Chapter 6

Contra D-continuous maps

6.1 Introduction

Covering spaces with closed sets have a historical background in general topology. In 1996, Dontchev [19] considered spaces where every cover by closed sets has a finite subcover. Such spaces are called strongly S-closed. It is natural to ask which class of generalized continuity transforms strongly S-closed spaces onto compact spaces. Such functions are called contra continuous functions.

The notion of contra semi-continuous functions has been introduced by Dontchev and Noiri [20], Jafari and Noiri [27, 28] introduced contra $\alpha$-continuous functions and contra pre-continuous functions in topological spaces.

Here contra D-continuous maps, strongly D-closed spaces and almost contra D-continuous maps are introduced. By an example it is shown that the composition of two contra D-continuous maps need
not be contra D-continuous. Some basic properties of these maps are investigated. For example, conditions under which the almost contra D-continuous image of a space is nearly compact, nearly Lindelof and nearly countably compact are investigated.

### 6.2 Contra D-continuous functions

In this section, a weaker form of contra continuous maps called contra D-continuous maps are considered and studied in detail.

**Definition 6.2.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is called contra D-continuous if $f^{-1}(F)$ is D-open (resp. D-closed) in $(X, \tau)$ for every closed (resp. open) set $F$ in $(Y, \sigma)$.

**Example 6.2.2.** Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is contra D-continuous function. It is observed for the closed (resp. open) set $F = \{a\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{a\}$ is D-open (resp. D-closed) in $(X, \tau)$.

**Theorem 6.2.3.** Every contra continuous function is a contra D-continuous function.

**Proof.** Let $f : (X, \tau) \to (Y, \sigma)$ be a contra continuous function. Let $V$ be an open set of $(Y, \sigma)$. Since $f$ is contra continuous, $f^{-1}(V)$ is closed in $(X, \tau)$. Hence by theorem 2.2.2, $f^{-1}(V)$ is D-closed in $(X, \tau)$. Thus $f$ is a contra D-continuous function. \qed
**Remark 6.2.4.** Converse of this theorem need not be true as seen from the following example:

**Example 6.2.5.** Let $X = \{a, b, c\} = Y, \tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b, f(b) = c$ and $f(c) = a$. Then $f$ is contra D-continuous but not contra continuous. It is observed for the open (resp. closed) set $U = \{b, c\}$ in $(Y, \sigma)$, $f^{-1}(U) = \{a, b\}$ is D-closed (resp. D-open) in $(X, \tau)$ but it is not closed.

**Remark 6.2.6.** contra D-continuous and contra g-continuous are independent. It is shown by the following examples:

**Example 6.2.7.** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is contra D-continuous but not contra g-continuous. It is observed for the closed set $F = \{b, c\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{b, c\}$ is D-open but not g-open in $(X, \tau)$.

**Example 6.2.8.** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b, f(b) = a; f(c) = b$. Then $f$ is contra g-continuous but not contra D-continuous. It is observed for the closed set $F = \{b\}$ in $(Y, \sigma)$, $f^{-1}(F) = \{a, c\}$ is g-open but not D-open in $(X, \tau)$.

**Remark 6.2.9.** contra D-continuous and contra $\alpha$-continuous are independent. It is shown by the following examples:

**Example 6.2.10.** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = \ldots$
c; \( f(b) = a; \) \( f(c) = b. \) Then \( f \) is contra D-continuous but not contra \( \alpha \)-continuous. It is observed for the closed set \( F = \{c\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{a\} \) is D-open but not \( \alpha \)-open in \( (X, \tau) \).

**Example 6.2.11.** Let \( X = Y = \{a, b, c\}, \) \( \tau = \{\emptyset, \{a\}, \{a, c\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{a, b\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b; \) \( f(b) = c; \) \( f(c) = a. \) Then \( f \) is contra \( \alpha \)-continuous but not contra D-continuous. It is observed for the closed set \( F = \{b, c\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{a, b\} \) is \( \alpha \)-open but not D-open in \( (X, \tau) \).

**Remark 6.2.12.** contra D-continuous and contra pre-semicontinuous are independent. It is shown by the following examples:

**Example 6.2.13.** Let \( X = Y = \{a, b, c\}, \) \( \tau = \{\emptyset, \{a\}, \{a, c\}, X\} \) and \( \sigma = \{\emptyset, \{b, c\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = f(c) = a \) and \( f(b) = b. \) Then \( f \) is contra D-continuous but not contra pre-semicontinuous. It is observed for the closed set \( F = \{a\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{a, c\} \) is D-open but not pre-semi-open in \( (X, \tau) \).

**Example 6.2.14.** Let \( X = Y = \{a, b, c\}, \) \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = a; \) \( f(b) = c; \) \( f(c) = b. \) Then \( f \) is contra pre-semicontinuous but not contra D-continuous. It is observed for the closed set \( F = \{b\} \) in \( (Y, \sigma) \), \( f^{-1}(F) = \{c\} \) is pre-semiopen but not D-open in \( (X, \tau) \).

