CHAPTER 1

INTRODUCTION

1.1 Introduction to Operator Theory

In mathematics, Operator Theory is a branch of Functional Analysis that focuses on bounded linear operators including closed operators, nonlinear operators and the algebras of operators. Self-adjoint operators, Hermitian operators and spectral theory play an important role in Quantum Physics and Time Frequency Analysis. Today many branches of analysis are inseparable from Operator Theory. We will review its development briefly in this section.

Let $X$ be a normed linear space and $T$ be a linear operator whose domain $D(T)$ and the range $T(X)$ lie in $X$. A Banach space is a complete normed linear space. A Hilbert space is a real or complex inner product space which is complete under the norm induced by the inner product $\langle \cdot, \cdot \rangle$ defined by $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ for all $x$. A linear operator $T$ on a Hilbert space $H$ is said to be bounded, if there exists a real number $c > 0$ such that $\|Tx\| \leq c \|x\|$ for all $x \in H$. Norm of $T$ is defined by $\|T\| = \inf \{c > 0 : \|Tx\| \leq c \|x\| \text{ for all } x \in H\}$.

Operator Theory analyses transformations between the vector spaces studied in Functional Analysis. The analysis might study the spectrum of an individual operator or the semigroup structure of a collection of operators.

In mathematics, spectral theory is an extension of the theory of eigen values and eigen vectors of a single square matrix to the theory of the structure of operators in a variety of mathematical spaces like Hilbert space, Lebesgue space, Sobolev space etc. It is a result of the studies of linear algebra, the solutions of systems of linear equations and their generalizations. The theory is connected to that of analytic functions because the spectral properties of an operator are related to analytic functions of the spectral parameter. The name spectral theory was introduced by
David Hilbert in his original formulation of Hilbert space theory. In spectral theory, an operator can be decomposed into a direct sum of operators corresponding to a partition of its spectrum.

Let $B(H)$ denote the algebra of all bounded linear operators defined on an infinite dimensional complex Hilbert space $H$. For $T \in B(H)$, the resolvent of $T$ is the set of all complex numbers $\lambda$ such that the transformation $R_\lambda = (\lambda I - T)^{-1}$ exists and is bounded. This set is often denoted by $\rho(T)$. The spectrum of a bounded linear operator $T$ acting on a Banach space $X$ is the set of all complex numbers $\lambda$ such that $\lambda I - T$ is a bounded linear operator and not invertible. If $\lambda I - T$ is invertible then the inverse is linear (this follows immediately from the linearity of $\lambda I - T$) and is bounded by the bounded inverse theorem. Therefore the spectrum consists precisely of those $\lambda$ where $\lambda I - T$ is not bijective. Often the spectrum of $T$ is denoted by $\sigma(T)$. The spectrum of $T$ is therefore the complement of the resolvent of $T$ in the complex plane. The spectrum of a bounded linear operator $T$ is always closed, bounded and non-empty subset of the complex plane. $\lambda \in \mathbb{C}$ is said to be an eigenvalue of the operator $T$ if $Tx = \lambda x$ for a nonzero vector $x \in H$. Every eigenvalue of $T$ belongs to $\sigma(T)$, but $\sigma(T)$ may contain non-eigen values.

According to Riesz representation theorem [19], there exists a unique operator $T^* \in B(H)$ such that $(Tx, y) = (x, T^*y)$ for all $x, y \in H$ and $T \in B(H)$. $T^*$ is called the adjoint of $T$. An operator $T \in B(H)$ is said to be Hermitian if $T = T^*$. An operator $T \in B(H)$ is normal if $T^*T = TT^*$. Many properties of these two operators have been studied by many authors. Later many non-normal operators were introduced by several authors.

