CHAPTER 2

k*-PARANORMAL AND ALGEBRAICALLY k*-PARANORMAL OPERATORS

2.1 Introduction

The investigation of operators obeying Weyl’s theorem was initialized by Hermann Weyl who proved that for every Hermitian operator $T$ on a complex Hilbert space $H$, $w(T) = \sigma(T) - \pi_{00}(T)$ [90].

We consider $B(H)$ as the Banach algebra of operators on a complex infinite dimensional Hilbert space $H$. Let $\sigma(T)$ denote the spectrum of $T \in B(H)$. In this chapter, necessary and sufficient conditions for an operator $T$ in $B(H)$ to be $k^*$-paranormal operator are found out. Some properties of $k^*$-paranormal operators are discussed. It is also proved that the Riesz projection $E_{\lambda}$ associated with $\lambda \in \text{iso}\sigma(T)$ satisfies $E_{\lambda} H = \ker(T - \lambda) = \ker(T - \lambda)^*$ and $E_{\lambda}$ is self-adjoint. It is shown that $(H)$ property and Weyl’s theorem hold for $k^*$-paranormal operators. Spectral mapping theorem and spectral mapping theorem for essential approximate point spectrum for algebraically $k^*$-paranormal operators are proved. Generalized Weyl’s theorem is proved for algebraically $k^*$-paranormal operators. Other Weyl type theorems are also discussed.

2.2 $k^*$-Paranormal operators

In this section we characterize $k^*$-paranormal operators and using matrix representation, it is proved that the restriction of $k^*$-paranormal operators to an invariant subspace is also $k^*$-paranormal operator.

Definition 2.2.1 [29] A bounded linear operator $T$ on a complex Hilbert space $H$ is said to be $k^*$-paranormal if $\|T^* x\|^k \leq \|T^k x\|$ for every unit vector $x \in H$, $k$ being a positive integer.
This class of operators is an extension of hyponormal and \( * \)-paranormal operators and has many interesting properties. \( k \)-*-paranormal operators have Bishop’s property \((\beta)\) [24]. They are normaloids [29]. There is no inclusion between different \( k \)-*-paranormal operators for different values of \( k \).

**Example 2.2.2** [29]

Let \( H \) be the direct sum of a denumerable number of copies of two dimensional Hilbert space \( R \times R \). Let \( A \) and \( B \) be two positive operators on \( R \times R \).

For any fixed positive integer \( n \), we define an operator \( T = T_{A,B,n} \) on \( H \) as follows:

\[
T(x_1, x_2, x_3, \ldots) = (0, Ax_1, Ax_2, \ldots, Ax_n, Bx_{n+1}, \ldots)
\]

Its adjoint \( T^* \) is given by

\[
T^*(x_1, x_2, x_3, \ldots) = (Ax_2, Ax_3, \ldots, Ax_n, Bx_{n+1}, \ldots).
\]

Let \( A \) and \( B \) are positive operators satisfying \( A^2 = C \) and \( B^4 = D \), where

\[
C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix},
\]

then \( T = T_{A,B,n} \) is of \( k \)-*-paranormal for \( k = 1 \).

**Theorem 2.2.3** [29] For \( k \geq 3 \), there exists a \( k \)-*-paranormal operator which is not \( * \)-paranormal operator.

**Theorem 2.2.4** [29] If \( T \) is a \( k \)-*-paranormal operator, then \( T \) is normaloid.

**Theorem 2.2.5** An operator \( T \in B(H) \) is \( k \)-*-paranormal for a positive integer \( k \) if and only if for any \( \mu > 0 \),

\[
T^{*k}T^k - k \mu^{k-1}TT^* + (k-1)\mu^k \geq 0.
\]

**Proof.** Let \( \mu > 0 \) and \( x \in H \) with \( \|x\| = 1 \). Let \( T \in B(H) \) be \( k \)-*-paranormal. Using generalized arithmetic and geometric mean inequality, we get
\[
\frac{1}{k} \langle \mu^{k+1} | T^k |^2 x, x \rangle + \frac{k-1}{k} \langle \mu x, x \rangle = \langle \mu^{k+1} | T^k |^2 x, x \rangle \frac{1}{k} \langle \mu x, x \rangle \frac{k+1}{k} \\
= \mu^{\frac{k+1}{k}} \langle T^k T^k x, x \rangle \frac{1}{k} \langle \mu x, x \rangle \frac{k+1}{k} \\
= \langle T^k x, T^k x \rangle \frac{1}{k} \| x \|^2 \frac{2(k-1)}{k}
\]

Therefore,
\[
\frac{1}{k} \langle \mu^{k+1} | T^k |^2 x, x \rangle + \frac{k-1}{k} \langle \mu x, x \rangle = \| T^k x \|^2 \frac{2(k-1)}{k} \\
= \| T^k x \|^2 \quad [\because x \text{ is a unit vector}]
\geq \| T^* x \|^2 \\
= \langle TT^* x, x \rangle.
\]