**Remark 6.2.15.** contra D-continuous and contra semi-continuous are independent. It is shown by the following examples:

**Example 6.2.16.** Let \( X = Y = \{a, b, c\}, \) \( \tau = \{\emptyset, \{c\}, \{a, c\}, X\} \) and \( \sigma = \{\emptyset, \{a, c\}, Y\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = b; \) \( f(b) =
Then \( f \) is contra D-continuous but not contra semi-continuous. It is observed for the closed set \( F = \{ b \} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{ a \} \) is D-open but not semi-open in \((X, \tau)\).

**Example 6.2.17.** Let \( X = Y = \{ a, b, c \} \), \( \tau = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \) and \( \sigma = \{ \emptyset, \{ b \}, \{ a, b \}, Y \} \). Then the identity function \( f \) is contra semi-continuous but not contra \( D \)-continuous. It is observed for the closed set \( F = \{ a, c \} \) in \((Y, \sigma)\), \( f^{-1}(F) = \{ a, c \} \) is semi-open but not \( D \)-open in \((X, \tau)\).

**Remark 6.2.18.** The composition of two contra \( D \)-continuous functions need not be contra \( D \)-continuous and this is shown by the following example:

**Example 6.2.19.** Let \( X = \{ a, b, c \} = Y = Z \), \( \tau = \{ \emptyset, \{ a \}, X \} \), \( \sigma = \{ \emptyset, \{ b, c \}, Y \} \) and \( \eta = \{ \emptyset, \{ a, c \}, Z \} \). Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = a; f(b) = b \) and \( f(c) = b \). Then \( f \) is contra D-continuous, since for the closed set \( V = \{ a \} \) in \((Y, \sigma)\), \( f^{-1}(V) = \{ a \} \) is \( D \)-open in \((X, \tau)\). Define \( g : (Y, \sigma) \rightarrow (Z, \eta) \) by \( g(x) = x \). Then \( g \) is contra D-continuous. It is observed for the closed set \( V = \{ b \} \) in \((Z, \eta)\), \( g^{-1}(V) = \{ b \} \) is \( D \)-open in \((Y, \sigma)\). But their composition is not contra D-continuous. It is observed for the closed set \( V = \{ b \} \) in \((Z, \eta)\), \( f^{-1}(g^{-1}(V)) = f^{-1}(\{ b \}) = \{ b, c \} \) is not \( D \)-open in \((X, \tau)\).

**Theorem 6.2.20.** The following are equivalent for a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) Assume that \( DO(X) \) (resp. \( DC(X) \)) is closed under any union (resp. intersection)

1. \( f \) is contra D-continuous

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2. The inverse image of a closed set $F$ of $Y$ is $D$-open in $X$.

3. For each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in DO(X, x)$ such that $f(U) \subseteq F$.

4. $f(D\text{-}cl(A)) \subseteq \text{Ker}(f(A))$ for every subset $A$ of $X$.

5. $D\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(\text{Ker}(B))$ for every subset $B$ of $Y$.

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2) Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in DO(X, x)$ such that $f(U_x) \subset F$. Hence $f^{-1}(F) = \bigcup\{U_x : x \in f^{-1}(F)\}$ is obtained and by assumption $f^{-1}(F)$ is $D$-open.

(2) $\Rightarrow$ (4) Let $A$ be any subset of $X$. Suppose that $y \notin \text{Ker}(f(A))$. Then by Lemma 1.1.8, there exists $F \in C(X, x)$ such that $f(A) \cap F = \emptyset$. Thus $A \cap f^{-1}(F) = \emptyset$ and $D\text{-}cl(A) \cap f^{-1}(F) = \emptyset$. Hence $f(D\text{-}cl(A)) \cap F = \emptyset$ and $y \notin f(D\text{-}cl(A))$ are obtained. Thus $f(D\text{-}cl(A)) \subseteq \text{Ker}(f(A))$.

(4) $\Rightarrow$ (5) Let $B$ be any subset of $Y$. By (4) and Lemma 1.1.8, $f(D\text{-}cl(f^{-1}(B))) \subset \text{Ker}(f(f^{-1}(B))) \subset \text{Ker}(B)$ and $D\text{-}cl(f^{-1}(B)) \subset f^{-1}(\text{Ker}(B))$.

(5) $\Rightarrow$ (1) Let $U$ be any open set of $Y$. Then by lemma 1.1.8, $D\text{-}cl(f^{-1}(U)) \subset f^{-1}(\text{Ker}(U)) = f^{-1}(U)$ and $D\text{-}cl(f^{-1}(U)) = f^{-1}(U)$. By assumption, $f^{-1}(U)$ is $D$-closed in $X$. Hence $f$ is contra $D$-continuous. \qed
Theorem 6.2.21. If $f : (X, \tau) \to Y, \sigma$ is $D$-irresolute (resp. contra $D$-continuous) and $g : (Y, \sigma) \to (Z, \eta)$ in contra $D$-continuous (resp. continuous) then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra $D$-continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is contra $D$-continuous (resp. continuous) then $g^{-1}(V)$ is $D$-closed (resp. open) in $(Y, \sigma)$ and since $f$ is $D$-irresolute (resp. contra $D$-continuous) then $f^{-1}(g^{-1}(V))$ is $D$-closed in $(X, \tau)$. Hence $g \circ f$ is contra $D$-continuous. \qed

