CHAPTER 5
COMPOSITION OPERATORS

5.1 Introduction

There are several ways of producing new functions under certain conditions when two functions \( f \) and \( T \) are given and one of them is to compose them. This new function is called the composite of \( f \) and \( T \) and is denoted by the symbol \( f \circ T \), whenever the range of \( T \) is a subset of the domain of \( f \). If \( f \) varies in a linear space of functions on the range of \( T \) with pointwise linear operations, then the mapping taking \( f \) into \( f \circ T \) is a linear transformation.

Let \( X \) be a non-empty set and \( V(X) \) be a vector space of complex valued functions on \( X \) under the pointwise operations of addition and scalar multiplication. If \( T \) is a mapping of \( X \) into \( X \) such that the composite \( f \circ T \) of \( f \) with \( T \) is in \( V \) whenever \( f \) is in \( V \), then \( T \) induces a linear transformation \( C_T \) on \( V \) defined by \( C_T f = f \circ T \) for every \( f \) in \( V(X) \). If \( V(X) \) is a topological vector space and \( C_T \) is bounded, then we call it a composition operator on \( V(X) \). We are interested in the case in which \( V(X) \) is a Hilbert space.

Another way of producing a new function when two functions \( \pi \) and \( f \) are given is to multiply them whenever it makes sense. This linear transformation is known as multiplication transformation induced by \( \pi \).

The multiplication transformations and the composite transformations under suitable situations breed another class of transformation known as the weighted composite transformation which is denoted by \( W_{\pi,T} \) and defined as \( W_{\pi,T}(f) = \pi \cdot f \circ T \).
5.2 Preliminaries

Let \( X \) be a set. Then a \( \sigma \)-algebra \( \Sigma \) is non-empty collection of subsets of \( X \) satisfying,

(i) the empty set is in \( \Sigma \) (ii) if \( A \) is in \( \Sigma \), then so is the complement of \( A \) and

(iii) if \( A_n \) is a sequence of elements of \( \Sigma \), then the countable union of \( A_n \) is also in \( \Sigma \).

If \( \Sigma \) is a \( \sigma \)-algebra and \( A \) is a subset of \( X \), then \( A \) is called measurable if \( A \) is a member of \( \Sigma \).

An ordered pair \((X, \Sigma)\), where \( X \) is a set and \( \Sigma \) is a \( \sigma \)-algebra over \( X \), is called a measurable space.

If \( \lambda \) is a measure on the \( \sigma \)-algebra \( \Sigma \), then the members of \( \Sigma \) are called \( \lambda \)-measurable sets or the measurable sets.

A set \( X \) together with \( \sigma \)-algebra \( \Sigma \) on \( X \) and a measure \( \lambda \) on \( \Sigma \) is called a measure space.

A measure space \( X \) is called finite, if \( \lambda \) is a finite real number. It is called \( \sigma \)-finite, if \( X \) is the countable union of measurable sets of finite measure.

If \( X \) is a \( \sigma \)-algebra over \( S \) and \( Y \) is a \( \sigma \)-algebra over \( T \), then a function \( f : S \to T \) is measurable, if the pre image of every set in \( Y \) is in \( X \).

One important notion in Lebesgue’s theory of integration and measure is that of “almost everywhere” (a.e). The notion says that a certain property is true over an entire set except over a subset that has measure zero.

Let \((X, \Sigma, \lambda)\) be a sigma finite measure space. Let \( L^2(X, \Sigma, \lambda) \) denote the set of all complex valued measurable functions \( f \) on \( X \) such that \( |f|^2 \) is integrable with respect to \( \lambda \), i.e., \( L^2(X, \Sigma, \lambda) = \left\{ f \mid f : X \to \mathbb{C} \text{ is measurable and } \int_X |f|^2 d\lambda < \infty \right\} \).
The relation of being almost everywhere (a.e) is an equivalence relation in $L^2(X, \Sigma, \lambda)$ and this equivalence relation splits $L^2(X, \Sigma, \lambda)$ into equivalence classes. The set of all equivalence classes in $L^2(X, \Sigma, \lambda)$ is usually denoted by $L^2(\lambda)$. If we take a norm on $L^2(\lambda)$ as $\|f\| = \left( \int_{\lambda} |f|^2 \, d\lambda \right)^{1/2}$ for every $f \in L^2(\lambda)$, then under this norm, the space $L^2(\lambda)$ is a Banach space. It becomes Hilbert space under the inner product defined as $\langle f, g \rangle = \int_{\lambda} f \bar{g} \, d\lambda$ for every $f, g \in L^2(\lambda)$.

The measurable transformation $T : X \to X$ is said to be non-singular if $\lambda T^{-1}(E) = 0$ whenever $\lambda(E) = 0$ for $E \in \Sigma$.