Hence,
\[
\frac{\mu^{\frac{k+1}{k}}}{k} \langle | T^k |^2 x, x \rangle + \frac{k-1}{k} \mu \langle x, x \rangle - \langle TT^* x, x \rangle \geq 0.
\Rightarrow \mu^{\frac{k+1}{k}} \langle | T^k |^2 x, x \rangle + (k-1) \mu \langle x, x \rangle - \langle TT^* x, x \rangle \geq 0.
\Rightarrow \langle | T^k |^2 x, x \rangle + (k-1) \mu k^k TT^* x, x \rangle \geq 0.
\Rightarrow \langle T^k T^k x, x \rangle + (k-1) \mu k^k TT^* x, x \rangle \geq 0.
\Rightarrow T^k T^k - k \mu k TT^* + (k-1) \mu k \geq 0.
\]

Conversely assume that \( T^k T^k - k \mu k TT^* + (k-1) \mu k \geq 0 \) for any \( \mu > 0 \).

If \( \| T^* x \| = 0 \), then \( k^* - \)paranormality condition is trivially satisfied.

If \( x \in H \) with \( \| x \| = 1 \) and \( \| T^* x \| \neq 0 \) then,
\[
\left\langle \left( T^k T^k - k \mu k TT^* + (k-1) \mu k \right) x, x \right\rangle \geq 0
\Rightarrow \langle T^k x, T^k x \rangle - k \mu k^{k-1} \langle T^* x, T^* x \rangle + (k-1) \mu k \langle x, x \rangle \geq 0
\Rightarrow \| T^k x \|^2 - k \mu k^{k-1} \| T^* x \|^2 + (k-1) \mu k \| x \|^2 \geq 0.
\]
Taking $\mu = \left\| T^* x \right\|^2$, 

\[ \Rightarrow \left\| T^k x \right\|^2 - k \left\| T^{*} x \right\|^{2k} + (k - 1) \left\| T^{*} x \right\|^{2k} \geq 0 \]

\[ \Rightarrow \left\| T^k x \right\|^2 - k \left\| T^{*} x \right\|^{2k} + k \left\| T^{*} x \right\|^{2k} - \left\| T^{*} x \right\|^{2k} \geq 0. \]

\[ \Rightarrow \left\| T^k x \right\|^2 - \left\| T^{*} x \right\|^{2k} \geq 0 \]

i.e., $\left\| T^k x \right\| \geq \left\| T^{*} x \right\|^k$ for any unit vector $x \in H$.

Therefore $T$ is $k^*$-paranormal.

**Corollary 2.2.6** If an operator $T \in B(H)$ is $k^*$-paranormal for a positive integer $k$ then $\alpha T$ is $k^*$-paranormal for $\alpha \in \mathbb{C}$.

**Theorem 2.2.7** If $T \in B(H)$ is a $k^*$-paranormal operator for a positive integer $k$, $T$ does not have a dense range and $T$ has the following representation:

\[ T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \] on $H = \overline{\text{ran}T} \oplus \ker T^*$, then $T_1$ is also $k^*$-paranormal operator on $\overline{\text{ran}T}$ and $T_3 = 0$. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$ where $\sigma(T)$ denotes the spectrum of $T$.

**Proof.** Let $T \in B(H)$ be $k^*$-paranormal operator.

Let $P$ be the orthogonal projection onto $\overline{\text{ran}T}$.

Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$.

Since $T$ is $k^*$-paranormal, by theorem 2.2.5,

\[ \left( T^{*k} T^k - k \mu^{k-1} TT^* + (k - 1) \mu^k \right) P \geq 0. \]

Hence,

\[ \begin{pmatrix} T_1^{*k} T_1^k - k \mu^{k-1} (T_1 T_1^* + T_2 T_2^*) + (k - 1) \mu^k & 0 \\ 0 & 0 \end{pmatrix} \geq 0. \]

Therefore $T_1^{*k} T_1^k - k \mu^{k-1} (T_1 T_1^* + T_2 T_2^*) + (k - 1) \mu^k \geq 0.$
\[ i.e, \quad T_1^{*k} T_1^k - k\mu^{k-1} T_1 T_1^* + (k-1)\mu^k \geq k\mu^{k-1} T_2^* T_2 \]
\[ = k\mu^{k-1}\left| T_2^* \right|^2 \]
\[ \geq 0. \]

Hence \( T_1 \) is also \( k^* \)-paranormal on \( \text{ran} \ T \).

Also for any \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \),
\[ \langle T_3 x_2, x_2 \rangle = \langle (I - P)x, (I - P)x \rangle = \langle (I - P)x, T^* (I - P)x \rangle = 0. \]

Hence \( T_3 = 0 \).

By ( [56], Corollary 7 ), \( \sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \tau \), where \( \tau \) is the union of certain of the holes in \( \sigma(T) \) which happens to be a subset of \( \sigma(T_1) \cap \sigma(T_3) \) and \( \sigma(T_1) \cap \sigma(T_3) \) has no interior points.