Theorem 6.2.22. If $f : (X, \tau) \to (Y, \sigma)$ is contra continuous and $g : (Y, \tau) \to (Z, \eta)$ is continuous then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra $D$-continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is continuous, $g^{-1}(U)$ is open in $(Y, \sigma)$. Since $f$ is contra continuous, $f^{-1}(g^{-1}(U))$ is closed in $(X, \tau)$. Hence by theorem 2.2.2, $(g \circ f)^{-1}(U)$ is $D$-closed in $(X, \tau)$. Hence $g \circ f$ is contra $D$-continuous. \qed

Theorem 6.2.23. If $f : (X, \tau) \to (Y, \sigma)$ is contra continuous and super-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is contra continuous then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra $D$-continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is contra continuous, $g^{-1}(U)$ is closed in $(Y, \sigma)$ and since $f$ is contra continuous and super-continuous then $f^{-1}(g^{-1}(U))$ is both open and regular closed in $(X, \tau)$. Hence by theorem 2.3.17, $(g \circ f)^{-1}(U)$ is $D$-closed in $(X, \tau)$. Hence $g \circ f$ is contra $D$-continuous. \qed
Theorem 6.2.24. Let \((X, \tau), (Y, \sigma)\) be any two topological spaces and \((Y, \sigma)\) be \(T_{1/2}\)-space (resp. \(T_\omega\)-space). Then the composition \(g \circ f : (X, \tau) \to (Z, \eta)\) of contra D-continuous function \(f : (X, \tau) \to (Y, \sigma)\) and the \(g\)-continuous (resp. \(\omega\)-continuous) function \(g : (Y, \sigma) \to (Z, \eta)\) is contra D-continuous.

Proof. Let \(F\) be any closed set of \((Z, \eta)\). Since \(g\) is \(g\)-continuous (resp. \(\omega\)-continuous), \(g^{-1}(F)\) is \(g\)-closed (resp. \(\omega\)-closed) in \((Y, \sigma)\) and \((Y, \sigma)\) is \(T_{1/2}\)-space (resp. \(T_\omega\)-space), hence \(g^{-1}(F)\) is closed in \((Y, \sigma)\). Since \(f\) is contra D-continuous, \(f^{-1}(g^{-1}(F))\) is D-open in \((X, \tau)\). Hence \(g \circ f\) is contra D-continuous. \(\square\)

Theorem 6.2.25. If \(f : (X, \tau) \to (Y, \sigma)\) is a surjective \(D^*\)-open function and \(g : (Y, \sigma) \to (Z, \eta)\) is a function such that \(g \circ f : (X, \tau) \to (Z, \eta)\) is contra D-continuous then \(g\) is contra D-continuous.

Proof. Let \(F\) be any closed set of \((Z, \eta)\). Since \(g \circ f\) is contra D-continuous then \((g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))\) is D-open in \((X, \tau)\) and since \(f\) is surjective and \(D^*\)-open, then \(f(f^{-1}(g^{-1}(F))) = g^{-1}(F)\) is D-open in \((Y, \sigma)\). Hence \(g\) is contra D-continuous. \(\square\)

Theorem 6.2.26. Let \(\{X_i/i \in I\}\) be any family of topological spaces. If \(f : X \to \prod X_i\) is a contra D-continuous function. Then \(\pi_i \circ f : X \to X_i\) is contra D-continuous function for each \(i \in I\), where \(\pi_i\) is the projection of \(\prod X_i\) onto \(X_i\).

Proof. It follows from theorem 6.2.21 and the fact that the projection is continuous. \(\square\)
Theorem 6.2.27. If $f : (X, \tau) \to (Y, \sigma)$ is strongly $D$-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is contra $D$-continuous then $g \circ f : (X, \tau) \to (Z, \eta)$ is contra continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is contra $D$-continuous, then $g^{-1}(U)$ is $D$-closed in $(Y, \sigma)$. Since $f$ is strongly $D$-continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is closed in $(X, \tau)$. Hence $g \circ f$ is contra continuous. $\square$

Theorem 6.2.28. If $f : (X, \tau) \to (Y, \sigma)$ is pre-$D$-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is contra pre-continuous then $g \circ f : (X, \tau) \to (Z, \eta)$ is contra $D$-continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is contra pre-continuous, then $g^{-1}(U)$ is pre-closed in $(Y, \sigma)$ and since $f$ is pre-$D$-continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $D$-closed in $(X, \tau)$. Hence $g \circ f$ is contra $D$-continuous. $\square$

Theorem 6.2.29. If $f : (X, \tau) \to (Y, \sigma)$ is strongly $D$-continuous and $g : (Y, \sigma) \to (Z, \eta)$ is contra $D$-continuous then $g \circ f : (X, \tau) \to (Z, \eta)$ is contra $D$-continuous.

