Chapter 7

Remarks on D-homeomorphisms

7.1 Introduction

The notion of homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces \((X, \tau)\) and \((Y, \sigma)\) is a bijective map \(f: (X, \tau) \rightarrow (Y, \sigma)\) where both \(f\) and \(f^{-1}\) are continuous. In the course of generalization of the notion of homeomorphism Maki et al. [37] introduced \(g\)-homeomorphisms and \(gc\)-homeomorphisms in topological spaces.

In this chapter D-homeomorphisms which is weaker than homeomorphisms and \(D^r\)-homeomorphisms are introduced. It turns out that the set of all \(D^r\)-homeomorphisms forms a group under the operation composition of functions. D-regular spaces and D-normal spaces in topological spaces are introduced. Also several properties of D-regular spaces and D-normal spaces and some preservation theorems for D-regular spaces and D-normal spaces are obtained.
7.2 D-homeomorphisms

In this section D-homeomorphisms is defined and the relationship between semi-homeomorphisms, pre-homeomorphisms and $g$-homeomorphisms are investigated. The fact that composition of two D-homeomorphisms need not be D-homeomorphism is established.

Definition 7.2.1. A bijection map $f : (X, \tau) \to (Y, \sigma)$ is called D-homeomorphism if $f$ is both D-continuous and D-open.

Example 7.2.2. Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{b, c\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c; f(b) = b$ and $f(c) = a$. Then $f$ is a D-homeomorphism

Theorem 7.2.3. Every D-homeomorphism is a gp-homeomorphism (resp. gpr-homeomorphism, gsp-homeomorphism, $\pi$gp-homeomorphism, $\rho$-homeomorphism)

Proof. By proposition 3.2.9, every D-continuous map is gp-continuous (resp. gpr-continuous, gsp-continuous, $\pi$gp-continuous, $\rho$-continuous) and also by theorem 2.2.6, every D-open is gp-open (resp. gpr-open, gsp-open, $\pi$gp-open, $\rho$-open), the proof follows. \qed

Remark 7.2.4. The converse of the above theorem need not be true as seen from the following example:

Example 7.2.5.

(i) Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a; f(b) = c$ and $f(c) =$
b. Then \( f \) is gp-homeomorphism but not D-homeomorphism. It is observed for the closed set \( F = \{c\} \) in \((Y, \sigma), f^{-1}(F) = \{b\}\) is gp-closed but not D-closed in \((X, \tau)\). Therefore \( f \) is not D-continuous.

(ii) By example 7.2.5(i), \( f \) is gpr-homeomorphism but not D-homeomorphism. It is observed for the closed set \( F = \{c\} \) in \((Y, \sigma), f^{-1}(F) = \{b\}\) is gpr-closed but not D-closed. Therefore \( f \) is not D-continuous.

(iii) By example 7.2.5(i), \( f \) is gsp-homeomorphism but not D-homeomorphism. It is observed for the closed set \( F = \{c\} \) in \((Y, \sigma), f^{-1}(F) = \{b\}\) is gsp-closed but not D-closed in \((X, \tau)\). Thus \( f \) is not D-continuous.

(iv) By example 7.2.5(i), \( f \) is gp-homeomorphism but not D-homeomorphism. It is observed for the closed set \( F = \{c\} \) in \((Y, \sigma), f^{-1}(F) = \{b\}\) is gp-closed but not D-closed in \((X, \tau)\). Therefore \( f \) is not D-continuous.

(v) Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be an identity map. Then \( f \) is \( \rho \)-homeomorphism but not D-homeomorphism. It is observed for the closed set \( F = \{b\}, f^{-1}(F) = \{b\}\) is \( \rho \)-closed but not D-closed. Therefore \( f \) is not D-continuous.

**Theorem 7.2.6.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a contra open, contra continuous and gp-homeomorphism. Then \( f \) is a D-homeomorphism.
Proof. Let $U$ be an open set of $(X, \tau)$. Then $f(U)$ is gp-open in $(Y, \sigma)$. Hence $Y - f(U)$ is gp-closed in $(Y, \sigma)$. Since $f$ is contra open, $f(U)$ is closed in $(Y, \sigma)$ and so $Y - f(U)$ is open in $(Y, \sigma)$. By theorem 1.1.23, $Y - f(U)$ is pre-closed in $(Y, \sigma)$ and by theorem 2.2.3, $Y - f(U)$ is D-closed in $(Y, \sigma)$, that is $f(U)$ is D-open in $(Y, \sigma)$. Hence $f$ is D-open. Let $F$ be closed in $(Y, \sigma)$. Then $f^{-1}(F)$ is gp-closed in $(X, \tau)$. Since $f$ is contra continuous, $f^{-1}(F)$ is open in $(X, \tau)$. By theorem 1.1.23 and by theorem 2.2.3, $f^{-1}(F)$ is D-closed in $(X, \tau)$. Hence $f$ is D-continuous. Since $f$ is D-continuous and D-open, $f$ is a D-homeomorphism.

Theorem 7.2.7. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a contra $\pi$-open, contra $\pi$-continuous and $\pi$gp-homeomorphism. Then $f$ is a D-homeomorphism.

Proof. Let $U$ be an open set of $(X, \tau)$. Then $f(U)$ is $\pi gp$-open in $(Y, \sigma)$. Hence $Y - f(U)$ is $\pi gp$-closed in $(Y, \sigma)$. Since $f$ is contra $\pi$-open, $f(U)$ is $\pi$-closed in $(Y, \sigma)$ and so $Y - f(U)$ is $\pi$-open in $(Y, \sigma)$. By theorem 1.1.10, $Y - f(U)$ is preclosed in $(Y, \sigma)$ and since every $\pi$-open is open and by theorem 2.2.3, $Y - f(U)$ is D-closed in $(Y, \sigma)$, that is $f(U)$ is D-open in $(Y, \sigma)$. Hence $f$ is D-open. Let $F$ be closed in $(Y, \sigma)$. Then $f^{-1}(F)$ is $\pi gp$-closed in $(X, \tau)$. Since $f$ is contra $\pi$-continuous, $f^{-1}(F)$ is $\pi$-open in $(X, \tau)$. By theorem 1.1.10, $f^{-1}(F)$ is pre-closed in $(X, \tau)$ and since every $\pi$-open is open and by theorem 2.2.3, $f^{-1}(F)$ is D-closed in $(X, \tau)$. Hence $f$ is D-continuous. Since $f$ is D-continuous and D-open, $f$ is a D-homeomorphism.

Theorem 7.2.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a contra regular open, RC-
continuous and gpr-homeomorphism. Then \( f \) is a D-homeomorphism.