Therefore \( \sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\} \).

**Theorem 2.2.8** If \( T \in B(H) \) is \( k^* \)-paranormal operator for a positive integer \( k \) and \( M \) is an invariant subspace of \( T \), then the restriction \( T \mid_M \) is \( k^* \)-paranormal.

**Proof.** Let \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) be the orthogonal projection of \( H \) onto an invariant subspace \( M \) of \( T \). Then \( \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP \). Since \( T \) is \( k^* \)-paranormal operator, by theorem 2.2.5,
\[ P \left( T_1^{*k} T_1^k - k\mu^{k-1} T T^* + (k-1)\mu^k \right) P \geq 0. \]

Hence \( T_1^{*k} T_1^k - k\mu^{k-1} (T_1 T_1^* + T_2 T_2^*) + (k-1)\mu^k \geq 0. \)

i.e, \( T_1^{*k} T_1^k - k\mu^{k-1} T_1 T_1^* + (k-1)\mu^k \geq k\mu^{k-1} T_2^* T_2 \)
\[ = k\mu^{k-1}\left| T_2^* \right|^2 \]
\[ \geq 0. \]

Hence \( T_1 \) (i.e) \( T \mid_M \) is also \( k^* \)-paranormal operator on \( M \).
Theorem 2.2.9 If $T \in B(H)$ is of $k^*$-paranormal operator for a positive integer $k$, $0 \neq \lambda \in \sigma_p(T)$ and $T$ is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \text{ran}(T - \lambda)^*$, then

1. $T_2 = 0$ and
2. $T_3$ is $k^*$-paranormal.

Proof. Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \text{ran}(T - \lambda)^*$.

Without loss of generality assume that $\lambda = 1$. Then by theorem 2.2.5 for $\mu = 1$,

$$0 \leq T^{-k} T^k - k TT^* + (k-1)I = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix},$$

where $X = -T_2 T_2^*$, $Y = T_2 + T_2 T_3 + \cdots + T_2 T_3^{k-1} - k T_3^* T_3^*$ and $Z = Y^* Y + T_3^* T_3^k - k T_3 T_3^* -(k-1)$.

A matrix of the form $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0$, $Z \geq 0$ and $Y = X^{1/2} W Z^{1/2}$, for some contraction $W$. Therefore $T_2 = 0$ and $T_3$ is $k^*$-paranormal.

Corollary 2.2.10 If $T \in B(H)$ is $k^*$-paranormal operator for a positive integer $k$ and $(T - \lambda)x = 0$ for $\lambda \neq 0$ and $x \in H$, then $(T - \lambda)^* x = 0$.

Corollary 2.2.11 If $T \in B(H)$ is $k^*$-paranormal operator for a positive integer $k$, then $T$ is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \text{ran}(T - \lambda)^*$, where $T_3$ is $k^*$-paranormal and $\ker(T_3 - \lambda) = \{0\}$.

2.3 Spectral properties

Let $T$ be a bounded linear operator on a Hilbert space $H$ and let $\text{iso}\sigma(T)$ be the set of isolated points of the spectrum $\sigma(T)$ of $T$. If $\lambda \in \text{iso}\sigma(T)$, the Riesz idempotent $E_\lambda$ of $T$ with respect to $\lambda$ is defined by $E_\lambda = \frac{1}{2\pi i} \oint_{\mathcal{D}} (zI - T)^{-1} dz$. 

20
where $D$ is a closed disk with centre at $\lambda$ and radius small enough such that $D \cap \sigma(T) = \{\lambda\}$.

It is well known that Riesz idempotent satisfies $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T|_{E_\lambda H}) = \{\lambda\}$ and $\ker(T - \lambda) \subset E_\lambda H$. Stampfli [83] showed that if $T$ satisfies the growth condition $G_1$, then $E_\lambda$ is self adjoint, and $E_\lambda H = \ker(T - \lambda)$. Recently, Cho and Tanahashi [32] obtained an improvement of Stampfli’s result under a stronger assumption: If $T$ is hyponormal then $E_\lambda$ is self adjoint and $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$ -----(#).

Moreover they showed that (#) holds if $T$ is either $p$-hyponormal or log-hyponormal. In the case $\lambda \neq 0$, the result (#) was further shown by Tanahashi and Uchiyama [85] to hold for $p$-quasihyponormal operators and by Uchiyama [87], [88] for class $A$ and paranormal operators. In this section it is shown that the same equality (#) holds for $k^*$-paranormal operators also.

In this section, it is also shown that, among other results, if $T \in B(H)$ is $k^*$-paranormal operator, then $T$ has $(H)$ property and Weyl’s theorem holds for $T$ and $T^*$. 

**Theorem 2.3.1** If $T \in B(H)$ is $k^*$-paranormal operator for a positive integer $k$ and $\lambda \in C$ and assume that $\sigma(T) = \{\lambda\}$, then $T = \lambda$.