Proof. Let $U$ be any open set of $(Z, \eta)$. Since $g$ is contra $D$-continuous, then $g^{-1}(U)$ is $D$-closed in $(Y, \sigma)$ and since $f$ is strongly $D$-continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is closed in $(X, \tau)$. By theorem 2.2.2, $(g \circ f)^{-1}(U)$ is $D$-closed in $(X, \tau)$. Hence $g \circ f$ is contra $D$-continuous. $\square$
Theorem 6.2.30. Let \( f : (X, \tau) \to (Y, \sigma) \) be surjective \( D \)-irresolute and \( D^* \)-open and \( g : (Y, \sigma) \to (Z, \eta) \) be any function. Then \( g \circ f : (X, \tau) \to (Z, \eta) \) is contra \( D \)-continuous if and only if \( g \) is contra \( D \)-continuous.

Proof. The ‘if’ part is easy to prove. To prove the ‘only if’ part, let \( F \) be any closed set of \((Z, \eta)\). Since \( g \circ f \) is contra \( D \)-continuous, then \((g \circ f)^{-1}(F)\) is \( D \)-open in \((X, \tau)\) and since \( f \) is \( D^* \)-open surjection, then \( f((g \circ f)^{-1}(F)) = g^{-1}(F) \) is \( D \)-open in \((Y, \sigma)\). Hence \( g \) is contra \( D \)-continuous. \( \square \)

Theorem 6.2.31. Let \( f : (X, \tau) \to (Y, \sigma) \) be a contra \( D \)-continuous function and \( H \) an open \( D \)-closed subset of \((X, \tau)\). Assume that \( DC(X, \tau) \) (the class of all \( D \)-closed sets of \((X, \tau)\)) is \( D \)-closed under finite intersections. Then the restriction \( f|H : (H, \tau_H) \to (Y, \sigma) \) is contra \( D \)-continuous.

Proof. Let \( U \) be any open set of \((Y, \sigma)\). By hypothesis and assumption, \( f^{-1}(U) \cap H = H_1 \)(say) is \( D \)-closed in \((X, \tau)\). Since \( f|H^{-1}(U) = H_1 \), it is sufficient to show that \( H_1 \) is \( D \)-closed in \( H \). By theorem 3.3.20, \( H_1 \) is \( D \)-closed in \( H \). Thus \( f|H \) is contra \( D \)-continuous. \( \square \)

Theorem 6.2.32. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function and \( g : X \to X \times Y \) the graph function given by \( g(x) = (x, f(x)) \) for every \( x \in X \). Then \( f \) is contra \( D \)-continuous if \( g \) is contra \( D \)-continuous.

Proof. Let \( F \) be a closed subset of \( Y \). Then \( X \times F \) is a closed subset of \( X \times Y \). Since \( g \) is contra \( D \)-continuous, then \( g^{-1}(X \times F) \) is a \( D \)-
open subset of $X$. Also $g^{-1}(X \times F) = f^{-1}(F)$. Hence $f$ is contra D-continuous.

**Theorem 6.2.33.** If a function $f : (X, \tau) \to (Y, \sigma)$ is contra D-continuous and $Y$ is regular, then $f$ is D-continuous.

**Proof.** Let $x$ be an arbitrary point of $X$ and $N$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $U$ in $Y$ containing $f(x)$ such that $cl(U) \subseteq N$. Since $f$ is contra D-continuous, by theorem 6.2.20, there exists $W \in DO(X, x)$ such that $f(W) \subseteq cl(U)$. Then $f(W) \subseteq N$. Hence by theorem 3.3.15, $f$ is D-continuous. \hfill \Box

**Theorem 6.2.34.** Every continuous and RC-continuous function is contra D-continuous.

**Proof.** Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Let $U$ be an open set of $(Y, \sigma)$. Since $f$ is continuous and RC-continuous, $f^{-1}(U)$ is open and regular closed in $(X, \tau)$. Hence by theorem 2.3.17, $f$ is contra D-continuous. \hfill \Box

**Theorem 6.2.35.** Every continuous and contra D-continuous (resp. contra continuous and D-continuous) function is a super-continuous (resp. RC-continuous) function.

**Proof.** Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Let $U$ be an open (resp. closed) set of $(Y, \sigma)$. Since $f$ is continuous and contra D-continuous (resp. contra continuous and D-continuous), $f^{-1}(U)$ is open.
and D-closed in \((X, \tau)\). Hence by theorem 2.3.24, \(f^{-1}(U)\) is regular open in \((X, \tau)\). This shows that \(f\) is a super-continuous (resp. RC-continuous) function.

\[\square\]

**Theorem 6.2.36.** Let \(f : (X, \tau) \to (Y, \sigma)\) be a function and \(X\) a \(D-T_s\) space. Then the following are equivalent.

1. \(f\) is contra \(D\)-continuous.

