) defined by \((f \Delta g)(x) = f(x)\) if \(x \in G\) and \((f \Delta g)(x) = g(x)\) if \(x \in H\) is D-continuous.

Proof. Let \(F\) be a closed subset of \((Y, \sigma)\). Then \(f^{-1}(F)\) is D-closed in \((G, \tau_G)\) and \(g^{-1}(F)\) is D-closed in \((H, \tau_H)\). Since \(G\) and \(H\) are both open and D-closed subsets of \((X, \tau)\), by theorem 3.3.21, \(f^{-1}(F)\) and \(g^{-1}(F)\) are both D-closed sets in \((X, \tau)\). By theorem 2.3.23, \(f^{-1}(F) \cup g^{-1}(F)\) is D-closed in \((X, \tau)\). By definition, \((f \Delta g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)\) is D-closed in \((X, \tau)\). Hence \(f \Delta g\) is D-continuous. □

3.4 Strongly D-continuous and Perfectly D-continuous functions

The different forms of continuous functions namely Strongly continuous functions [31] and Perfectly continuous functions [49] have been introduced in this section. The concepts of Strongly D-continuous
maps, Perfectly D-continuous maps in topological spaces are introduced and some of their basic properties are studied.

**Definition 3.4.1.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called

1. Perfectly D-continuous if \( f^{-1}(F) \) is clopen in \((X, \tau)\) for every D-closed set (resp. D-open set) \( F \) of \((Y, \sigma)\).

2. Strongly D-continuous if \( f^{-1}(F) \) is closed (resp. open) in \((X, \tau)\) for every D-closed set (resp. D-open set) \( F \) of \((Y, \sigma)\).

3. Pre-D-continuous if \( f^{-1}(F) \) is D-closed in \((X, \tau)\) for every pre-closed set \( F \) of \((Y, \sigma)\).

**Remark 3.4.2.** From the above definition and the results that closed set \( \rightarrow \) D-closed set \( \rightarrow \) gp-closed (resp. \( \pi \) gp-closed) we have the following

```
Strongly gp-continuous
                     /           \\
Perfectly D-continuous --- Strongly D-continuous --- D-continuous
                     \\
Strongly \( \pi \) gp-continuous
```

**Theorem 3.4.3.** 1. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is perfectly D-continuous then \( f \) is strongly D-continuous and also D-irresolute.

2. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is pre-D-continuous then \( f \) is D-continuous.

3. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is strongly D-continuous and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is D-continuous then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is continuous.
4. If \( f : (X, \tau) \to (Y, \sigma) \) is strongly D-continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is perfectly D-continuous then \( g \circ f : (X, \tau) \to (Z, \eta) \) is strongly D-continuous.

5. If \( f : (X, \tau) \to (Y, \sigma) \) is perfectly D-continuous and \( g : (Y, \sigma) \to (Z, \eta) \) is pre-D-continuous then \( g \circ f : (X, \tau) \to (Z, \eta) \) is D-continuous.

**Proof.**

1. Let \( F \) be D-closed in \((Y, \sigma)\). Then \( f^{-1}(F) \) is clopen in \((X, \tau)\) and hence \( f^{-1}(F) \) is closed in \((X, \tau)\) and so \( f \) is strongly D-continuous. By theorem 2.2.2, closed set implies D-closed, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Thus \( f \) is D-irresolute.

2. Since every closed set is pre-closed, the proof is obvious.

3. Let \( F \) be closed in \((Z, \eta)\). Then \( g^{-1}(F) \) is D-closed in \((Y, \sigma)\) and \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is closed in \((X, \tau)\). Then \( g \circ f \) is continuous.

4. Let \( F \) be D-closed in \((Z, \eta)\). Then \( g^{-1}(F) \) is clopen in \((Y, \sigma)\). Since every closed set is pre-closed and by theorem 2.2.2, \( g^{-1}(F) \) is D-closed in \((Y, \sigma)\). Hence \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is closed in \((X, \tau)\). Then \( g \circ f \) is strongly D-continuous.