**Proof.** Let \( U \) be an open set of \((X, \tau)\). Then \( f(U) \) is gpr-open in \((Y, \sigma)\). Hence \( Y - f(U) \) is gpr-closed in \((Y, \sigma)\). Since \( f \) is contra regular open, \( f(U) \) is regular closed in \((Y, \sigma)\) and so \( Y - f(U) \) is regular open in \((Y, \sigma)\). By theorem 1.1.9, \( Y - f(U) \) is preclosed in \((Y, \sigma)\) and since every regular open is open and by theorem 2.2.3, \( Y - f(U) \) is D-closed in \((Y, \sigma)\), that is \( f(U) \) is D-open in \((Y, \sigma)\). Hence \( f \) is D-open. Let \( F \) be closed in \((Y, \sigma)\). Then \( f^{-1}(F) \) is gpr-closed in \((X, \tau)\). Since \( f \) is RC-Continuous, \( f^{-1}(F) \) is regular open in \((X, \tau)\). By theorem 1.1.9, \( f^{-1}(F) \) is pre-closed in \((Y, \sigma)\) and since every regular open is open by theorem 2.2.3, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( f \) is D-continuous. Since \( f \) is D-continuous and D-open, \( f \) is a D-homeomorphism. \( \square \)

**Theorem 7.2.9.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a contra open, contra continuous and pre-homeomorphism. Then \( f \) is a D-homeomorphism.

**Proof.** Let \( U \) be an open set of \((X, \tau)\). Then \( f(U) \) is pre-open in \((Y, \sigma)\). Hence \( Y - f(U) \) is pre-closed in \((Y, \sigma)\). Since \( f \) is contra open, \( f(U) \) is closed in \((Y, \sigma)\) and so \( Y - f(U) \) is open in \((Y, \sigma)\). By theorem 2.2.3, \( Y - f(U) \) is D-closed in \((Y, \sigma)\), that is \( f(U) \) is D-open in \((Y, \sigma)\). Hence \( f \) is D-open. Let \( F \) be closed in \((Y, \sigma)\). Then \( f^{-1}(F) \) is pre-closed in \((X, \tau)\). Since \( f \) is contra continuous, \( f^{-1}(F) \) is open in \((X, \tau)\). By theorem 2.2.3, \( f^{-1}(F) \) is D-closed in \((X, \tau)\). Hence \( f \) is D-continuous. Since \( f \) is D-continuous and D-open, \( f \) is a D-homeomorphism. \( \square \)
Theorem 7.2.10. *Every homeomorphism is a D-homeomorphism.*

**Proof.** By theorems 2.2.2 and 3.2.3, the proof follows. \(\square\)

**Remark 7.2.11.** The converse of the above theorem need not be true as seen from the following example:

**Example 7.2.12.** Let \(X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}\) and \(\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}\). Then the identity map \(f : (X, \tau) \rightarrow (Y, \sigma)\) is D-homeomorphism but not a homeomorphism, because it is not continuous. Observe that for the closed set \(F = \{b, c\}\) in \((Y, \sigma)\), \(f^{-1}(F) = \{b, c\}\) is not closed in \((X, \tau)\).

**Remark 7.2.13.** D-homeomorphism and \(g\)-homeomorphism are independent as seen from the following example:

**Example 7.2.14.** (i) Let \(X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}\) and \(\sigma = \{\emptyset, \{b\}, \{a, b\}, Y\}\). Define \(f : (X, \tau) \rightarrow (Y, \sigma)\) by \(f(a) = b; f(b) = a;\) and \(f(c) = c\). Then \(f\) is a D-homeomorphism but not \(g\)-homeomorphism, because it is not \(g\)-open. It is observed for the open set \(V = \{a, c\}\) in \((X, \tau)\), \(f(V) = \{b, c\}\) is not \(g\)-open in \((Y, \sigma)\).

(ii) Let \(X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}\) and \(\sigma = \{\emptyset, \{c\}, \{a, b\}, Y\}\). Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) defined by \(f(a) = a; f(b) = c; f(c) = b\). Then \(f\) is \(g\)-homeomorphism but not D-homeomorphism because it is not D-open. It is observed for the open set \(V = \{a\}\) in \((X, \tau)\), \(f(V) = \{a\}\) is not D-open in \((Y, \sigma)\).
Remark 7.2.15. D-homeomorphism and semi-homeomorphism are independent as seen from the following example:

(i) Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}. \) Let \( f : (X, \tau) \to (Y, \sigma) \) be an identity map. Then \( f \) is a D-homeomorphism but not semi-homeomorphism, because it is not semi-continuous. It is observed for the closed set \( F = \{b, c\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{b, c\} \) is not semi-closed in \( (X, \tau) \). Therefore \( f \) is not a semi-continuous map.

(ii) Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}. \) Then the identity map \( f : (X, \tau) \to (Y, \sigma) \) is a semi-homeomorphism but not a D-homeomorphism because it is not D-open. It is observed for the open set \( V = \{a, c\} \) in \( (X, \tau) \), \( f(V) = \{a, c\} \) is not D-open in \( (Y, \sigma) \). Therefore \( f \) is not a D-open map.

Remark 7.2.16. D-homeomorphism and pre-homeomorphism are independent as seen from the following example:

Example 7.2.17. (i) Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}. \) Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b; \ f(b) = a; \ f(c) = c. \) Then \( f \) is a D-homeomorphism but not a pre-homeomorphism, because it is not pre-continuous. It is observed for the closed set \( F = \{b, c\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{a, c\} \) is not pre-closed in \( (X, \tau) \). Therefore \( f \) is not a pre-continuous map.
(ii) Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{c\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, c\}, Y\}$. Then the identity map $f : (X, \tau) \rightarrow (Y, \sigma)$ is a pre-homeomorphism but not a D-homeomorphism because it is not D-open. It is observed for the open set $V = \{c\}$ in $(X, \tau)$, $f(V) = \{c\}$ is not D-open in $(Y, \sigma)$. Therefore $f$ is not a D-open map.

**Remark 7.2.18.** We have the following relationship between D-homeomorphism and other related generalized homeomorphisms. $A \rightarrow B (A \nleftrightarrow B)$ represent $A$ implies $B$ but not conversely ($A$ and $B$ are independent of each other).

![Diagram of D-homeomorphism and related generalized homeomorphisms]

**Theorem 7.2.19.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an injection D-continuous map. Then the following statements are equivalent:

1. $f$ is a D-open map.
2. $f$ is a D-homeomorphism.
3. $f$ is a D-closed map.

**Proof.** (1) $\Rightarrow$ (2). By hypothesis and by assumption, proof is obvious.
(2) ⇒ (3). Let \( F \) be a closed set of \((X, \tau)\). Then \( F^c \) is open in \((X, \tau)\). By hypothesis \( f(F^c) = (f(F))^c \) is D-open in \((Y, \sigma)\). That is \( f(F) \) is D-closed in \((Y, \sigma)\). Therefore \( f \) is a D-closed map.