2. \(f\) is contra continuous

**Proof.** (1) \(\Rightarrow\) (2). Let \(U\) be an open set of \((Y, \sigma)\). Since \(f\) is contra \(D\)-continuous, \(f^{-1}(U)\) is D-closed in \((X, \tau)\) and since \(X\) is \(D-T_s\) space, \(f^{-1}(U)\) is closed in \((X, \tau)\). Hence \(f\) is contra continuous.

(2) \(\Rightarrow\) (1) Let \(U\) be an open set of \((Y, \sigma)\). Since \(f\) is contra continuous, \(f^{-1}(U)\) is closed in \((X, \tau)\). Hence by theorem 2.2.2, \(f^{-1}(U)\) is D-closed in \((X, \tau)\). Hence \(f\) is contra \(D\)-continuous.

\[\square\]

### 6.3 Contra \(D\)-closed and Strongly \(D\)-closed

In this section, contra \(D\)-closed graph and strongly \(D\)-closed spaces are introduced. Several properties are proved.

**Definition 6.3.1.** A graph \(G(f)\) of a function \(f : (X, \tau) \to (Y, \sigma)\) is said to be contra \(D\)-closed in \(X \times Y\) if for each \((x, y) \in (X \times Y) - G(f)\) there exist \(U \in DO(X, x)\) and \(V \in C(Y, y)\) such that \((U \times V) \cap G(f) = \emptyset\).
Lemma 6.3.2. A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is contra D-closed if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in DO(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Theorem 6.3.3. If $f : (X, \tau) \to (Y, \sigma)$ is contra D-continuous and $Y$ is Urysohn then $G(f)$ is contra D-closed in $X \times Y$.

**Proof.** Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$ and there exist open sets $V, W$ such that $f(x) \in V, y \in W$ and $cl(V) \cap cl(W) = \emptyset$. Since $f$ is contra D-continuous and by theorem 6.2.20, there exists $U \in DO(X, x)$ such that $f(U) \subseteq cl(V)$. Hence $f(U) \cap cl(W) = \emptyset$. Thus by lemma 6.3.2, $G(f)$ is contra D-closed in $X \times Y$. \hfill $\square$

Definition 6.3.4. A topological space $(X, \tau)$ is said to be strongly D-closed if every D-closed cover of $X$ has a finite subcover.

Example 6.3.5. A $D-T_s$ strongly $S$-closed space is strongly D-closed.

Theorem 6.3.6. Let $(X, \tau)$ be $D-T_s$ space. If $f : (X, \tau) \to (Y, \sigma)$ has a contra D-closed graph, then the inverse image of a strongly $S$-closed set $K$ of $Y$ is closed in $(X, \tau)$.

**Proof.** Let $K$ be a strongly $S$-closed set of $Y$ and $x \in f^{-1}(K)$. For each $k \in K, (x, k) \notin G(f)$. By Lemma 6.3.2, there exist $U_k \in DO(X, x)$ and $V_k \in C(Y, k)$ such that $f(U_k) \cap V_k = \emptyset$. Since $\{K \cap V_k/k \in K\}$ is a closed cover of the subspace $K$, there exists a finite subset $K_0 \subset K$ such that $K \subset \bigcup\{V_k/k \in K_0\}$.

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Then $U$ is open, since $X$ is a $D$-$T_s$ space. Therefore $f(U) \cap K = \emptyset$ and $U \cap f^{-1}(K) = \emptyset$. This shows that $f^{-1}(K)$ is closed in $(X, \tau)$.

**Theorem 6.3.7.** If a space $(X, \tau)$ is strongly $D$-closed then the space is strongly $S$-closed.

**Proof.** This proof follows from the definitions of 1.3.4(7) and 6.3.4 and by theorem 2.2.2.

**Theorem 6.3.8.** Let $(X, \tau)$ be $D$-connected and $(Y, \sigma)$ be a $T_1$-space. If $f : (X, \tau) \to (Y, \sigma)$ is contra $D$-continuous then $f$ is constant.

**Proof.** Since $(Y, \sigma)$ is a $T_1$ space, $\land = \{f^{-1}(y) / y \in Y\}$ is a disjoint $D$-open partition of $X$. If $|\land| \geq 2$, then $X$ is the union of two non-empty $D$-open sets. Since $(X, \tau)$ is $D$-connected, $|\land| = 1$. Hence $f$ is constant.

**Theorem 6.3.9.** Let $f : (X, \tau) \to (Y, \sigma)$ be a contra $D$-continuous and pre-closed surjection. If $(X, \tau)$ is a $D$-$T_s$ space, then $(Y, \sigma)$ is a locally indiscrete space.

**Proof.** Let $U$ be any open set of $(Y, \sigma)$. Since $f$ is contra $D$-continuous and $(X, \tau)$ is a $D$-$T_s$ space, $f^{-1}(U)$ is closed in $(X, \tau)$. Since $f$ is a pre-closed surjection, then $U$ is pre-closed in $(Y, \sigma)$. Therefore $cl(U) = cl(int(U)) \subset U$. Hence $U$ is closed in $(Y, \sigma)$. Thus $(Y, \sigma)$ is a locally indiscrete space.
Theorem 6.3.10. If every closed subset of a space $X$ is $D$-open then the following are equivalent.