An operator $T$ is said to be positive (denoted by $T \geq 0$), if $(Tx, x) \geq 0$ for all $x \in H$ and is said to be strictly positive (denoted by $T > 0$), if $T$ is positive and invertible. Among non-normal operators, the hyponormal operator was introduced by P. R. Halmos. An operator $T \in B(H)$ is called hyponormal if $T^*T \geq TT^*$. Several conditions for a hyponormal operator to be normal were given by
G. S. Stampfli [82]. Istratescu et al [59] studied the hyponormal operators and introduced a new class of operators called class(N) operators. Later this operator was renamed as paranormal operator by T. Furuta. An operator \( T \in \mathcal{B}(H) \) is called paranormal, if \( \|T^*x\|^2 \leq \|T^2x\| \|x\| \) for every \( x \in H \). It was proved that hyponormal operators form a subclass of paranormal operators and the paranormal operators are normaloids, that is, they satisfy \( \|T^n\| = \|T\|^n \) for all non negative integers \( n \).

For \( 0 < p < 1 \), an operator \( T \) is said to be \( p \)-hyponormal, if \( (T^*T)^p \geq (TT^*)^p \). If \( p = \frac{1}{2} \), \( T \) is called semi-hyponormal. The class of \( p \)-hyponormal operators is more general than the class of hyponormal operators. Aluthge [10] has defined and studied \( p \)-hyponormal operators. Every \( p \)-hyponormal operator is \( q \)-hyponormal for \( p \geq q > 0 \) by Lowner-Heinz theorem “\( A \succeq B \succeq 0 \) ensures that \( A^\alpha \succeq B^\alpha \) for every \( \alpha \in [0,1] \)”.

Tanahashi [84] introduced log-hyponormal operators. An operator \( T \) is said to be log-hyponormal, if \( T \) is invertible and satisfy the inequality \( \log T^*T \geq \log TT^* \). Log-hyponormal operators are extensions of hyponormal ones. Every log-hyponormal operator is paranormal [14].

In order to discuss the relation between paranormal, \( p \)-hyponormal and log-hyponormal operators, Furuta et al [51] introduced class A operators defined by \( |T^2| \geq |T|^2 \), where the polar decomposition of \( T \) is \( T = U |T| \), which is a generalisation of \( p \)-hyponormal, log-hyponormal operators and a subclass of paranormal operators. As a generalization of class A and paranormal operators Furuta et al [51] introduced class A(k) and absolute-k-paranormal operators for \( k > 0 \) respectively. An operator \( T \) belongs to class A(k) if \( (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2 \) and absolute-k-paranormal if \( \|T^kTx\| \geq \|Tx\|^{k+1} \) for every unit vector \( x \in H \).

Mahmoud M. Kutkut [70] introduced and studied parahyponormal operators. An
operator \( T \in B(H) \) is said to be parahyponormal if \( \|Tx\|^2 \leq \|TT^*x\| \) for every unit vector \( x \in H \). Patel [76] studied the paranormal operators and introduced a new class of operators called \(*\)-paranormal operators. An operator \( T \) is called \(*\)-paranormal if \( \|T^*x\|^2 \leq \|T^{2}x\| \|x\| \) for each \( x \) in \( H \). An extension of \(*\)-paranormal operators is quasi \(*\)-paranormal introduced by Arora and Thukral [15]. Veluchamy and Devika [89] proved that in a finite dimensional Hilbert space every quasi-\(*\)-paranormal operator \( T \) is normal. C. S. Ryoo and P. Y. Sik [29] defined \( k\)-\(*\)-paranormal operators, \( k \) being a positive integer which generalizes the class of \(*\)-paranormal operators. They also showed many examples to prove the inclusion relation, Normal \( \subset k\)-hyponormal \( \subset k\)-\(*\)-paranormal \( \subset \) Normaloid. Several extensions of paranormal operators have been considered till now, for example, class \( A(k) \), absolute \( k\)-paranormal class, \( k\)-paranormal operators. Panayappan and Radharamani [75] introduced and characterized a new class of operators named as quasi parahyponormal operators. Also, they introduced a new class of operators [74] namely \( p\)-\(*\)-paranormal and absolute \( k\)-\(*\)-paranormal operators and showed that every hyponormal operator is an absolute \( k\)-\(*\)-paranormal operator and for each \( k > 0 \), every class \( A(k^*) \) operator is an absolut \( k\)-\(*\)-paranormal operator.