**Proof.** We consider two cases:

Case (i) : Let $\lambda = 0$. Since $T$ is $k^*$-paranormal, by theorem 2.2.4, $T$ is normaloid. Therefore $T = 0$.

Case(ii): Assume that $\lambda \neq 0$. Since $\sigma(T) = \lambda$, $T$ is an invertible normaloid operator. Then $T_1 = \frac{1}{\lambda}T$ is an invertible normaloid operator with $\sigma(T_1) = \{1\}$. Hence $T_1$ is similar to an invertible isometry $B$ (on an equivalent normed linear space) with
\[ \sigma(B) = 1 \] (by theorem 2, [61]). \( T_1 \) and \( B \) being similar, 1 is an eigenvalue of \( T_1 = \frac{1}{\lambda} T \) (by theorem 5, [61]). Therefore by theorem 1.2.4, \( T_1 = I \). Hence \( T = \lambda \).

The following theorem characterises isolated points of the spectrum of \( T \) where \( T \) is \( k^* \)-paranormal operator, as the poles of the resolvent of \( T \).

**Theorem 2.3.2** If \( T \in B(H) \) is \( k^* \)-paranormal operator for some positive integer \( k \), then \( T \) is polaroid.

**Proof.** Case (i) : Let \( \lambda = 0 \) and \( T_1 = T |_{E_{\lambda}H} \). By theorem 2.2.8, \( T_1 \) is \( k^* \)-paranormal. Hence \( T_1 \) is normaloid and \( E_{\lambda}H = \ker T \). Therefore 0 is simple pole of \( T \) and so \( T \) is polaroid.

Case (ii): Let \( \lambda(\neq 0) \in \text{iso} \sigma(T) \). Then using the spectral projection of \( T \) with respect to \( \lambda \), we can write \( T = T_1 \oplus T_2 \), where \( \sigma(T_1) = \{ \lambda \} \) and \( \sigma(T_2) = \sigma(T) - \{ \lambda \} \). For \( T_1 \) is \( k^* \)-paranormal operator and \( \sigma(T_1) = \{ \lambda \} \), by theorem 2.3.1 \( T_1 = \lambda \). Since \( \lambda \notin \sigma(T_2) \), \( T_2 - \lambda \) is invertible. Hence both \( T_1 - \lambda I \) and \( T_2 - \lambda I \) have finite ascent and descent and therefore \( T - \lambda I \) have finite ascent and descent. So \( \lambda \) is a pole of the resolvent of \( T \). Hence \( T \) is polaroid.

Evidently, the polaroid property implies reguloid property and reguloid property implies isoloid property. Hence the following corollaries are immediate consequences of the above theorem.

**Corollary 2.3.3** If \( T \) is \( k^* \)-paranormal operator for some positive integer \( k \), then \( T \) is reguloid.

**Corollary 2.3.4** \( k^* \)-paranormal operators are isoloids.

**Theorem 2.3.5** Let \( T \) be \( k^* \)-paranormal for a positive integer \( k \) and \( \lambda \in \text{iso} \sigma(T) \). Then the Riesz idempotent operator \( E_{\lambda} \) with respect to \( \lambda \) satisfies \( E_{\lambda}H = \ker (T - \lambda) \). Hence \( \lambda \) is an eigenvalue of \( T \).
Proof. Let \( T \in B(H) \) be \( k^* \)-paranormal. Let \( \lambda \in \text{iso} \sigma(T) \) and \( E_\lambda = \frac{1}{2\pi i} \int_{\partial D} (\lambda I - T)^{-1} d\lambda \) be the associated Riesz idempotent where \( D \) is a closed disk centered at \( \lambda \) which contains no other points of \( \sigma(T) \). By definition \( \ker(T - \lambda) \subset E_\lambda(H) \) is clear. We shall show that \( E_\lambda H \subset \ker(T - \lambda) \).

By definition of Riesz idempotent \( \sigma(T \mid_{E_\lambda H}) = \{ \lambda \} \) and \( T \big|_{E_\lambda H} \) is \( k^* \)-paranormal. Therefore by theorem 2.3.1, \( T \big|_{E_\lambda H} = \lambda I \), where I is the identity operator on \( E_\lambda H \).

Hence \( E_\lambda H = \ker(T - \lambda) \). Hence \( \lambda \) is an eigen value of \( T \).

Theorem 2.3.6 \([24]\) \( k^* \)-paranormal operators have Bishop’s property (\( \beta \)) and hence have SVEP.

Theorem 2.3.7 Let \( T \in B(H) \) be a \( k^* \)-paranormal operator for a positive integer \( k \). Let \( \lambda \in \sigma(T) \) be an isolated point in \( \sigma(T) \). Then,

\[
\chi_T(\{\lambda\}) = \left\{ x \in H : \left\| (T - \lambda)^n x \right\|^\frac{1}{n} \to 0 \text{ as } n \to \infty \right\} = E_\lambda(H),
\]

where \( E_\lambda \) denotes the Riesz idempotent for \( \lambda \) and \( \chi_T(F) \) denotes the analytic subspace with respect to the closed subset \( F \subseteq C \).