1. $X$ is $S$-closed
2. $X$ is strongly $S$-closed

Proof. $(1) \Rightarrow (2)$ Let $\{G_{\alpha}/\alpha \in I\}$ be a closed cover of $X$. Then by hypothesis and by theorem 2.3.24, $\{G_{\alpha}/\alpha \in I\}$ is a regular closed cover of $X$. Since $X$ is $S$-closed, it has a finite subcover of $X$. Hence $X$ is strongly $S$-closed.

$(2) \Rightarrow (1)$ Let $\{G_{\alpha}/\alpha \in I\}$ be a regular closed cover of $X$. Since every regular closed is closed and $X$ is strongly $S$-closed, it has a finite subcover of $X$. Hence $X$ is $S$-closed. \hfill \Box

Definition 6.3.11. A topological space $(X, \tau)$ is said to be D-Ultra Hausdorff if for each pair of distinct points $x$ and $y$ in $X$ there exist disjoint D-clopen sets $U$ and $V$ of $x$ and $y$ respectively.

Theorem 6.3.12. If $f : (X, \tau) \to (Y, \sigma)$ is contra $D$-continuous injection, where $Y$ is Urysohn then the topological space $(X, \tau)$ is a $D$-Hausdorff.

Proof. Let $x_1$ and $x_2$ be two distinct points of $(X, \tau)$. Suppose $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $f$ is injective and $x_1 \neq x_2$ then $y_1 \neq y_2$. Since the space $Y$ is Urysohn, there exist open sets $V$ and $W$ such that $y_1 \in V, y_2 \in W$ and $\text{cl}(V) \cap \text{cl}(W) = \emptyset$. Since $f$ is contra $D$-continuous and by theorem 6.2.20, there exist $D$-open sets $Ux_1 \in DO(X, x_1)$ and
Ux_2 \in DO(X, x_2) such that f(Ux_1) \subset \text{cl}(V) and f(Ux_2) \subset \text{cl}(W). Thus we have Ux_1 \cap Ux_2 = \emptyset, since \text{cl}(V) \cap \text{cl}(W) = \emptyset. Hence (X, \tau) is a D-Hausdorff.

\textbf{Theorem 6.3.13.} If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a contra D-continuous injection, where \( Y \) is D-ultra Hausdorff then the topological space \( (X, \tau) \) is D-Hausdorff.

\textbf{Proof.} Let \( x_1 \) and \( x_2 \) be two distinct points of \( (X, \tau) \). Since \( f \) is injective and \( Y \) is D-ultra Hausdorff, then \( f(x_1) \neq f(x_2) \) and also there exist clopen sets \( U \) and \( W \) in \( Y \) such that \( f(x_1) \in U \) and \( f(x_2) \in W \), where \( U \cap W = \emptyset \). Since \( f \) is contra D-continuous, \( x_1 \) and \( x_2 \) belong to D-open sets \( f^{-1}(U) \) and \( f^{-1}(W) \) respectively, where \( f^{-1}(U) \cap f^{-1}(W) = \emptyset \). Hence \( (X, \tau) \) is D-Hausdorff.

\textbf{Theorem 6.3.14.} If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is continuous and \( (X, \tau) \) is a locally indiscrete space, then \( f \) is contra D-continuous.

\textbf{Proof.} Let \( U \) be any open set of \( (Y, \sigma) \). Since \( f \) is continuous, \( f^{-1}(U) \) is open in \( (X, \tau) \) and since \( (X, \tau) \) is locally indiscrete, \( f^{-1}(U) \) is closed in \( (X, \tau) \). Hence by theorem 2.2.2, \( f^{-1}(U) \) is D-closed in \( (X, \tau) \). Thus \( f \) is contra D-continuous.

\textbf{Corollary 6.3.15.} If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is continuous and \( (X, \tau) \) is mildly Hausdorff strongly S-closed space then \( f \) is contra D-continuous.

\textbf{Proof.} It follows from Lemma 1.1.22 and theorem 6.3.14.
Theorem 6.3.16. A contra D-continuous image of a D-connected space is connected.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a contra D-continuous function of a D-connected space onto a topological space $Y$. If possible, assume that $Y$ is not connected. Then $Y = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$, where $A$ and $B$ are clopen sets in $Y$. Since $f$ is contra D-continuous, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty D-open sets in $X$. Hence $X$ is not D-connected, which is a contradiction. Therefore $Y$ is connected. 

\[\square\]

Theorem 6.3.17. The image of a strongly D-closed space under a contra D-continuous surjective function is compact.

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a contra D-continuous surjection. Let $\{V_\alpha/\alpha \in I\}$ be any open cover of $Y$. Since $f$ is contra D-continuous, then $\{f^{-1}(V_\alpha)/\alpha \in I\}$ is a D-closed cover of $X$. Since $X$ is strongly D-closed, then there exists a finite subset $I_0$ of $I$ such that $X = \bigcup\{f^{-1}(V_\alpha)/\alpha \in I_0\}$. Thus we have $Y = \bigcup\{V_\alpha/\alpha \in I_0\}$. Hence $Y$ is compact. 

\[\square\]

Theorem 6.3.18. Every strongly D-closed space $(X, \tau)$ is a compact $S$-closed space.