An operator \( T \in B(H) \) is \( n\)-paranormal for positive integer \( n \) such that \( n \geq 2 \) if \( \|T^n x\| \geq \|Tx\| \) for every unit vector \( x \in H \). This operator has been studied in [49], [50] and [58]. It is well known that every paranormal operator is \( n\)-paranormal for \( n = 2,3,\ldots \). Later \( n\)-paranormal operator was renamed as \( k\)-paranormal operator. C. S. Kubrusly and B. P. Duggal [64] showed that \( k\)-paranormal operators are normaloids. Also in [43] they proved that they satisfy Bishop’s property, have finite ascent \( \leq 1 \) and single valued extension property. Ito [71] showed that if \( T \) satisfies
\[
\|T^n\|^2 \geq \|T\|^2
\]
for some positive integer \( n \) such that \( n \geq 2 \), then \( T \) is \( n\)-paranormal. The class of these operators is denoted by class \( A(n) \) which is a generalization of class \( A \). Ito showed that class \( A(n) \) is a subclass of \( n\)-paranormal operators [71].
The inclusion relations between some of the various classes of operators discussed in this chapter are shown below.

\[
\{ \text{hyponormal} \} \subseteq \{ \log \text{- hyponormal} \} \\
\subseteq \{ \text{classA} \} \\
\subseteq \{ \text{paranormal} \} \\
\subseteq \{ k \text{- paranormal} \} \\
\subseteq \{ \text{normaloid} \}
\]

\[
\{ \text{hyponormal} \} \subseteq \{ * \text{- paranormal} \} \\
\subseteq \{ k^* \text{- paranormal} \} \\
\subseteq \{ \text{normaloid} \}
\]

1.2 A brief survey on Weyl’s theorem and Weyl type theorems

In 1909, while writing about differential equations, Hermann Weyl noticed something about the essential spectrum of a self adjoint operator on a Hilbert space. When we take it away from the spectrum, we are left with the isolated eigen values of finite multiplicity. This was soon generalized to normal operators and then to more and more classes of operators, bounded and unbounded on Hilbert and on Banach spaces.

In Functional Analysis, compact operators are linear operators that map bounded sets to precompact sets. The set of all compact operators acting on a Hilbert space \( H \) is the closure of the set of finite rank operators in the uniform operator topology. In general, operators on infinite dimensional space feature properties that do not appear in the finite dimensional case, i.e., for matrices. The spectral properties of compact operators resemble those of square matrices.
Let $B(H)$ be the algebra of all bounded operators on an infinite dimensional complex Hilbert space $H$ and $K(H)$ be the closed ideal of compact operators. H. Weyl [90] examined the spectra of all compact perturbations $T + K$ of a single Hermitian operator $T$ and $K \in K(H)$ and discovered that $\lambda \in \sigma(T + K)$ for every compact operator $K$ if and only if $\lambda$ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. This result is known as Weyl’s theorem. The Weyl spectrum has the property of being invariant under perturbations by compact operators, that is $w(T + K) = w(T)$ for all $T \in B(H)$ and $K \in K$. This property implies that $w(T) = \bigcap \sigma(T + K)$, where the intersection is taken over all $K \in K(H)$ and that $w(T) = \{0\}$ when $T$ is compact and the space is finite dimensional.

For $T \in B(H)$, domain, null space and range of $T$ are denoted by $D(T), N(T)$ and $\text{ran} T$ respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and deficiency of $T \in B(H)$ defined as $\alpha(T) = \dim(T^{-1}(0)) < \infty$ and $\beta(T) = \dim(H / \text{ran} T) < \infty$. The class of all upper semi-Fredholm operators denoted by $\Phi_+(H)$ and the class of lower semi-Fredholm operators denoted by $\Phi_-(H)$ are defined by $\Phi_+(H) := \{T \in B(H) : \alpha(T) < \infty \text{ and } \text{ran} T \text{ is closed}\}$ and $\Phi_-(H) := \{T \in B(H) : \beta(T) < \infty \}$.