Proof. Let \( T \in B(H) \) be \( k^* \)-paranormal. By theorem 2.3.6 \( T \) has SVEP. Then by [Corollary 2.4 \([65]\), \( \chi_T(\{\lambda\}) = \left\{ x \in H : \left\| (T - \lambda)^n x \right\|^\frac{1}{n} \to 0 \text{ as } n \to \infty \right\} \). The second half of the equality has been proved by F. Riesz and B. Sz. Nagy \([80]\).

Theorem 2.3.8 Let \( T \in B(H) \) be a \( k^* \)-paranormal operator for a positive integer \( k \) and \( \lambda \) be a nonzero isolated point in \( \sigma(T) \). Then the Riesz idempotent operator \( E_\lambda \) with respect to \( \lambda \) is self adjoint and satisfies \( E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^* \).

Proof. Let \( T \in B(H) \) be a \( k^* \)-paranormal operator and let \( \lambda \in \text{iso} \sigma(T) \).

Without loss of generality let us assume that \( \lambda = 1 \).
Let \( T = \begin{pmatrix} 1 & T_2 \\ 0 & T_3 \end{pmatrix} \) on \( H = \ker(T - \lambda) \oplus \overline{\text{ran}(T - \lambda)^*} \).

By theorem 2.2.9, \( T_2 = 0 \) and \( T_3 \) is \( k^* \)- paranormal. Since \( 1 \in \sigma(T) \), we see that either \( 1 \notin \sigma(T_3) \) or \( 1 \in \sigma(T_3) \).

If \( 1 \in \sigma(T_3) \), then since \( T_3 \) is isoloid, \( 1 \) is an eigenvalue of \( T_3 \). However by corollary 2.2.11 we have \( \ker(T_3 - 1) = \{0\} \). Therefore \( 1 \notin \sigma(T_3) \) and hence \( T_3 - 1 \) is invertible on \( \overline{\text{ran}(T - \lambda)^*} \). Since \( T - 1 = 0 \oplus (T_3 - 1) \) and \( T_3 - 1 \) is invertible, we have \( \ker(T - 1) = \ker(T - 1)^* \). Also we have,

\[
E_\lambda = \frac{1}{2\pi i} \int_{\partial D} (zI - T)^{-1} \, dz
\]

\[
= \frac{1}{2\pi i} \int_{\partial D} \begin{pmatrix} (z - 1)^{-1} & 0 \\ 0 & (z - T_3)^{-1} \end{pmatrix} \, dz
\]

\[
= \begin{pmatrix} \frac{1}{2\pi i} \int_{\partial D} (z - 1)^{-1} \, dz & 0 \\ 0 & \frac{1}{2\pi i} \int_{\partial D} (z - T_3)^{-1} \, dz \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Therefore \( E_\lambda \) is the orthogonal projection onto \( \ker(T - 1) \) and it is selfadjoint.

**Definition 2.3.9** [8] Let \( T \in L(X) \) be a bounded operator. \( T \) is said to have property (H), if \( H_0(\lambda I - T) = \ker(\lambda I - T) \), where \( H_0(T) = \left\{ x \in X : \lim_{n \to x} \left\| T^n x \right\| \frac{1}{n} = 0 \right\} \).

**Theorem 2.3.10** \( k^* \)- paranormal operators have (H) property.

**Proof.** Let \( T \in B(H) \) be \( k^* \)- paranormal operator. If \( \lambda \in \sigma(T) \) with the spectral projection \( E_\lambda \), then by ( Theorem 3.1, [63] ), \( E_\lambda(H) = H_0(\lambda I - T) \).
Then by theorem 2.3.5, \( E_\lambda(H) = \ker(\lambda I - T) \).

Hence by Theorem 1.2.5, Theorem 1.2.6 and Theorem 1.2.7, we get the following results.

**Theorem 2.3.11** If \( T \in B(H) \) is \( k^* \)-paranormal operator for some positive integer \( k \), then \( T \) has SVEP, \( p(\lambda I - T) \leq 1 \) for all \( \lambda \in C \) and \( T^* \) is reguloid.

**Theorem 2.3.12** If \( T \in B(H) \) is \( k^* \)-paranormal operator for some positive integer \( k \), then Weyl’s theorem holds for \( T \) and \( T^* \). If in addition, \( T^* \) has SVEP then a-Weyl’s theorem holds for both \( T \) and \( T^* \).

**Theorem 2.3.13** Let \( T \in B(H) \) be \( k^* \)-paranormal operator for some positive integer \( k \). If \( T^* \) has SVEP, then a-Weyl’s theorem holds for \( f(T) \) for every \( f \in H(\sigma(T)) \), where \( H(\sigma(T)) \) denotes the space of all analytic functions \( f : U \to C \) on an open neighbourhood \( U \) of \( \sigma(T) \) containing \( \sigma(T) \).