Proof. Let $\{V_\alpha/\alpha \in I\}$ be a cover of $X$ such that for every $\alpha \in I$, $V_\alpha$ is open and regular closed due to assumption. Then by theorem 2.3.17,
each $V_\alpha$ is D-closed in $X$. Since $X$ is strongly D-closed, there exists a finite subset $I_0$ of $I$ such that $X = \cup\{V_\alpha / \alpha \in I_0\}$. Hence $(X, \tau)$ is a compact S-closed space. 

\[ \square \]

**Theorem 6.3.19.** The image of a D-compact space under a contra D-continuous surjective function is strongly S-closed.

**Proof.** Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is a contra D-continuous surjection. Let $\{G_\alpha / \alpha \in I\}$ be any closed cover of $Y$. Since $f$ is contra D-continuous, then $\{f^{-1}(G_\alpha) / \alpha \in I\}$ is a D-open cover of $X$. Since $X$ is D-compact, there exists a finite subset $I_0$ of $I$ such that $X = \cup\{f^{-1}(G_\alpha) / \alpha \in I_0\}$. Thus we have $Y = \cup\{G_\alpha / \alpha \in I_0\}$. Hence $Y$ is strongly S-closed. 

\[ \square \]

**Theorem 6.3.20.** The image of a D-compact space in any D-$T_s$ space under a contra D-continuous surjective function is strongly D-closed.

**Proof.** Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is a contra D-continuous surjection. Let $\{G_\alpha / \alpha \in I\}$ be any D-closed cover of $Y$. Since $Y$ is D-$T_s$ space, then $\{G_\alpha / \alpha \in I\}$ is a closed cover of $Y$. Since $f$ is contra D-continuous, then $\{f^{-1}(G_\alpha) / \alpha \in I\}$ is a D-open cover of $X$. Since $X$ is D-compact, there exists a finite subset $I_0$ of $I$ such that $X = \cup\{f^{-1}(G_\alpha) / \alpha \in I_0\}$. Thus we have $Y = \cup\{G_\alpha / \alpha \in I_0\}$. Hence $Y$ is strongly D-closed. 

\[ \square \]

**Theorem 6.3.21.** The image of a strongly D-closed space under a D-irresolute surjective function is strongly D-closed.
Proof. Suppose that \( f : (X, \tau) \to (Y, \sigma) \) is a D-irresolute surjection. Let \( \{ G_\alpha/\alpha \in I \} \) be any D-closed cover of \( Y \). Since \( f \) is D-irresolute then \( \{ f^{-1}(G_\alpha)/\alpha \in I \} \) is a D-closed cover of \( X \). Since \( X \) is strongly D-closed, then there exists a finite subset \( I_0 \) of \( I \) such that \( X = \bigcup \{ f^{-1}(G_\alpha)/\alpha \in I_0 \} \). Thus, we have \( Y = \bigcup \{ G_\alpha/\alpha \in I_0 \} \). Hence \( Y \) is strongly D-closed.

Lemma 6.3.22. The product of two D-open sets is D-open.

Theorem 6.3.23. Let \( f : (X_1, \tau) \to (Y, \sigma) \) and \( g : (X_2, \tau) \to (Y, \sigma) \) be two functions where \( Y \) is a Urysohn space and \( f \) and \( g \) are contra D-continuous function. Then \( \{ (x_1, x_2)/f(x_1) = g(x_2) \} \) is D-closed in the product space \( X_1 \times X_2 \).

Proof. Let \( V \) denotes the set \( \{ (x_1, x_2)/f(x_1) = g(x_2) \} \). In order to show that \( V \) is D-closed, we show that \( (X_1 \times X_2) - V \) is D-open. Let \( (x_1, x_2) \notin V \). Then \( f(x_1) \neq g(x_2) \). Since \( Y \) is Urysohn, there exist open sets \( U_1 \) and \( U_2 \) of \( f(x_1) \) and \( g(x_2) \) such that \( cl(U_1) \cap cl(U_2) = \emptyset \). Since \( f \) and \( g \) are contra D-continuous, \( f^{-1}(cl(U_1)) \) and \( g^{-1}(cl(U_2)) \) are D-open sets containing \( x_1 \) and \( x_2 \) in \( X_1 \) and \( X_2 \). Hence by Lemma 6.3.22, \( f^{-1}(cl(U_1)) \times g^{-1}(cl(U_2)) \) is D-open. Further \( (x_1, x_2) \in f^{-1}(cl(U_1)) \times g^{-1}(cl(U_2)) \subset ((X_1 \times X_2) - V) \). If follows that \( (X_1 \times X_2) - V \) is D-open. Thus \( V \) is D-closed in the product space \( X_1 \times X_2 \).