(3) ⇒ (1). Let \( V \) be an open set of \((X, \tau)\). Then \( V^c \) is closed in \((X, \tau)\). By hypothesis \( f(V^c) = (f(V))^c \) is D-closed in \((Y, \sigma)\). That is \( f(V) \) is D-open in \((Y, \sigma)\). Therefore \( f \) is a D-open map.

\[\square\]

**Remark 7.2.20.** The composition of two D-homeomorphism maps need not be D-homeomorphism as seen from the following example:

**Example 7.2.21.** Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{c\}, \{a, b\}, X\} \), \( \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\} \) and \( \eta = \{\emptyset, \{b\}, Z\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b; \ f(b) = c; \ f(c) = a \) and \( g : (Y, \sigma) \to (Z, \eta) \) is the identity map. Then both \( f \) and \( g \) are D-homeomorphisms but their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is not a D-homeomorphism. It is observed for the closed set \( F = \{a, c\} \) in \((Z, \eta)\), \((g \circ f)^{-1}(F) = \{b, c\}\) is not D-closed in \((X, \tau)\). Hence \( g \circ f \) is not a D-continuous map and so \( g \circ f \) is not a D-homeomorphism.

**Theorem 7.2.22.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a D-homeomorphism. Let \( A \) be an open D-closed subset of \( X \) and let \( B \) be a closed subset of \( Y \) such that \( f(A) = B \). Assume that \( DC(X, \tau) \) (the class of all D-closed sets of \((X, \tau)\)) is closed under finite intersections. Then the restriction \( f|A : (A, \tau_A) \to (B, \sigma_B) \) is a D-homeomorphism.
Proof. It is shown that $f|A$ is a bijection, $f|A$ is a D-open map and $f|A$ is a D-continuous map.

(i) Since $f$ is one-one, $f|A$ is also one-one. Also since $f(A) = B$, $f|A(A) = B$ so that $f|A$ is onto and hence $f|A$ is a bijection.

(ii) Let $U$ be an open set of $(A, \tau_A)$. Then $U = A \cap H$ for some open set $H$ in $(X, \tau)$. Since $f$ is one-one, then $f(U) = f(A \cap H) = f(A) \cap f(H) = B \cap f(H)$. Since $f$ is D-open and $H$ is an open set in $(X, \tau)$ then $f(H)$ is a D-open set in $(Y, \sigma)$. Therefore $f(U)$ is a D-open set in $(B, \sigma_B)$. Hence $f|A$ is a D-open map.

(iii) Let $F$ be a closed set of $(B, \sigma_B)$. Then $F = B \cap K$ for some closed set $K$ in $(Y, \sigma)$. Since $B$ is a closed set in $(Y, \sigma)$, $F$ is a closed set in $(Y, \sigma)$. By hypothesis and assumption, $f^{-1}(F) \cap A = H_1$ (say) is a D-closed set in $(X, \tau)$. Since $(f|A)^{-1}(F) = H_1$, it is sufficient to show that $H_1$ is a closed set in $(A, \tau_A)$. Let $G_1$ be $\omega$-open in $(A, \tau_A)$ such that $H_1 \subseteq G_1$. Then by hypothesis and by Lemma 1.1.19(2), $G_1$ is $\omega$-open in $X$. Since $H_1$ is a D-closed set in $(X, \tau)$, we have $pcl_X(H_1) \subseteq int(G_1)$. Since $A$ is open and by lemma 1.1.15, $pcl_A(H_1) = pcl_X(H_1) \cap A \subseteq int(G_1) \cap A = int(G_1)$ and so $H_1 = (f|A)^{-1}(F)$ is a D-closed in $(A, \tau_A)$. Therefore $f|A$ is a D-continuous map. Hence $f|A$ is a D-homeomorphism.

$\square$
**Theorem 7.2.23.** Let $(X, \tau)$ be a topological space and let $(Y, \sigma)$ be a D-Hausdorff space. Let $f : (X, \tau) \to (Y, \sigma)$ be a one-one D-irresolute map. Then $(X, \tau)$ is also a D-Hausdorff space.

**Proof.** Let $x_1, x_2$ be any two distinct points of $X$. Since $f$ is one-one, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1), y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $(Y, \sigma)$ is D-Hausdorff, there exist D-open sets $U_1$ and $U_2$ of $(Y, \sigma)$ such that $y_1 \in U_1, y_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Since $f$ is D-irresolute, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are D-open sets of $(X, \tau)$. Now $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = f^{-1}(\emptyset) = \emptyset$ and $y_1 \in U_1$ implies $f^{-1}(y_1) \in f^{-1}(U_1)$ implies $x_1 \in f^{-1}(U_1)$. Also $y_2 \in U_2$ implies $f^{-1}(y_2) \in f^{-1}(U_2)$ implies $x_2 \in f^{-1}(U_2)$. This shows that for every pair of distinct points $x_1, x_2$ of $X$, there exist disjoint D-open sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ such that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$. Hence the space $(X, \tau)$ is a D-Hausdorff space. \qed

**Theorem 7.2.24.** Every D-compact subspace $A$ of a D-Hausdorff space $X$ is D-closed. Assume that $DO(X, \tau)$ (the class of all D-open sets of $(X, \tau)$) be closed under finite intersection.

**Proof.** It is shown that $X - A$ is a D-open subset of $(X, \tau)$. Let $x \in X - A$. Since $X$ is D-Hausdorff, there exist disjoint D-open sets $U_y$ and $V_y$ of $x$ and $y$ such that $U_y \cap V_y = \emptyset$. Now the collection \( \{V_y : y \in A\} \) is a D-open cover of $A$. Since $A$ is compact, there exists a finite subcover \( \{V_{y_i}, i = 1, 2, \ldots, n\} \) such that $A \subseteq \cup \{V_{y_i}, i = 1, 2, \ldots, n\}$.
1, 2, \ldots, n}. Let \( U = \cap \{U_y, i = 1, 2, \ldots, n\} \) and \( V = \cup \{V_y, i = 1, 2, \ldots, n\} \). Then by assumption, \( U \) is a D-open set of \( X \). Clearly \( U \cap V = \emptyset \). Hence \( U \cap A = \emptyset \). Thus \( U \subseteq X - A \) and hence \( X - A \) is D-open. Therefore \( A \) is D-closed.

\[ \square \]

**Theorem 7.2.25.** Let \((X, \tau)\) be a topological space and let \((Y, \sigma)\) be a D-Hausdorff space. Assume that \( DO(X, \tau) \) (the class of all D-open sets of \((X, \tau)\)) be closed under finite intersections. If \( f, g \) are D-irresolute maps of \( X \) into \( Y \), then the set \( A = \{x \in X : f(x) = g(x)\} \) is a D-closed subset of \((X, \tau)\).