### 2.4 Algebraically \( k^* \)-paranormal operators

In this section, spectral mapping theorem and the essential approximate point spectral theorem for algebraically \( k^* \)-paranormal operators are proved. It is also shown that algebraically \( k^* \)-paranormal operators are polaroids.

**Definition 2.4.1** An operator \( T \in B(H) \) is defined to be algebraically \( k^* \)-paranormal for a positive integer \( k \), if there exists a non-constant complex polynomial \( p(t) \) such that \( p(T) \) is of class \( k^* \)-paranormal.

If \( T \in B(H) \) is algebraically \( k^* \)-paranormal operator for some positive integer \( k \), then there exists a non-constant polynomial \( p(t) \) such that \( p(T) \) is \( k^* \)-paranormal. By the theorem 2.3.11, \( p(T) \) is of finite ascent. Hence \( p(T) \) has SVEP and hence \( T \) has SVEP (Theorem 3.3.6, [66]).
**Theorem 2.4.2** If \( T \in B(H) \) is algebraically \( k^* \)-paranormal operator for some positive integer \( k \) and \( \sigma(T) = \{ \mu_0 \} \), then \( T - \mu_0 \) is nilpotent.

**Proof.** If \( T \in B(H) \) is algebraically \( k^* \)-paranormal, then there exists a non-constant polynomial \( p(t) \) such that \( p(T) \) is \( k^* \)-paranormal for some positive integer \( k \). Since for any \( T \in B(H) \), \( \sigma(p(T)) = p(\sigma(T)) \) and since \( \sigma(T) = \mu_0 \), we have \( \sigma(p(T)) = p(\mu_0) \). By theorem 2.3.1, \( p(T) = p(\mu_0) \).

Let \( p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1} \cdots (z - \mu_s)^{k_s} \) where \( \mu_j \neq \mu_s \) for \( j \neq s \). Then \( 0 = p(T) - p(\mu_0) = a(T - \mu_0)^{k_0}(T - \mu_1)^{k_1} \cdots (T - \mu_s)^{k_s} \). Since \( T - \mu_1, T - \mu_2, \ldots, T - \mu_s \) are invertible, \( (T - \mu_0)^{k_0} = 0 \). Hence \( T - \mu_0 \) is nilpotent.

We know from [Theorem 3.3.9, [66]], that if an operator \( T \in B(H) \) satisfies SVEP then \( f(T) \) satisfies SVEP for every \( f \) which is analytic on an open neighbourhood of \( \sigma(T) \). Conversely, if \( f \in H(\sigma(T)) \), then SVEP for \( f(T) \) implies SVEP for \( T \).

Our next theorem provides spectral mapping theorem on the Weyl spectra of algebraically \( k^* \)-paranormal.

**Theorem 2.4.3** [4] For every \( T \in B(H) \) and \( f \in H(\sigma(T)) \), \( f(T) \) has SVEP at \( \lambda \in C \) if and only if \( T \) has SVEP at every \( \mu \in \sigma(T) \) such that \( f(\mu) = \lambda \).

**Theorem 2.4.4** If \( T \) is algebraically \( k^* \)-paranormal operator for some positive integer \( k \), then \( \omega(f(T)) = f(\omega(T)) \) for every \( f \in H(\sigma(T)) \).

**Proof.** Suppose that \( T \in B(H) \) is algebraically \( k^* \)-paranormal for some positive integer \( k \), then there exists a non-constant complex polynomial \( p(t) \) such that \( p(T) \) is \( k^* \)-paranormal. Then by theorem 2.3.6, \( p(T) \) has SVEP and hence \( T \) has SVEP. Then by [ Proposition 38.5, [57]], \( \text{ind}(T - \lambda) \leq 0 \) for all \( \lambda \in C \).
We next show that \( w(f(T)) = f(w(T)) \) for all \( f \in H(\sigma(T)) \). Let \( f \in H(\sigma(T)) \).

Since it generally holds \( w(f(T)) \subseteq f(w(T)) \), it suffices to show that \( f(w(T)) \subseteq w(f(T)) \).

Suppose if \( \lambda \notin w(f(T)) \), then \( f(T) - \lambda I \) is Weyl and hence \( \text{ind}(f(T) - \lambda) = 0 \).

Let \( f(T) - \lambda = c(T - \lambda_1)(T - \lambda_2) \cdots (T - \lambda_n) g(T) \) where \( c, \lambda_1, \lambda_2, \cdots, \lambda_n \in C \) and \( g(T) \) is invertible. Since the operators \( T - \lambda_1, T - \lambda_2, \cdots T - \lambda_n \) commute, every \( T - \lambda_i \) is Fredholm for \( 1 \leq i \leq n \).

Also \( \text{ind}(f(T) - \lambda) = 0 = \text{ind}(T - \lambda_1) + \text{ind}(T - \lambda_2) + \cdots + \text{ind}(T - \lambda_n) + \text{ind}(g(T)) \).