An operator $T \in B(H)$ is said to be semi-Fredholm, $T \in \Phi_{\pm}(H)$, if $T \in \Phi_+(H) \cup \Phi_-(H)$ and Fredholm, $T \in \Phi(H)$, if $T \in \Phi_+(H) \cap \Phi_-(H)$. The index of a semi-Fredholm operator is an integer defined as $\text{ind}(T) = \dim \ker T - \dim \ker T^*$. An upper semi-Fredholm operator with index less than or equal to zero is called upper semi-Weyl operator and is denoted by $T \in \Phi_+(H)$. A lower semi-Fredholm operator with index greater than or equal to zero is called lower semi-Weyl operator and is denoted by $T \in \Phi_-(H)$. A Fredholm operator of index zero is called Weyl operator.

The spectrum of an operator $T$ is denoted by $\sigma(T)$, where $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$. The Weyl spectrum of $T$ is defined as
\( w(T) = \{ \lambda \in C : T - \lambda \text{ is not Weyl} \} \). The set of all isolated eigen values of \( T \) with finite geometric multiplicity is denoted by \( \pi_{\infty}(T) \).

According to Coburn [27], we say that Weyl’s theorem holds for an operator \( T \) in \( B(H) \) if \( \sigma(T) - w(T) = \pi_{\infty}(T) \). Weyl’s theorem has been extended from Hermitian operators to several classes of operators including hyponormal, \( p \)-hyponormal [30], \( M \)-hyponormal, log-hyponormal [39], [72] , paranormal [33], algebraically paranormal [35], posinormal [41] , quasi class A operators [42], algebraically hyponormal [54], class A [86] and class of operators that satisfy property \( (H_p) \) [1].

The point spectrum of an operator \( T \) is defined by \( \sigma_p(T) = \{ \lambda \in C : \lambda \text{ is an eigen value of } T \} \). The approximate point spectrum of \( T \) is denoted by \( \sigma_a(T) \), where \( \sigma_a(T) = \{ \lambda \in C : T - \lambda I \text{ is not bounded below} \} \). The essential spectrum of \( T \) is defined as \( \sigma_e(T) = \{ \lambda \in C : T - \lambda I \text{ is not Fredholm} \} \). The essential spectrum is always closed and it is a subset of \( \sigma(T) \). When \( T \) is self-adjoint, \( \sigma(T) \subseteq R \). The essential spectrum is invariant under compact perturbations. For self-adjoint and more general, for normal operators, the Weyl and the Fredholm spectra coincide. Every normal Fredholm operator has index zero.

The ascent (length of the null chain) of an operator \( T \in B(H) \) denoted by \( \text{asc } T \), is the smallest non-negative integer \( p = p(T) \) such that \( \ker T^p = \ker T^{p+1} \). If such integer does not exist, we put \( p(T) = \infty \). The descent (length of the image chain) of \( T \in B(H) \) denoted by \( \text{dsc } T \), is the least non-negative integer \( q = q(T) \) such that \( \text{ran } T^q = \text{ran } T^{q+1} \) and if such integer does not exist, we put \( q(T) = \infty \). It is well known that if \( p(T) \) and \( q(T) \) are both finite then \( p(T) = q(T) \) (Proposition 38.3,[57]). Every operator with finite ascent has SVEP. Moreover \( 0 < p(\lambda I - T) = q(\lambda I - T) < \infty \) precisely when \( \lambda \) is a pole of the resolvent of \( T \) [Proposition 50.2,[57]].
The inter relationship among descent, ascent, \( \alpha(T) \) and \( \beta(T) \) is given in the following theorem.

**Theorem 1.2.1** \([1, 57]\) If \( T \) is a linear operator on a vector space \( X \), then the following properties hold:

(i) \( \text{if } p(T) < \infty, \text{ then } \alpha(T) \leq \beta(T). \)

(ii) \( \text{if } q(T) < \infty, \text{ then } \beta(T) \leq \alpha(T). \)

(iii) \( \text{if } p(T) = q(T) < \infty, \text{ then } \alpha(T) = \beta(T) \) (possibly infinite).