5. Let \( F \) be closed in \((Z, \eta)\). Since every closed set is pre-closed, \( g^{-1}(F) \) is D-closed in \((Y, \sigma)\). Hence \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is clopen in \((X, \tau)\). By theorem 2.2.2, \( (g \circ f)^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( g \circ f \) is D-continuous.
Theorem 3.4.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective, $D$-irresolute and $M$-preclosed. If $(X, \tau)$ is a $D\text{-}T\frac{1}{2}$ space, then $(Y, \sigma)$ is also $D\text{-}T\frac{1}{2}$ space.

Proof. Let $A$ be $D$-closed in $(Y, \sigma)$. Since $f$ is $D$-irresolute, $f^{-1}(A)$ is $D$-closed in $(X, \tau)$. Since $(X, \tau)$ is a $D\text{-}T\frac{1}{2}$ space, $f^{-1}(A)$ is pre-closed in $(X, \tau)$. Since $f$ is $M$-preclosed then $f(f^{-1}(A)) = A$ is pre-closed in $(Y, \sigma)$. Hence $(Y, \sigma)$ is a $D\text{-}T\frac{1}{2}$ space.

3.5 D-compactness and D-connectedness

In this section D-Compactness and D-Connectedness are defined by using D-closed sets and some of their properties are studied.

Definition 3.5.1. A topological space $(X, \tau)$ is D-compact if every D-open cover of $X$ has a finite subcover.

Theorem 3.5.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective, $D$-continuous function. If $X$ is D-compact then $Y$ is compact.

Proof. Let $\{A_i : i \in I\}$ be an open cover of $Y$. Then $\{f^{-1}(A_i) : i \in I\}$ is a D-open cover of $X$. Since $X$ is D-compact, it has a finite subcover say $\{f^{-1}(A_1), \ldots, f^{-1}(A_n)\}$. Since $f$ is surjective, $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $Y$. Hence $Y$ is compact.
Definition 3.5.3. A subset $A$ of a space $X$ is called D-compact relative to $X$ if every collection $\{U_i : i \in I\}$ of D-open subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in I\}$, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup \{U_i : i \in I_0\}$.

Theorem 3.5.4. Every D-closed subset of a D-compact space $X$ is D-compact relative to $X$.

Proof. Let $A$ be a D-closed subset of a D-compact space $X$. Let $\{U_i : i \in I\}$ be a cover of $A$ by D-open subsets of $X$. So, $A \subseteq \bigcup \{U_i : i \in I\}$ and then $A^c \cup (\bigcup \{U_i : i \in I\}) = X$. Since $X$ is D-compact, there exists a finite subset $I_0$ of $I$ such that $A^c \cup (\bigcup \{U_i : i \in I_0\}) = X$. Then $A \subseteq \bigcup \{U_i : i \in I_0\}$. Hence $A$ is D-compact relative to $X$. \qed

Theorem 3.5.5. If $f : (X, \tau) \to (Y, \sigma)$ is an injective D-irresolute and a subset $A$ of $X$ is D-compact relative to $X$ then its image $f(A)$ is D-compact relative to $Y$.

Proof. Let $\{f(U_i) : i \in I\}$ be a cover of $f(A)$ by D-open subsets of $(Y, \sigma)$. Since $f$ is D-irresolute, $\{U_i : i \in I\}$ is a cover of $A$ by D-open subsets of $(X, \tau)$. Since $A$ is compact relative to $X$, there exists a finite subset $I_0$ of $I$ such that $A \subseteq \bigcup \{U_i : i \in I_0\}$. Hence $f(A) \subseteq \bigcup \{f(U_i) : i \in I_0\}$. Thus $f(A)$ is compact relative to $Y$. \qed

Theorem 3.5.6. If $p : X \times Y \to X$ is a projection, then $p$ is D-irresolute.