**Proof.** We shall show that \( X - A \) is a D-open subset of \((X, \tau)\). Now \( X - A = \{x \in X : f(x) \neq g(x)\} \). Let \( p \in X - A \). Set \( y_1 = f(p) \) and \( y_2 = g(p) \). By the definition of \( X - A \), it is \( y_1 \neq y_2 \). Thus \( y_1 \) and \( y_2 \) are two distinct points of \( Y \). Since \((Y, \sigma)\) is a D-Hausdorff space, there exist D-open sets \( U_1, U_2 \) of \((Y, \sigma)\) such that \( y_1 = f(p) \in U_1 \), \( y_2 = g(p) \in U_2 \) and \( U_1 \cap U_2 = \emptyset \). Therefore \( p \in f^{-1}(U_1) \), \( p \in g^{-1}(U_2) \) so that \( p \in f^{-1}(U_1) \cap g^{-1}(U_2) = W \) (say). Since \( f \) and \( g \) are D-irresolute maps, \( f^{-1}(U_1) \) and \( g^{-1}(U_2) \) are D-open sets of \((X, \tau)\) and by assumption \( W \) is a D-open set containing \( p \). We will now show that \( W \subseteq X - A \). Let \( y \in W \). Since \( U_1 \cap U_2 = \emptyset \), \( f(y) \neq g(y) \) and hence from the definition of \( X - A \), \( y \in X - A \). Therefore \( W \subseteq X - A \) and hence \( X - A \) is D-open set. Therefore \( A \) is a D-closed subset of \((X, \tau)\). \[ \square \]
7.3 $D^*$-homeomorphisms

In this section a new class of maps called $D^*$-homeomorphisms is introduced. This class of maps forms a group under the operation composition of maps and $D^*$-homeomorphism is an equivalence relation between topological spaces have been proved.

**Definition 7.3.1.** A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be $D^*$-homeomorphism if both $f$ and $f^{-1}$ are $D$-irresolute.

**Example 7.3.2.** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c; f(b) = a; f(c) = b$. Then $f$ is a $D^*$-homeomorphism.

**Theorem 7.3.3.** A bijective $D$-irresolute map of a $D$-compact space $X$ onto a $D$-Hausdorff space $Y$ is a $D^*$-homeomorphism.

**Proof.** Let $(X, \tau)$ be a $D$-compact space and $(Y, \sigma)$ be a $D$-Hausdorff space. Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective $D$-irresolute map. It is shown that $f$ is a $D^*$-homeomorphism. It is only to show that $f^{-1}$ is a $D$-irresolute map. Let $F$ be a $D$-closed set of $(X, \tau)$. Since $(X, \tau)$ is a $D$-compact space, by theorem 3.5.4, $F$ is a $D$-compact subset of $(X, \tau)$. Since $f$ is $D$-irresolute and by theorem 3.5.5, $f(F)$ is a $D$-compact subset of $(Y, \sigma)$. Since $(Y, \sigma)$ is a $D$-Hausdorff space and assume that $DO(X, \tau)$ is closed under finite intersections, then by theorem 7.2.24, $f(F)$ is a $D$-closed set in $(Y, \sigma)$. Hence $f$ is a $D^*$-homeomorphism. □

**Theorem 7.3.4.** If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are
D*-homeomorphisms then their composition \( g \circ f : (X, \tau) \to (Z, \eta) \) is also D*-homeomorphism.

**Proof.** Let \( F \) be a D-closed set of \((Z, \eta)\). Since \( g \) is a D*-homeomorphism, \( g^{-1}(F) \) is D-closed in \((Y, \sigma)\) and since \( f \) is a D*-homeomorphism, \( f^{-1}(g^{-1}(F)) \) is D-closed in \((X, \tau)\). Hence \( g \circ f \) is D-irresolute. Also for a D-closed set \( G \) in \((X, \tau)\), we have \((g \circ f)(G) = g(f(G))\). Since \( f \) is a D*-homeomorphism, \( f(G) \) is a D-closed set in \((Y, \sigma)\) and since \( g \) is a D*-homeomorphism, \( g(f(G)) \) is a D-closed set in \((Z, \eta)\). Hence \((g \circ f)^{-1}\) is D-irresolute. Hence \( g \circ f \) is a D*-homeomorphism. \( \square \)

Let \( \Gamma \) be a collection of all topological spaces. A relation is introduced, say “\( \equiv D^* \)” into the family \( \Gamma \) as follows: For two elements \((X, \tau)\) and \((Y, \sigma)\) of \( \Gamma \), \((X, \tau)\) is D*-homeomorphic to \((Y, \sigma)\) say \((X, \tau) \equiv D^*(Y, \sigma)\) if there exists a D*-homeomorphism \( f : (X, \tau) \to (Y, \sigma) \). Then the following theorem on the relation “\( \equiv D^* \)” is obtained.

**Theorem 7.3.5.** The relation \( \equiv D^* \) above is an equivalence relation in the collection \( \Gamma \) of all topological spaces.

**Proof.** (i) For any element \((X, \tau) \in \Gamma\), \((X, \tau) \equiv D^*(X, \tau)\) holds. Indeed, the identity function \( I_x : (X, \tau) \to (X, \tau) \) is a D*-homeomorphism.

(ii) Suppose \((X, \tau) \equiv D^*(Y, \sigma)\), where \((X, \tau)\) and \((Y, \sigma) \in \Gamma\). Then there exists a D*-homeomorphism \( f : (X, \tau) \to (Y, \sigma) \). By defini-
tion it is seen that \( f^{-1} : (Y, \sigma) \to (X, \tau) \) is a \(D^*\)-homeomorphism and hence \((Y, \sigma) \cong D^*(X, \tau)\).

(iii) Suppose that \((X, \tau) \cong D^*(Y, \sigma)\) and \((Y, \sigma) \cong D^*(Z, \eta)\), where \((X, \tau), (Y, \sigma)\) and \((Z, \eta) \in \Gamma\). By theorem 7.3.4, it is shown that \((X, \tau) \cong D^*(Z, \eta)\).

\[ \square \]

The family of all \(\cong D^*\)-homeomorphisms of a topological space \((X, \tau)\) onto itself by \(D^*-h(X, \tau)\) is denoted.

**Theorem 7.3.6.** The set \(D^*-h(X, \tau)\) is a group under the composition of maps.

**Proof.** Define a binary operation \(\gamma : D^*-h(X, \tau) \times D^*-h(X, \tau) \to D^*-h(X, \tau)\) by \(\gamma(f, g) = g \circ f\) (the composition of \(f\) and \(g\)) for all \(f, g \in D^*-h(X, \tau)\). Then by theorem 7.3.4, \(g \circ f \in D^*-h(X, \tau)\). It is known that the composition of maps is associative and the identity map \(I : (X, \tau) \to (X, \tau)\) belongs to \(D^*-h(X, \tau)\) serves as the identity element. If \(f \in D^*-h(X, \tau)\) then \(f^{-1} \in D^*-h(X, \tau)\) such that \(f \circ f^{-1} = f^{-1} \circ f = I\) and so inverse exists for each element of \(D^*-h(X, \tau)\). Therefore \((D^*-h(X, \tau), \circ)\) is a group under the of composition of maps.

\[ \square \]

**Theorem 7.3.7.** Let \(f : (X, \tau) \to (Y, \sigma)\) be a \(D^*\)-homeomorphism. Then \(f\) induces an isomorphism from the group \(D^*-h(X, \tau)\) onto the group \(D^*-h(Y, \sigma)\).
Proof. It is defined that a map \( k_f : D^* \cdot h(X, \tau) \rightarrow D^* \cdot h(Y, \sigma) \) by \( k_f(\theta) = f \circ \theta \circ f^{-1} \), for every \( \theta \in D^* \cdot h(X, \tau) \), where \( f \) is a given map. By theorem 7.3.4, \( k_f \) is well-defined in general, because \( f \circ \theta \circ f^{-1} \) is a \( D^* \)-homeomorphism for every \( D^* \)-homeomorphism \( \theta : (X, \tau) \rightarrow (Y, \sigma) \).