(iv) \( \text{if } \alpha(T) = \beta(T) < \infty \text{ and if one chain is finite then } p(T) = q(T). \)

An upper semi-Fredholm operator with finite ascent is called upper semi-Browder operator and is denoted by \( T \in B_{\alpha}(H) \), while a lower semi-Fredholm operator with finite descent is called lower semi-Browder operator and is denoted by \( T \in B_{\beta}(H) \). A Fredholm operator with finite ascent and descent is called Browder operator. The class all Browder operators is defined by \( B(H) = B_{\alpha}(H) \cap B_{\beta}(H) = \{ T \in \Phi(H) : p(T), q(T) < \infty \} \). Clearly, the class of all Browder operators is contained in the class of all Weyl operators. Similarly, the class of all upper semi-Browder operators is contained in the class of all upper semi-Weyl operators and the class of all lower semi-Browder operators is contained in the class of lower semi-Weyl operators.

These classes of operators motivate the definition of several spectra. The upper semi-Browder spectrum of \( T \in B(H) \) is defined by \( \sigma^{ub}(T) = \{ \lambda \in C : T - \lambda I \notin B_{\alpha}(H) \} \), the lower semi-Browder spectrum of \( T \in B(H) \) is defined by \( \sigma^{lb}(T) = \{ \lambda \in C : T - \lambda I \notin B_{\beta}(H) \} \), while the Browder spectrum of an operator \( T \in B(H) \) is defined as \( \sigma_{b}(T) = \{ \lambda \in C : T - \lambda I \text{ is not Browder} \} \). Browder’s theorem, a weaker version of Weyl’s theorem was introduced by Harte and Lee in [53]. Browder’s theorem holds for \( T \), if \( \sigma(T) - w(T) = p_{00}(T) \), \( p_{00}(T) \) is defined as \( p_{00}(T) = \sigma(T) - \sigma_{b}(T) \).
The set of all isolated eigen values of finite multiplicity of $T$ in $\sigma_a(T)$ is denoted by $\pi^0_{00}(T)$ and the essential approximate point spectrum of an operator $T$ is defined as $\sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \not\in \Phi^*_{ea}(H) \}$. Analogously, to conduct a similar study where the spectrum is replaced by the approximate point spectrum, the concept of a-Weyl’s and a-Browder’s theorems were introduced by V. Rakočevič in [77] and in [79].

We say that a-Weyl’s theorem holds for $T$, if $T$ satisfies the equality $\sigma_a(T) - \sigma_{ea}(T) = \pi^0_{00}(T)$ and a-Browder’s theorem holds for $T$ if $\sigma_{ea}(T) = \sigma_{ab}(T)$. It is well known that the following implications hold [36], [53], [77].

\[
\begin{align*}
\text{a-Weyl's theorem} & \implies \text{a-Browder's theorem} \\
\implies & \\
\implies & \text{Weyl's theorem} \implies \text{Browder's theorem}
\end{align*}
\]

We say that $T$ satisfies property (w) if $\sigma_a(T) - \sigma_{ea}(T) = \pi^0_{00}(T)$ and $T$ satisfies property (b) if $\sigma_a(T) - \sigma_{ea}(T) = p_{00}(T)$. The following diagram exhibits the relationships between Weyl’s theorems, a-Weyl’s theorem, a-Browder’s theorem and property (w) [6], [79].

\[
\begin{align*}
\text{Property}(w) & \implies \text{a-Browder's theorem} \\
\implies & \\
\implies & \text{Weyl's theorem} \Leftarrow \text{a - Weyl's theorem}
\end{align*}
\]

Evidently, if $\text{acc } \sigma(T)$ is the set of accumulation points of $\sigma(T)$ then $w(T) \subseteq \sigma_b(T) = \sigma_v(T) \cup \text{acc } \sigma(T)$.