Proof. Let $A$ be a D-closed subset of $X$. Since $p$ is a projection, $p^{-1}(A) = A \times Y$ is a subset of $X \times Y$. Now to show that $p^{-1}(A) = A \times Y$
is $D$-closed in $X \times Y$. Let $U$ be an $\omega$-open subset of $X \times Y$ such that $A \times Y \subseteq U$. Then $V \times Y = U$, for some open set $V$ of $X$ containing $A$. Since $A$ is $D$-closed in $X$, we have $pcl_X(A) \subseteq \text{int}(V)$ and $pcl_X(A) \times Y \subseteq \text{int}(V) \times Y$. That is $pcl_X \times Y (A \times Y) \subseteq \text{int}(V \times Y) = \text{int}(U)$. Hence $p^{-1}(A) = A \times Y$ is $D$-closed in $X \times Y$. 

\[\square\]

**Theorem 3.5.7.** If the product space $X \times Y$ is $D$-compact then each of the spaces $X$ and $Y$ is $D$-compact.

**Proof.** Let $X \times Y$ be $D$-compact. By theorem 3.5.6, the projection $p : X \times Y \to X$ is $D$-irresolute and then by theorem 3.5.5, $p(X \times Y) = X$ is $D$-compact. The proof for the space $Y$ is similar to the case of $X$. 

\[\square\]

**Theorem 3.5.8.** Let $A$ be any subset of $Y$.

1. If $X \times A$ is $D$-closed in the product space $X \times Y$ and $Y$ is $T_\omega$-space then $A$ is $D$-closed in $Y$.

2. If $X$ is compact and $A$ is $D$-closed in $Y$ and $X \times Y$ is $T_\omega$-space then $X \times A$ is $D$-closed in $X \times Y$.

**Proof.** 1. Let $U$ be an $\omega$-open set of $Y$ such that $A \subseteq U$. Then $X \times A \subseteq X \times U$. Since $Y$ is $T_\omega$-space, $U$ is open in $Y$ and $X \times U$ is open in $X \times Y$. Hence $X \times U$ is $\omega$-open in $X \times Y$. Since $X \times A$ is $D$-closed in $X \times Y$, $pcl(X \times A) \subseteq \text{int}(X \times U) = X \times U$. By Proposition 1.1.23, $X \times pcl(A) \subseteq X \times \text{int}(U)$. Thus $pcl(A) \subseteq \text{int}(U)$. Hence $A$ is $D$-closed in $Y$. 

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2. Let $U$ be an $\omega$-open set of $X \times Y$ such that $X \times A \subseteq U$. Since $X$ is compact and $X \times Y$ is $T\omega$-space and by the generalization of lemma 1.1.20, there exists an open set $V$ in $Y$ containing $A$ such that $X \times V \subseteq U$. Since $A$ is D-closed in $Y$, $pcl(A) \subseteq int(V)$. Therefore $X \times pcl(A) \subseteq X \times int(V)$. This implies $X \times pcl(A) \subseteq int(X) \times int(V) \subseteq int(X \times V)$. By proposition 1.1.23, $pcl(X \times A) \subseteq int(X \times V) \subseteq int(U)$. Hence $X \times A$ is D-closed in $X \times Y$.

Definition 3.5.9. A topological space $(X, \tau)$ is D-connected if $X$ cannot be written as the disjoint union of two non-empty D-open sets.

Theorem 3.5.10. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjective, D-continuous (resp. D-irresolute) function. If $X$ is D-connected then $Y$ is connected (resp. D-connected)

Proof. Suppose $Y$ is not connected (resp. not D-connected). Then $Y = A \cup B$, where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and $A, B$ are open (resp.D-open ) sets in $Y$. Since $f$ is surjective, $f(X) = Y$ and since $f$ is D-continuous (resp.D-irresolute), $X = f^{-1}(A) \cup f^{-1}(B)$ is the disjoint union of two non-empty D-open sets, which is a contradiction to $X$ is D-connected.

Theorem 3.5.11. If the product space $X \times Y$ is D-connected then each of the spaces $X$ and $Y$ is D-connected.
**Proof.** Let $X \times Y$ be D-connected. By theorem 3.5.6, the projection $p : X \times Y \to X$ is D-irresolute and then by theorem 3.5.10, $p(X \times Y) = X$ is D-connected. The proof for the space $Y$ is similar to the case of $X$. \hfill \Box