Then \( k_f \) is a bijection. Further, for all \( \theta_1, \theta_2 \in D^* \cdot h(X, \tau) \), \( K_f(\theta_1 \circ \theta_2) = f \circ (\theta_1 \circ \theta_2) \circ f^{-1} = (f \circ \theta_1 \circ f^{-1}) \circ (f \circ \theta_2 \circ f^{-1}) = k_f(\theta_1) \circ k_f(\theta_2) \).

Hence \( k_f \) is a homeomorphism and hence it is an isomorphism induced by \( f \). \( \square \)

7.4 D-regular and D-normal Spaces

Munshi [46] introduced \( g \)-regular and \( g \)-normal spaces using \( g \)-closed sets in topological spaces. In this section some new spaces namely D-regular and D-normal spaces are introduced and some of their characterizations are obtained.

Definition 7.4.1. A topological space \((X, \tau)\) is said to be D-regular if for every D-closed set \( F \) and each point \( x \notin F \) there exist disjoint preopen sets \( U \) and \( V \) such that \( F \subseteq U \) and \( x \in V \).

Example 7.4.2. Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{b\}, \{a, c\}, X\} \). Then the D-closed sets of \( X \) are \( \emptyset, \{b\}, \{a, c\} \) and \( X \). And the preopen sets of \( X \) are \( P(X) \). Hence \((X, \tau)\) is D-regular.

Theorem 7.4.3. If \((X, \tau)\) is a D-regular space and \( Y \) is open and D-closed subset of \((X, \tau)\) then the subspace \( Y \) is D-regular.
Proof. Let $F$ be any D-closed subset of $Y$ and $y \in F^c$. By theorem 3.3.21, $F$ is D-closed in $(X, \tau)$. Since $(X, \tau)$ is D-regular, there exist disjoint preopen subsets $U$ and $V$ of $(X, \tau)$ such that $y \in U$ and $F \subseteq V$. By Lemma 1.1.14, $U \cap Y$ and $V \cap Y$ are the disjoint preopen subsets of the subspace $Y$ such that $y \in U \cap Y$ and $F \subseteq V \cap Y$. Hence the subspace $Y$ is D-regular.  $lacksquare$

Theorem 7.4.4. Let $(X, \tau)$ be a topological space. Then the following statements are equivalent:

1. $(X, \tau)$ is D-regular.

2. For each point $x \in X$ and for each D-open neighborhood $W$ of $x$, there exists a preopen set $V$ of $x$ such that $pcl(V) \subseteq W$.

3. For each point $x \in X$ and for each D-closed set $F$ not containing $x$, there exists a preopen set $V$ of $x$ such that $pcl(V) \cap F = \emptyset$.

Proof. (1) $\Rightarrow$ (2) Let $W$ be any D-open neighborhood of $x$. Then there exists a D-open set $G$ such that $x \in G \subseteq W$. Since $G^c$ is D-closed and $x \notin G^c$, by hypothesis there exist preopen sets $U$ and $V$ such that $G^c \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ and so $V \subseteq U^c$. Now, $pcl(V) \subseteq pcl(U^c) = U^c$ and $G^c \subseteq U$ implies $U^c \subseteq G \subseteq W$. Hence $pcl(V) \subseteq W$.

(2) $\Rightarrow$ (1) Let $F$ be any D-closed set and $x \notin F$. Then $x \in F^c$ and $F^c$ is D-open and so $F^c$ is a D-open neighborhood of $x$ and by
hypothesis, there exists a preopen set $V$ of $x$ such that $x \in V$ and $pcl(V) \subseteq F^c$ which implies $F \subseteq (pcl(V))^c$. Then $(pcl(V))^c$ is a pre-open set containing $F$ and $V \cap (pcl(V))^c = \emptyset$. Therefore $X$ is D-regular.

(2) $\Rightarrow$ (3) Let $x \in X$ and $F$ be a D-closed set such that $x \notin F$. Then $F^c$ is a D-open neighborhood of $x$ and by hypothesis, there exists a preopen set $V$ of $x$ such that $pcl(V) \subseteq F^c$ and hence $pcl(V) \cap F = \emptyset$.

(3) $\Rightarrow$ (2) Let $x \in X$ and $W$ be a D-open neighborhood of $x$. Then there exists a D-open set $G$ such that $x \in G \subseteq W$. Since $G^c$ is D-closed and $x \notin G^c$, by hypothesis there exists a preopen set $V$ of $x$ such that $pcl(V) \cap G^c = \emptyset$. Therefore $pcl(V) \subseteq G \subseteq W$. $\square$

**Theorem 7.4.5.** Assume that $DC(X, \tau)$ (the set of all D-closed sets of $(X, \tau)$) be closed under any intersection. Then the following are equivalent:

1. $(X, \tau)$ is D-regular

2. $pcl_\theta(A) = Dcl(A)$ for every subset $A$ of $(X, \tau)$.

3. $pcl_\theta(A) = A$ for every D-closed set $A$.

**Proof.** (1)$\Rightarrow$(2) For any subset $A$ of $(X, \tau)$, it is $A \subseteq Dcl(A) \subseteq pcl_\theta(A)$. Let $x \in (Dcl(A))^c$. Then there exists a D-closed set $F$ such that $x \in F^c$ and $A \subseteq F$. By assumption, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$. Now,
\( x \in U \subseteq pcl(U) \subseteq F^c \subseteq A^c \) and therefore \( pcl(U) \cap A = \emptyset \). Thus \( x \in (pcl_\theta(A))^c \) and hence \( pcl_\theta(A) = Dcl(A) \).

(2)\( \Rightarrow \) (3) it is trivial follows by lemma 2.3.20 and by assumption.

(3)\( \Rightarrow \) (1) Let \( F \) be any D-closed set and \( x \in F^c \). Since \( F \) is D-closed, by assumption \( x \in (pcl(F))^c \) and so there exists a pre-open set \( U \) such that \( x \in U \) and \( pcl(U) \cap F = \emptyset \). Then \( F \subseteq (pcl(U))^c \). Let \( V = (pcl(U))^c \). Then \( V \) is a pre-open set such that \( F \subseteq V \). Also the sets \( U \) and \( V \) are disjoint and hence \((X, \tau)\) is D-regular.

\textbf{Theorem 7.4.6.} If \( f : (X, \tau) \to (Y, \sigma) \) is bijective \( \omega^* \)-open and pre-irresolute then \( f \) is D-irresolute.