A stronger version of Weyl’s theorem, generalized Weyl’s theorem was introduced by Berkani [20]. Recently, M. Berkani and J. J. Koliha [23] studied the concepts of generalized Weyl’s theorem and generalized Browder’s theorem and they showed that $T$ satisfies the generalized Weyl’s theorem whenever $T$ is a normal operator on Hilbert space. Raul E. Curto and Young Min Han [34] extended
generalized Weyl’s theorem to several classes of operators much larger than that of normal operators.

For an operator $T \in B(H)$ and a non-negative integer $n$, define $T_{[n]}$ to be the restriction of $T$ to $\text{ran}(T^n)$ viewed as a map from $\text{ran}(T^n)$ into $\text{ran}(T^n)$. In particular $T_{[0]} = T$. If, for some positive integer $n$, $\text{ran}(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then $T$ is called an upper (resp. a lower) semi-$B$-Fredholm operator. Moreover if $T_{[n]}$ is a Fredholm operator, then $T$ is called a $B$-Fredholm operator. A semi-$B$-Fredholm operator is an upper or a lower semi-$B$-Fredholm operator. The index of a semi-$B$-Fredholm operator $T$ is the index of semi-Fredholm operator $T_{[d]}$, where $d$ is the degree of the stable iteration of $T$ and is defined as $d = \inf \left\{ n \in N : \text{for all } m \in N, m \geq n \Rightarrow (\text{ran}T^m \cap N(T)) \subset (\text{ran}T^m \cap N(T)) \right\}$. $T$ is called a $B$-Weyl operator if it is $B$-Fredholm of index zero.

The concept of Drazin invertibility plays an important role for the class of $B$-Fredholm operators. Let $A$ be unital algebra. We say that $x \in A$ is Drazin invertible of degree $k$ if there exists an element $a \in A$ such that $x^k ax = x^k$, $axa = a$ and $xa = ax$.

For $a \in A$, the Drazin spectrum is defined as $\sigma_d(a) = \left\{ \lambda \in C : a - \lambda \text{ is not Drazin invertible} \right\}$. In the case of $T \in B(H)$, $T$ is Drazin invertible if and only if $T$ has finite ascent and descent which is also equivalent to having $T$ decomposed as $T_1 \oplus T_2$ where $T_1$ is invertible and $T_2$ is nilpotent.

Let $E(T)$ denote the isolated eigenvalues of $T$ with no restriction on multiplicity. The $B$-Weyl spectrum $\sigma_{bw}(T)$ of $T$ is defined by $\sigma_{bw}(T) = \left\{ \lambda \in C : T - \lambda I \text{ is not } B\text{-Weyl operator} \right\}$. 

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A bounded operator $T \in B(H)$ is said to satisfy generalized Weyl’s theorem if \( \sigma_{gw}(T) = \sigma(T) - E(T) \).

Recently in [22], Berkani showed that if $T$ is a hyponormal operator, then $T$ satisfies generalized Weyl’s theorem and the B-Weyl spectrum \( \sigma_{bw}(T) \) of $T$ satisfies spectral mapping theorem. It has been extended from hyponormal operators to $p$-hyponormal operators and $M$-hyponormal operators [26], class of operators satisfying SVEP [13] and algebraically paranormal operators [92]. In general, generalized Weyl’s theorem implies Weyl’s theorem but the converse is not true [22].

Now let us state one of the basic notions of spectral theory. The single valued extension property was introduced by Dunford [44], [45]. It plays an important role in local spectral theory. We consider the following local version of this property, which has been studied in recent papers [3, 4, 5, 7] and previously by Finch [47]. The basic role of single valued extension property arises in local spectral theory, since every decomposable operator enjoys this property.

Let $X$ be a complex Banach space and $T \in B(X)$. An operator $T$ is said to have single valued extension property at $\lambda_0 \in C$ (abbreviated as SVEP at $\lambda_0$), if for every open neighbourhood $U$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation \( (\lambda I - T)f(\lambda) = 0 \) for all $\lambda \in U$ is the constant function $f \equiv 0$. An operator $T \in B(X)$ is said to have SVEP, if $T$ has SVEP at every $\lambda \in C$.