Since each of \( \text{ind}(T - \lambda_i) \leq 0 \), it follows that \( \text{ind}(T - \lambda_i) = 0 \) for all \( i = 1, 2, \cdots, n \).

Therefore \( T - \lambda_i \) is Weyl for each \( i = 1, 2, \cdots, n \). Hence \( \lambda_i \notin w(T) \) for \( i = 1, 2, \cdots, n \).

So \( \lambda \notin f(w(T)) \) and therefore \( f(w(T)) \subseteq w(f(T)) \). So \( w(f(T)) = f(w(T)) \).

**Theorem 2.4.5** If \( T \) is algebraically \( k^* \)-paranormal operator for some positive integer \( k \), then \( \pi_{\infty}(T) \subseteq \sigma(T) - w(T) \) and \( \pi_{\infty}(p(T)) \subseteq \sigma(p(T)) - w(p(T)) \).

**Proof.** If \( \lambda \in \pi_{\infty}(T) \), then \( \lambda \in \text{iso}\sigma(T) \) and \( 0 < \dim \ker(T - \lambda) < \infty \). The hypothesis \( \lambda \in \text{iso}\sigma(T) \) implies that ascent of \( (T - \lambda) = \text{descent of} \ (T - \lambda) < \infty \).

Therefore \( \dim \ker(T - \lambda) = \dim \ker(T - \lambda)^* \), which implies that \( T - \lambda \in \Phi(H) \) and \( \text{ind}(T - \lambda) = 0 \). Hence \( \lambda \in \sigma(T) - w(T) \). Similarly we can prove the other inclusion.

**Theorem 2.4.6** If \( T \) or \( T^* \) is algebraically \( k^* \)-paranormal operator for some positive integer \( k \), then \( \sigma_{ed}(f(T)) = f(\sigma_{ed}(T)) \) for every \( f \in H(\sigma(T)) \).

**Proof.** For \( T \in B(H) \), in [78], Rakocevic shows the inclusion \( \sigma_{ed}(f(T)) \subseteq f(\sigma_{ed}(T)) \) for every \( f \in H(\sigma(T)) \) with no restrictions on \( T \). Thus to prove the theorem, it is enough to prove that \( \sigma_{ed}(T) \subseteq \sigma_{ed}(f(T)) \).

Suppose that \( \lambda \notin \sigma_{ed}(f(T)) \). Then \( f(T) - \lambda \in \Phi^+(H) \), that is \( f(T) - \lambda \) is upper semi-Fredholm operator with index less than or equal to zero.
Also $f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2) \cdots (T - \alpha_n) g(T)$ where $g(T)$ is invertible and $c, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$.

If $T$ is algebraically $k^*$-paranormal for some positive integer $k$, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is $k^*$-paranormal. Then $p(T)$ has SVEP and hence $T$ has SVEP. Therefore $\text{ind}(T - \alpha_i) \leq 0$ and hence $T - \alpha_i \in \Phi_+^-(H)$ for each $i = 1, 2, \cdots, n$. Therefore $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.

If $T^*$ is algebraically $k^*$-paranormal for some positive integer $k$, then there exists a non-constant polynomial $p(t)$ such that $p(T^*)$ is $k^*$-paranormal. Then $p(T^*)$ has SVEP and hence $T^*$ has SVEP. Therefore $\text{ind}(T - \alpha_i) \geq 0$ for each $i = 1, 2, \cdots, n$.

Therefore

$$0 \leq \sum_{i=1}^{n} \text{ind}(T - \alpha_i) = \text{ind}(f(T) - \lambda) \leq 0.$$  

So, $\text{ind}(T - \alpha_i) = 0$ for each $i = 1, 2, \cdots, n$. Hence $T - \alpha_i$ is Weyl for each $i = 1, 2, \cdots, n$. $(T - \alpha_i) \in \Phi_+^-(H)$ implies $\alpha_i \notin \sigma_{ea}(T)$. Then $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Therefore $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Hence we have $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

**Theorem 2.4.7** If $T$ is algebraically $k^*$-paranormal operator for some positive integer $k$, then $T$ is polaroid.

**Proof.** Let $\lambda \in \text{iso}\sigma(T)$ and let

$$E_\lambda := \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$$

be the associated Riesz idempotent, where $D$ is a closed disk of center $\lambda$ which contains no other points of $\sigma(T)$. We can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$.

Since $T$ is algebraically $k^*$-paranormal, $p(T)$ is $k^*$-paranormal for some non-constant polynomial $p(t)$. Since $T_1$ is algebraically $k^*$-paranormal with $\sigma(T_1) = \{\lambda\}$, by
Theorem 2.4.2, $T_i - \lambda I$ is nilpotent. Since $\lambda \not\in \sigma(T_2)$, $T_2 - \lambda I$ is invertible. Hence both $T_i - \lambda I$ and $T_2 - \lambda I$ and so $T - \lambda I$ have finite ascent and descent. Therefore $\lambda$ is a pole of the resolvent of $T$. Hence $T$ is polaroid.