\textbf{Proof.} Let \( F \) be any D-closed set of \((Y, \sigma)\). Let \( U \) be any \( \omega \)-open in \((X, \tau)\) such that \( f^{-1}(F) \subseteq U \). Then \( F \subseteq f(U) \). Since \( f \) is \( \omega^* \)-open and \( F \) is D-closed in \((Y, \sigma)\), \( pcl(F) \subseteq int(f(U)) \). Since \( f \) is bijective and pre-irresolute, \( f^{-1}(pcl(F)) \subseteq f^{-1}(int(f(U))) \subseteq int(U) \) and \( f^{-1}(pcl(F)) \) is a preclosed set in \((X, \tau)\). Now, \( pcl(f^{-1}(F)) \subseteq pcl(f^{-1}(pcl(F))) = f^{-1}(pcl(F)) \subseteq int(U) \). Therefore \( f^{-1}(F) \) is D-closed in \((X, \tau)\) and hence \( f \) is D-irresolute.

\textbf{Theorem 7.4.7.} If \((X, \tau)\) is D-regular and \( f : (X, \tau) \to (Y, \sigma) \) is bijective, \( \omega^* \)-open pre-irresolute and M-preopen then \((Y, \sigma)\) is D-regular.

\textbf{Proof.} Let \( F \) be any D-closed set of \((Y, \sigma)\) and \( y \notin F \). By theorem 7.4.6, the map \( f \) is D-irresolute and hence \( f^{-1}(F) \) is D-closed in \((X, \tau)\).
Let \( f(x) = y \). Since \( f \) is bijective, \( x \notin f^{-1}(F) \). By hypothesis there exist disjoint preopen sets \( U \) and \( V \) such that \( x \in U \) and \( f^{-1}(F) \subseteq V \). Since \( f \) is M-preopen and bijective we have \( y \in f(U) \), \( F \subseteq f(V) \) and \( f(U) \cap f(V) = \emptyset \). This shows that the space \((Y, \sigma)\) is D-regular. \( \square \)

**Theorem 7.4.8.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \omega \)-irresolute, M-preclosed, pre-irresolute injection and \((Y, \sigma)\) is D-regular, then \((X, \tau)\) is D-regular.

**Proof.** Let \( F \) be any D-closed set of \((X, \tau)\) and \( x \notin F \). Since \( f \) is \( \omega \)-irresolute and M-preclosed, by Proposition 3.3.16, \( f(F) \) is D-closed in \((Y, \sigma)\) and \( f(x) \notin f(F) \). Since \((Y, \sigma)\) is D-regular and so there exist disjoint preopen sets \( U \) and \( V \) in \((Y, \sigma)\) such that \( f(x) \in U \) and \( f(F) \subseteq V \). By hypothesis, \( f^{-1}(U) \) and \( f^{-1}(V) \) are preopen sets of \((X, \tau)\) such that \( x \in f^{-1}(U) \) and \( F \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Therefore \((X, \tau)\) is D-regular. \( \square \)

**Lemma 7.4.9.** If \( Y \) is a D-compact subspace of the \( D-T_2 \) D-Hausdorff space \( X \) and \( x_o \) is not in \( Y \) then there exist disjoint preopen sets \( U \) and \( V \) of \( X \) containing \( x_o \) and \( Y \) respectively. Assume that \( DO(X, \tau) \) (the set of all D-open sets of \((X, \tau)\)) be D-open under any union and be also D-open under finite intersection.

**Proof.** Since \( X \) is D-Hausdorff and \( x_o \notin Y \), for each \( x \in Y \) there exist disjoint D-open sets \( U_x \) and \( V_x \) such that \( x_o \in U_x \) and \( x \in V_x \). The collection \( \{V_x / x \in Y\} \) is a D-open cover of \( Y \). Since \( Y \) is a D-compact subspace of \( X \), then there exist finitely many points \( x_1, x_2, \ldots, x_n \) of \( Y \) such that \( Y \subseteq \cup \{V_{x_i} : i = 1, 2, \ldots, n\} \). Let \( U = \cap \{U_{x_i} : i = 1, 2, \ldots, n\} \). Then \( U \) is D-open.
1, 2, . . . , n} and \( V = \bigcup \{ Vx_i : i = 1, 2, . . . , n \} \). Then by assumption \( U \) and \( V \) are disjoint D-open sets of \( X \) such that \( x_o \in U \) and \( Y \subseteq V \). Since \( X \) is \( D-T_{\frac{1}{2}} \), then the sets \( U \) and \( V \) are disjoint preopen sets of \( X \) containing \( x_o \) and \( Y \) respectively.

\[ \square \]

**Theorem 7.4.10.** Every D-compact \( D-T_{\frac{1}{2}} \) D-Hausdorff space is D-regular.

**Proof.** Let \( X \) be a D-compact \( D-T_{\frac{1}{2}} \) D-Hausdorff space. Let \( x \) be a point of \( X \) and \( B \) be a D-closed set in \( X \) not containing \( x \). Then by theorem 3.5.4, \( B \) is D-compact. Thus \( B \) is a D-compact subspace of the \( D-T_{\frac{1}{2}} \) D-Hausdorff space \( X \). Let us assume that the set of all D-open sets be D-open under any union and be also D-open under finite intersection. Then by Lemma 7.4.9, there exist disjoint preopen sets about \( x \) and \( B \) respectively. Hence \( X \) is D-regular. \[ \square \]

**Theorem 7.4.11.** Let \((X, \tau)\) be a \( D-T_s \) Submaximal space. Then the following are equivalent:

1. \((X, \tau)\) is D-regular.

2. For every D-closed set \( F \) and each \( x \in X - F \) there exist disjoint D-open sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \).

**Proof.** \((1) \Rightarrow (2)\) Let \((X, \tau)\) be a submaximal space. Let \( F \) be a D-closed set and \( x \) be a point of \( X - F \). Then by hypothesis, there exist disjoint preopen sets \( U \) and \( V \) such that \( x \in U \) and \( F \subseteq V \). Since the space \((X, \tau)\) is submaximal, then \( U \) and \( V \) are open sets. Since every
open set is D-open, $U$ and $V$ are D-open sets. Therefore for every D-closed set $F$ and each $x \in X - F$, there exist disjoint D-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

(2) $\Rightarrow$ (1) Let $(X, \tau)$ be a $D-T_s$-space. Let the point $x$ and the D-closed set $F$ not containing $x$ be given. Then by hypothesis there exist disjoint D-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$. Since the space $(X, \tau)$ is $D-T_s$ and every open set is preopen, then there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$. $\square$