Trivially an operator $T \in B(H)$ has SVEP at every point of the resolvent set $\rho(T) = C - \sigma(T)$. Moreover from the identity theorem for analytic function, it easily follows that $T \in B(H)$ has SVEP at every point of the boundary $\partial \sigma(T)$ of the spectrum of $T$. In particular, $T$ has SVEP at every isolated point of $\sigma(T)$.

The SVEP is inherited by the restrictions to closed invariant subspaces, that is, if $T \in B(H)$ has the SVEP at $\lambda$ and $M$ is a closed invariant subspace then $T|_M$ has SVEP at $\lambda$. 

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An operator $T \in B(H)$ satisfies Bishop’s property ($\beta$) if, for every open subset $U$ of the complex plane $C$ and every sequence of analytic functions $f_n : U \rightarrow H$ with the property that $(T - \lambda)f_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on all compact subsets of $U$, $f_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$ locally uniformly on compact subsets in $U$.

The quasinilpotent part $H_0(T - \lambda)$ and the analytic core $K(T - \lambda)$ of $T - \lambda$ are defined by $H_0(T - \lambda) = \left\{ x \in H : \lim_{n \rightarrow \infty} \left\| (T - \lambda)^n x \right\|^{\frac{1}{n}} = 0 \right\}$ and $K(T - \lambda) = \left\{ x \in H : \text{there exists a sequence } (x_n) \subset H \text{ and a constant } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \left\| x_n \right\| \leq \delta^n \left\| x \right\| \text{ for all } n = 1, 2, 3, \ldots \right\}$. It is proved that $H_0(T - \lambda)$ and $K(T - \lambda)$ are non-closed hyper invariant subspaces of $T - \lambda$ such that $(T - \lambda)^{-q}(0) \subseteq H_0(T - \lambda)$ for all $q = 1, 2, 3, \ldots$ and $(T - \lambda) K(T - \lambda) = K(T - \lambda)$.

An operator $T \in B(H)$ is said to satisfy the property ($H_p$) if $H_0(T - \lambda) = (T - \lambda)^{-p}(0)$ for all complex number $\lambda$ and for some integer $p \geq 1$, where $H_0(T - \lambda)$ denotes the quasinilpotent part of $T$. An invertible operator $T \in B(H)$ is said to be doubly power bounded if $\sup \left\{ \left\| T^n \right\| \mid n \in \mathbb{N} \right\} < \infty$.

**Theorem 1.2.2** [Theorem 1.5.13, [66]] All doubly power bounded operators and in particular, all invertible isometries are generalized scalars.

**Theorem 1.2.3** [Theorem 1.5.10 [66]] A generalized scalar operator with finite spectrum is algebraic. In particular, a quasinilpotent generalized scalar operator is nilpotent.

**Theorem 1.2.4** [Theorem 1.5.14 [66]] Let $T \in L(X)$ be doubly power bounded on a non trivial Banach space $X$. Then $\sigma(T) = \{1\}$ if and only if $T$ is an identity operator.
Let $\lambda$ be an isolated point of the spectrum $\sigma(T)$ of Hilbert space operator $T$. Let $D_\lambda$ be a closed disk centered at $\lambda$ which satisfies $D_\lambda \cap \sigma(T) = \{\lambda\}$. The operator $E_\lambda := \frac{1}{2\pi i} \int_{\partial D_\lambda} (\lambda I - T)^{-1} \, d\lambda$ is called the Riesz idempotent with respect to $\lambda$ which has properties $E_\lambda^2 = E_\lambda$, $E_\lambda T = TE_\lambda$, $\ker(T - \lambda) \subseteq E_\lambda H$ and $\sigma(T | E_\lambda H) = \{\lambda\}$. In [82] Stampfli proved that if $T$ is hyponormal and $\lambda \in \sigma(T)$ is isolated, then the Riesz idempotent $E_\lambda$ with respect to $\lambda$ is self adjoint and satisfies $E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*$. This result has been extended to p-hyponormal operators by Cho and Tanahashi [32], to $w$-hyponormal operator by Han et al [55], to classA operators when $\lambda \neq 0$ by A. Uchiyama and K. Tanahashi [88] and to paranormal operators by Uchiyama [87]. In this thesis this result is extended to $k$-paranormal and $k^*$-paranormal operators.