Corollary 6.3.24. If \( f : (X, \tau) \to (Y, \sigma) \) is contra D-continuous and \( Y \) is a Urysohn space, then \( V = \{ (x_1, x_2)/f(x_1) = f(x_2) \} \) is D-closed in the product space \( X_1 \times X_2 \).
Theorem 6.3.25. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a continuous function. Then \( f \) is RC-continuous if and only if it is contra D-continuous.

Proof. Suppose that \( f \) is RC-continuous. Since every RC-continuous function is contra continuous, by theorem 6.2.3, \( f \) is contra D-continuous. Conversely, let \( V \) be any open set of \( (Y, \sigma) \). Since \( f \) is continuous and contra D-continuous, \( f^{-1}(V) \) is open and D-closed in \( (X, \tau) \). By theorem 2.3.24, \( f^{-1}(V) \) is regular open in \( (X, \tau) \). That is, \( \text{int}(\text{cl}(f^{-1}(V))) = f^{-1}(V) \). Since \( f^{-1}(V) \) is open, \( \text{int}(\text{cl}(f^{-1}(V))) = \text{int}(f^{-1}(V)) \) and so \( \text{cl}(\text{int}(f^{-1}(V))) = f^{-1}(V) \). Therefore \( f^{-1}(V) \) is regular closed in \( (X, \tau) \). Hence \( f \) is RC-continuous.

\[ \square \]

Theorem 6.3.26. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be perfectly D-continuous function, \( X \) be locally indiscrete space and connected. Then \( Y \) has an indiscrete topology.

Proof. Suppose that there exists a proper open set \( U \) of \( Y \). Since \( Y \) is locally indiscrete, \( U \) is a closed set of \( Y \). Therefore by theorem 2.2.2, \( U \) is a D-closed set of \( Y \). Since \( f \) is perfectly D-continuous, \( f^{-1}(U) \) is a proper clopen set of \( X \). This shows that \( X \) is not connected, which is a contradiction. Therefore \( Y \) has an indiscrete topology.

\[ \square \]

Theorem 6.3.27. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is a function and \( (X, \tau) \) a locally indiscrete D-\( T_s \) space, then the following statements are equivalent:

1. \( f \) is perfectly continuous.
2. $f$ is continuous and contra continuous

3. $f$ is continuous and contra $D$-continuous.

4. $f$ is super-continuous.

**Proof.** (1)⇒(2) is obvious.

(2)⇒(3) by theorem 2.2.2, it is clear.

(3)⇒(4) by theorem 6.2.35, it is clear.

(4)⇒(1) Let $U$ be any open set of $(Y, \sigma)$. By assumption, $f^{-1}(U)$ is regular open in $(X, \tau)$. Since $(X, \tau)$ is a locally indiscrete space, $f^{-1}(U)$ is open and closed in $(X, \tau)$. Hence $f^{-1}(U)$ is clopen in $(X, \tau)$. Hence $f$ is perfectly continuous. ♦

**Theorem 6.3.28.** Let $f : (X, \tau) \to (Y, \sigma)$ be a contra $D$-continuous function. Let $A$ be an open $D$-closed subset of $X$ and let $B$ be an open subset of $Y$. Assume that $DC(X, \tau)$ (the class of all $D$-closed sets of $(X, \tau)$) be $D$-closed under finite intersections. Then the restriction $f|A : (A, \tau_A) \to (B, \sigma_B)$ is a contra $D$-continuous function.

**Proof.** Let $V$ be an open set of $(B, \sigma_B)$. Then $V = B \cap K$ for some open set $K$ in $(Y, \sigma)$. Since $B$ is an open set of $Y$, $V$ is an open set in $(Y, \sigma)$. By hypothesis and assumption, $f^{-1}(V) \cap A = H_1$(say) is a $D$-closed set in $(X, \tau)$. Since $(f|A)^{-1}(V) = H_1$, it is sufficient to show that $H_1$ is a $D$-closed set in $(A, \tau_A)$. Let $G_1$ be $\omega$-open in $(A, \tau_A)$ such that $H_1 \subseteq G_1$. Then by hypothesis and by Lemma 1.1.19(2), $G_1$ is $\omega$-open in $(X, \tau)$. Since $H_1$ is a $D$-closed set in $(X, \tau)$, we have $pcl_X(H_1) \subseteq int(G_1)$. Since $A$ is open and Lemma 1.1.15, $pcl_A(H_1) =$
\[ pcl_X(H_1) \cap A \subseteq \text{int}(G_1) \cap \text{int}(A) = \text{int}(G_1 \cap A) \subseteq \text{int}(G_1) \] and so \( H_1 = (f|A)^{-1}(V) \) is a \( D \)-closed set in \((A, \tau_A)\). Hence \( f|A \) is a contra \( D \)-continuous function.

\[ \square \]

**Theorem 6.3.29.** If a topological space \((X, \tau)\) is locally indiscrete space then compactness and strongly \( D \)-closedness are the same.

**Proof.** Let \((X, \tau)\) be a compact space. Since \((X, \tau)\) is a locally indiscrete space, then every open set is closed and by theorem 2.2.2, compactness and strongly \( D \)-closedness are the same in a locally indiscrete topological space. \( \square \)