**Theorem 7.4.12.** The closure of a compact open subset of a compact Hausdorff $D-T_s$ space is $D$-regular.

**Proof.** Let $X$ be a compact Hausdorff $D-T_s$ space and $A$ be a compact open subset of $X$. Since every compact subset of a Hausdorff space is closed, $A$ is closed in $X$. By theorem 2.2.2, $A$ is D-closed in $X$. Let $F$ be a D-closed set of $A$ and the point $x \not\in F$. By theorem 3.3.21, $F$ is D-closed in $X$. Since $X$ is $D-T_s$, then $F$ is closed in $X$. Hence $F$ is compact, since every closed subset of a compact Hausdorff space is compact. By theorem 1.1.16(1), there exist disjoint open sets $U$ and $V$ of $X$ containing $F$ and $x$ respectively. Since every open set is preopen, then there exist disjoint preopen sets $U$ and $V$ of $X$ containing $F$ and $x$ respectively. Therefore $A$ is D-regular. $\square$

**Theorem 7.4.13.** Let $(X, \tau)$ be a submaximal space. Then the closure of a strongly compact subset of a $D$-regular space is compact.
Proof. Let \((X, \tau)\) be a submaximal D-regular space. Let \(A\) be a strongly compact subset of \(X\). Let \(\xi\) be an open covering of \(\bar{A}\). Then for each \(x \in A \subseteq \bar{A}\), there exists an open set \(W_x\) in \(\xi\) such that \(x \in W_x\). Since \(X\) is D-regular and by theorem 7.4.4 there exists a preopen set \(V_x\) of \(x\) such that \(pcl(V_x) \subseteq W_x\) for each \(x\). Now the family \(\{V_x : x \in A\}\) is a preopen covering of \(A\). Since \(A\) is strongly compact, then there exist finitely many points \(x_1, x_2, \ldots, x_n\) such that \(A \subseteq \bigcup_{i=1}^n V_{x_i}\). Since \(X\) is submaximal and \(pcl(V_{x_i}) = cl(V_{x_i})\) for every preopen set \(V_{x_i}, A \subseteq \bigcup_{i=1}^n V_{x_i}\) implies \(cl(A) \subseteq cl(\bigcup_{i=1}^n V_{x_i}) \subseteq \bigcup_{i=1}^n cl(V_{x_i}) = \bigcup_{i=1}^n pcl(V_{x_i}) \subseteq \bigcup_{i=1}^n W_{x_i}\). This shows that every open covering of \(\bar{A}\) is a finite subcovering. Therefore \(\bar{A}\) is compact. \(\square\)

Definition 7.4.14. A topological space \((X, \tau)\) is said to be D-normal if for each pair of disjoint D-closed sets \(A\) and \(B\), there exist disjoint preopen sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

Example 7.4.15. As in example 7.4.2, \((X, \tau)\) is D-normal.

Theorem 7.4.16. If \((X, \tau)\) is a D-normal space and \(Y\) is an open and D-closed subset of \((X, \tau)\) Then the subspace \(Y\) is D-normal.

Proof. Let \(A\) and \(B\) be any two disjoint D-closed sets of \(Y\). By theorem 3.3.21, \(A\) and \(B\) are D-closed in \((X, \tau)\). Since \((X, \tau)\) is D-normal, there exist disjoint preopen sets \(U\) and \(V\) of \((X, \tau)\) such that \(A \subseteq U\) and \(B \subseteq V\). By Lemma 1.1.14, \(U \cap Y\) and \(V \cap Y\) are disjoint preopen sets in \(Y\) and also \(A \subseteq U \cap Y\) and \(B \subseteq V \cap Y\). Hence the subspace \(Y\) is D-normal. \(\square\)
Theorem 7.4.17. Let \((X, \tau)\) be a topological space. Then the following statements are equivalent:

1. \((X, \tau)\) is D-normal.

2. For each D-closed set \(F\) and for each D-open set \(U\) containing \(F\) there exists a preopen set \(V\) containing \(F\) such that \(pcl(V) \subseteq U\).

3. For each pair of disjoint D-closed sets \(A\) and \(B\) in \((X, \tau)\), there exists a preopen set \(U\) containing \(A\) such that \(pcl(U) \cap B = \emptyset\).

Proof. (1)\(\Rightarrow\) (2) Let \(F\) be a D-closed set and \(U\) be a D-open set such that \(F \subseteq U\). Then \(F \cap U^c = \emptyset\). By assumption, there exist preopen sets \(V\) and \(W\) such that \(F \subseteq V\) and \(U^c \subseteq W\) and \(V \cap W = \emptyset\) and hence \(pcl(V) \cap W = \emptyset\). Now, \(pcl(V) \cap U^c \subseteq pcl(V) \cap W = \emptyset\) and so \(pcl(V) \subseteq U\).

(2)\(\Rightarrow\) (3) Let \(A\) and \(B\) be disjoint D-closed sets of \((X, \tau)\). Since \(A \cap B = \emptyset\), \(A \subseteq B^c\) and \(B^c\) is D-open. By assumption, there exists a preopen set \(U\) containing \(A\) such that \(pcl(U) \subseteq B^c\) and so \(pcl(U) \cap B = \emptyset\).

(3) \(\Rightarrow\) (1) Let \(A\) and \(B\) be any two disjoint D-closed sets of \((X, \tau)\). Then by assumption, there exists a preopen set \(U\) containing \(A\) such that \(pcl(U) \cap B = \emptyset\). Again, by assumption there exists a preopen set \(V\) containing \(B\) such that \(pcl(V) \cap A = \emptyset\). Also, \(pcl(U) \cap pcl(V) = \emptyset\) and hence \(U \cap V = \emptyset\). Therefore \((X, \tau)\) is D-normal.

\(\square\)

Theorem 7.4.18. Let \((X, \tau)\) be a submaximal space. Then the following are equivalent:
1. \((X, \tau)\) is D-normal.

2. For each pair of disjoint D-closed sets \(A\) and \(B\) in \((X, \tau)\), there exist preopen sets \(U\) containing \(A\) and \(V\) containing \(B\) such that \(\text{pcl}(U) \cap \text{pcl}(V) = \emptyset\).

**Proof.** (1)⇒(2) Let \(A\) and \(B\) be any two disjoint D-closed sets of \((X, \tau)\). Then by theorem 7.4.17, there exists a preopen set \(U\) containing \(A\) such that \(\text{pcl}(U) \cap B = \emptyset\). Since \(\text{pcl}(U)\) is preclosed and since \((X, \tau)\) is submaximal then \(\text{pcl}(U) = cl(U)\) and \(\text{pcl}(U)\) is closed. By theorem 2.2.2, \(\text{pcl}(U)\) is D-closed and so \(B\) and \(\text{pcl}(U)\) are disjoint D-closed sets in \((X, \tau)\). Therefore, again by theorem 7.4.17, there exists a preopen set \(V\) containing \(B\) such that \(\text{pcl}(U) \cap \text{pcl}(V) = \emptyset\).

(2)⇒(1) \(\text{pcl}(U) \cap \text{pcl}(V) = \emptyset\) implies \(U \cap V = \emptyset\) and by assumption \((X, \tau)\) is D-normal. \qed

**Theorem 7.4.19.** If \((X, \tau)\) is a D-normal space and \(f : (X, \tau) \rightarrow (Y, \sigma)\) is bijective, \(\omega^{*}\)-open, pre-irresolute and M-preopen then \((Y, \sigma)\) is D-normal.