**Corollary 2.4.8** Suppose $T \in B(H)$ is algebraically $k*$-paranormal operator for some positive integer $k$, then $T$ is reguloid.

**Corollary 2.4.9** Suppose $T \in B(H)$ is algebraically $k*$-paranormal operator for some positive integer $k$, then $T$ is isoloid.

2.5 **Generalized Weyl’s theorem**

In this section, it is shown that generalized Weyl’s theorem holds for algebraically $k*$-paranormal operators and other Weyl type theorems are discussed.

**Theorem 2.5.1** Suppose $T \in B(H)$ is algebraically $k*$-paranormal operator for some positive integer $k$, then generalized Weyl’s theorem holds for $T$.

**Proof.** Let $T \in B(H)$ be algebraically $k*$-paranormal operator. Then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is $k*$-paranormal. Then $p(T)$ has SVEP and consequently $T$ has SVEP. Assume that $\lambda \in \sigma(T) - \sigma_{bw}(T)$ where $\sigma_{bw}(T)$ is B-Weyl spectrum of $T$. Then $T - \lambda$ is B-Weyl and not invertible.

We claim that $\lambda \in \partial \sigma(T)$. Assume the contrary that $\lambda$ is an interior point of $\sigma(T)$. Then there exists a neighborhood $U$ of $\lambda$ such that $\alpha(T - \mu) > 0$ for all $\mu$ in $U$. It follows from (Theorem 10, [47]), that $T$ does not have SVEP. We have a contradiction. Therefore $\lambda \in \partial \sigma(T) - \sigma_{bw}(T)$. It follows that $\lambda \in E(T)$ where $E(T)$ denotes the set of all isolated eigenvalues of $T$ with no restriction on multiplicity.

Conversely assume that $\lambda \in E(T)$, then $\lambda$ is isolated in $\sigma(T)$. Using the Riesz idempotent $E_{\lambda}$ with respect to $\lambda$, we can represent $T$ as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. 

29
Since $T_1$ is algebraically $k^*$-paranormal, by theorem 2.4.2, $T_1 - \lambda$ is nilpotent. As $\lambda \notin \sigma(T_2)$, $T_2 - \lambda$ is invertible. Therefore $T - \lambda I$ is Drazin invertible (Proposition 6, [81]) and (Corollary 2.2, [67]). By (Lemma 4.1, [21]) $T - \lambda I$ is a B-Fredholm operator of index zero. Hence $\lambda \in \sigma(T) - \sigma_{aw}(T)$. So $\sigma(T) - \sigma_{aw}(T) = E(T)$.

Corollary 2.5.2 Suppose $T \in B(H)$ is algebraically $k^*$-paranormal operator for some positive integer $k$, then Weyl’s theorem holds for $T$.

By (Theorem 2.16, [6]) we get the following result.

Corollary 2.5.3 If $T \in B(H)$ is algebraically $k^*$-paranormal for some positive integer $k$ and $T^*$ has SVEP, then a-Weyl’s theorem and property (w) hold for $T$.

Theorem 2.5.4 If $T \in B(H)$ is algebraically $k^*$-paranormal operator for some positive integer $k$, then Weyl’s theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.

Proof. For every $f \in H(\sigma(T))$,

$$
\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)) \quad \text{by ([69], Lemma)}
$$

$$
= f(w(T)) \quad \text{by corollary 2.5.2}
$$

$$
= w(f(T)) \quad \text{by theorem 2.4.4}
$$

Hence Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

If $T^*$ has SVEP, then by (Lemma 2.5, [2]) $\sigma_{ea}(T) = w(T)$ and $\sigma(T) = \sigma_a(T)$. Hence we get the following results.

Corollary 2.5.5 Suppose $T \in B(H)$ is algebraically $k^*$-paranormal for some positive integer $k$ and if in addition $T^*$ has SVEP, then a-Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Corollary 2.5.6 If $T^* \in B(H)$ is algebraically $k^*$-paranormal operator for some positive integer $k$, then $w(f(T)) = f(w(T))$ for every $f \in H(\sigma(T))$. 

30
**Theorem 2.5.7** [2] Suppose that $T \in L(X)$ is polaroid. Then we have (i) if $T^*$ has SVEP then property (b), (or equivalently property (w), Weyl’s theorem, a-Weyl’s theorem) holds for $T$ (ii) if $T$ has SVEP then property (b), (or equivalently property (w), Weyl’s theorem, a-Weyl’s theorem) holds for $T^*$.

By the above theorem, we get the following results.

**Corollary 2.5.7** Suppose $T \in B(H)$ is algebraically $k^*$-paranormal for some positive integer $k$ and $T^*$ has SVEP, then property (b) holds for $T$.

**Corollary 2.5.8** Suppose $T \in B(H)$ is algebraically $k^*$-paranormal for some positive integer $k$, then Weyl’s theorem, a-Weyl’s theorem, property (w) and property (b) hold for $T^*$. 