An operator $T$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Berberain [17] proved that if every restriction $T |_M$ to its reducing subspace $M$ is isoloid and if every finite dimensional eigen space of $T$ reduces $T$, then Weyl’s theorem holds for $T$. Using Riesz idempotent, Stampfli [82] proved that every hyponormal operator is isoloid and this result is one of the most famous results in this theory. This result has been extended to several classes of operators, including $p$-hyponormal, log-hyponormal and paranormal [30, 31, 32, 85, 86] operators.

An operator $T \in B(H)$ is called normaloid, if $r(T) = \|T\|$, where $r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T) \}$ is spectral radius of $T$. Equivalently, $\|T^n\| = \|T\|^n$ for all non negative integers $n$.

An operator $T \in B(H)$ is called hereditarily normaloid, that is, $T \in \text{HN}$ if every part of it (including itself) is normaloid. An operator $T \in B(H)$ is totally hereditarily normaloid, denoted by $T \in \text{THN}$, if $T \in \text{HN}$ and $T^{-1}$ for every
invertible part $T_p$ of $T$ is normaloid. The class THN is introduced in [40] by Duggal and Djordjević and have since been studied in [37,40].

An operator $T \in B(H)$ is said to be quasinilpotent, if $\|T^n\|^{1/n} \to 0$ as $n \to \infty$ and is said to be nilpotent if $T^n = 0$ for some $n$. An operator $T \in B(H)$ is polaroid if $\pi(T) = \{\lambda \in \text{iso}\sigma(T)\}$, where $\pi(T)$ is the set of poles of the resolvent of $T$ and $\text{iso}\sigma(T)$ is the set of isolated points of $\sigma(T)$. A necessary and sufficient condition for $\lambda \in \pi(T)$ is that $\text{asc}(T - \lambda I) = \text{dsc}(T - \lambda I) < \infty$.

A bounded operator $T \in B(H)$ is said to be reguloid if for every isolated point $\lambda$ of $\sigma(T)$, $\lambda I - T$ is relatively regular, i.e., there exists $S_\lambda \in B(H)$ such that $(\lambda I - T)S_\lambda (\lambda I - T) = \lambda I - T$. It is well known that $T \in B(H)$ is relatively regular operator if and only if $\ker T$ and $T(H)$ are complemented. Obviously if $T \in B(H)$ is polaroid, then $T$ is reguloid and hence $T$ is isoloid.

For any bounded operator $T \in L(X)$ on an infinite-dimensional complex Banach space, the following theorems reveal the interrelationship among $(H)$ property, SVEP and a-Weyl’s theorem.

**Theorem 1.2.5** [ Theorem 2.58 [8] ] Suppose that the operator $T \in L(X)$ has property $(H)$. Then $T$ has SVEP and $p(\lambda I - T) \leq 1$ for all $\lambda \in C$.

**Theorem 1.2.6** [ Theorem 2.6 [8] ] Suppose that the operator $T \in L(X)$ has property $(H)$. Then Weyl’s theorem holds for $T$ and $T^*$. If in addition, $T^*$ has SVEP then a-Weyl’s theorem holds for both $T$ and $T^*$.

**Theorem 1.2.7** [Theorem 2.8 [8] ] Suppose that $T$ has property $(H)$ and $T^*$ has SVEP. If $f \in H(\sigma(T))$, where $H(\sigma(T))$ is the space of all analytic functions in an open neighbourhood of $\sigma(T)$, then a-Weyl’s theorem holds for $f(T)$. 