**Proof.** Let \(A\) and \(B\) be disjoint D-closed sets of \((Y, \sigma)\). By theorem 7.4.6, the map \(f\) is D-irresolute and hence \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint D-closed sets of \((X, \tau)\). Since \((X, \tau)\) is D-normal, there exist disjoint preopen sets \(U\) and \(V\) such that \(f^{-1}(A) \subseteq U\) and \(f^{-1}(B) \subseteq V\). Since \(f\) is M-preopen and bijective, \(A \subseteq f(U), B \subseteq f(V)\) and \(f(U) \cap f(V) = \emptyset\). Hence \((Y, \sigma)\) is D-normal. \qed
Theorem 7.4.20. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \omega \)-irresolute, \( M \)-preclosed pre-irresolute injection and \((Y, \sigma)\) is D-normal then \((X, \tau)\) is D-normal.

**Proof.** Let \( A \) and \( B \) be any two disjoint D-closed sets of \((X, \tau)\). Since \( f \) is \( \omega \)-irresolute, open, preclosed, by theorem 7.4.8, \( f(A) \) and \( f(B) \) are disjoint D-closed sets of \((Y, \sigma)\). Since \((Y, \sigma)\) is D-normal, there exist disjoint preopen sets \( U \) and \( V \) in \((Y, \sigma)\) such that \( f(A) \subseteq U \) and \( f(B) \subseteq V \). By hypothesis, \( f^{-1}(U) \) and \( f^{-1}(V) \) are preopen sets of \((X, \tau)\) such that \( A \subseteq f^{-1}(U) \) and \( B \subseteq f^{-1}(V) \) and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Therefore \((X, \tau)\) is D-normal. \( \square \)

Theorem 7.4.21. Let \((X, \tau)\) be a D-Ts submaximal space. Then the following are equivalent:

1. \((X, \tau)\) is D-normal

2. For every disjoint D-closed sets \( A \) and \( B \), there exist disjoint D-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

**Proof.** (1) \( \Rightarrow \) (2) Let \((X, \tau)\) be a submaximal space. Let \( A \) and \( B \) be disjoint D-closed sets of \((X, \tau)\). Then by hypothesis, there exist disjoint preopen sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \). Since the space \((X, \tau)\) is submaximal, then \( U \) and \( V \) are open sets. By theorem 2.2.2, for every disjoint D-closed sets \( A \) and \( B \), there exist disjoint D-open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \).

(2) \( \Rightarrow \) (1) Let \((X, \tau)\) be a D-Ts space. Let \( A \) and \( B \) be disjoint D-closed sets of \((X, \tau)\). Then by hypothesis, there exist disjoint D-open
sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. Since $U$ and $V$ are D-open sets of $X$ and $X$ is a $D-T_s$ space, then $U$ and $V$ are disjoint open sets of $X$. Since every open set is preopen, therefore for any pair of disjoint D-closed sets $A$ and $B$, there exist disjoint preopen sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

\[\Box\]

**Theorem 7.4.22.** The closure of a compact open subset of a compact Hausdorff $D-T_s$ space is $D$-normal.

**Proof.** Let $X$ be a compact Hausdorff $D-T_s$ space and $A$ be a compact open subset of $X$. Since every compact subset of a Hausdorff space is closed, $A$ is closed in $X$. By theorem 2.2.2, $A$ is D-closed in $X$. Now to prove $A = \bar{A}$ is D-normal. Suppose for any pair of disjoint D-closed sets $F_1$ and $F_2$ of $A$ are given. By theorem 3.3.21, $F_1$ and $F_2$ are D-closed sets in $X$. Since $X$ is $D-T_s$, then $F_1$ and $F_2$ are closed sets in $X$. Since every closed subset of a compact Hausdorff space is compact, $F_1$ and $F_2$ are compact sets in $X$. By theorem 1.1.16(2), there exist disjoint open sets $U$ and $V$ of $X$ containing $F_1$ and $F_2$ respectively. Since every open set is preopen, then there exist disjoint preopen sets $U$ and $V$ of $X$ containing $F_1$ and $F_2$ respectively. Hence $A$ is D-normal. \[\Box\]

**Theorem 7.4.23.** Let $(Y, \sigma)$ be a submaximal extremely disconnected space. If $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous D-closed injection and $(Y, \sigma)$ is $D$-normal, then $(X, \tau)$ is normal.

**Proof.** Let $A$ and $B$ be any two disjoint closed sets of $(X, \tau)$. Since $f$ is injective and D-closed, $f(A)$ and $f(B)$ are disjoint D-closed sets
of \((Y, \sigma)\). Since \((Y, \sigma)\) is D-normal, by theorem 7.4.18, there exist preopen sets \(U\) and \(V\) such that \(f(A) \subseteq U\) and \(f(B) \subseteq V\) and \(pcl(U) \cap pcl(V) = \emptyset\). Since the space \((Y, \sigma)\) is submaximal, then there exist open sets \(U\) and \(V\) such that \(f(A) \subseteq U\) and \(f(B) \subseteq V\) and \(cl(U) \cap cl(V) = \emptyset\). Since \(f\) is weakly continuous, it follows that \([77]\), theorem 1, \(A \subseteq f^{-1}(U) \subseteq int(f^{-1}(cl(U))), B \subseteq f^{-1}(V) \subseteq int(f^{-1}cl(V))\) and \(int(f^{-1}(cl(U))) \cap int(f^{-1}(cl(V))) = \emptyset\). Therefore \((X, \tau)\) is normal.

\[\textbf{Theorem 7.4.24.} \text{Let} (X, \tau) \text{be a D-Ts space and let} (Y, \sigma) \text{be sub-
maximal extremely disconnected space. If} f : (X, \tau) \to (Y, \sigma) \text{is pre-
irresolute D-closed injection and} (Y, \sigma) \text{is D-normal, then} (X, \tau) \text{is D-
normal.} \]

\[\textbf{Proof.} \text{Let} A \text{and} B \text{be any two disjoint D-closed sets of} (X, \tau). \text{Since} \(X, \tau) \text{is a D-Ts space and since} f \text{is injective and D-closed,} f(A) \text{and} f(B) \text{are disjoint D-closed sets of} (Y, \sigma). \text{Since} (Y, \sigma) \text{is D-normal, by theorem 7.4.18, there exist preopen sets} U \text{and} V \text{such that} f(A) \subseteq U \text{and} f(B) \subseteq V \text{and} pcl(U) \cap pcl(V) = \emptyset\). \text{Since} f \text{is pre-irresolute,} A \subseteq f^{-1}(U) \subseteq pint(f^{-1}(pcl(U))), B \subseteq f^{-1}(V) \subseteq pint(f^{-1}(pcl(V))) \text{and} pint(f^{-1}(pcl(U))) \cap pint(f^{-1}(pcl(V))) = \emptyset\). \text{Therefore} (X, \tau) \text{is D-normal.} \]

\[\square\]