A measure $\lambda$ is absolutely continuous with respect to another measure $\mu$ if $\lambda(E) = 0$ for every $E \in \Sigma$, with $\mu(E) = 0$. When a measure $\lambda$ is absolutely continuous with respect to a positive measure $\mu$, then it can be written as $\lambda(E) = \int_E f \, d\mu$. The function $f$ is called the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$. Sometimes it is denoted by $\frac{d\lambda}{d\mu}$.

Let $T$ be a non-singular measurable transformation from $X$ into itself. A bounded linear operator $C_T$ on $L^2(\lambda)$ defined as $C_T f = f \circ T$ for every $f$ in $L^2(\lambda)$ is said to be a composition operator induced by $T$, if (i) the measure $\lambda T^{-1}$ is absolutely continuous with respect to the measure $\lambda$ and (ii) the Radon-Nikodym derivative $\frac{d(\lambda T^{-1})}{d\lambda} = f_0$ is essentially bounded. Harrington and Whitley [52] have shown that if $C_T \in B(L^2(\lambda))$ and $C^*_T$ is its adjoint, then $C^*_T C_T f = f_0 f$ and $C_T C^*_T f = (f_0 \circ T) Pf$ for all $f \in L^2(\lambda)$, where $P$ denotes the projection of $L^2(\lambda)$ onto $\text{ran}(C_T)$ and $f_0 = \frac{d(\lambda T^{-1})}{d\lambda}$. Thus it follows that $C_T$ has dense range (i.e.)
$T^{-1}\Sigma = \Sigma$, if and only if $C_T C_T^* = f_0 \circ T$. Every essentially bounded complex valued measurable function $f_0$ induces a bounded operator $M_{f_0}$ on $L^2(\lambda)$ which is defined by $M_{f_0}f = f_0f$ for every $f \in L^2(\lambda)$. $M_{f_0}$ is called the multiplication operator induced by $f_0$. Further $C_T^* C_T = M_{f_0}$ and $C_T^* C_T^2 = M_{h_0}$ where $h_0 = \frac{d(\lambda(T \circ T))^{-1}}{d\lambda}$.

In this thesis, we denote the Radon-Nikodym derivative of the measure with respect to $\lambda$ by $h$, i.e., $f_0$ by $h$ and $\frac{d(\lambda T^{-k})}{d\lambda}$ by $h_k$ where $k$ is a positive integer greater than or equal to one and $T^k$ is obtained by composing $T$, $k$ times. Then $C_T^* C_T = M_h$ and $C_T^* C_T^2 = M_{h_2}$. In general, $C_T^* C_T^k = M_{h_k}$ where $M_{h_k}$ is the multiplication operator on $L^2(\lambda)$ induced by the complex valued measurable function $h_k$.

The following lemma due to the Harrington and Whitely [52] shows that if $C_T$ is a composition operator on $L^2(\lambda)$, then it turns out that $C_T^* C_T$ is a multiplication operator and $C_T^* C_T^*$ is close to a multiplication operator.

**Lemma 5.2.1** Let $P$ denote the projection of $L^2(\lambda)$ on $\overline{\text{ran}(C_T)}$ and $\overline{\text{ran}(C_T)} = \{f \in L^2 : f$ is $T^{-1}\Sigma$ measurable$\}$. Then $C_T C_T^* f = (h \circ T) P f$ and $C_T^* C_T f = h f$ for every $f \in L^2$.

If $w$ is a non negative complex valued $\Sigma$ measurable function, then the weighted composition operator $W$ on $L^2(\lambda)$ induced by a non singular measurable transformation $T$ from $X$ into itself with the conditions that (i) the measure $\lambda T^{-1}$ is absolutely continuous with respect to the measure $\lambda$ and (ii) the Radon-Nikodym
derivative $\frac{d(\lambda T^{-1})}{d\lambda} = h$ is essentially bounded is given by $Wf = w(f \circ T)$ for $f \in L^2(\lambda)$. In the case $w = 1$, we say $W$ is a composition operator. Let $w_k$ denote $w(w \circ T)(w \circ T^2)\ldots(w \circ T^{k-1})$. Then $W^k f = w_k(f \circ T)^k$ [73]. To examine the weighted composition operators effectively, Alan Lambert [9] associated conditional expectation operator $E$ with $T$ as $E(\bullet/T^{-1}\Sigma) = E(\bullet). E(f)$ is defined for each non-negative measure function $f \in L^p (p \geq 1)$ and is uniquely determined by the conditions:

1. $E(f)$ is $T^{-1}\Sigma$ measurable,

2. If $B$ is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges, we have

$$\int_B f d\lambda = \int_B E(f) d\lambda.$$ 

As an operator on $L^p$, $E$ is the projection onto the closure of range of $C_T$ and $E$ is the identity operator on $L^p$, if and only if, $T^{-1}\Sigma = \Sigma$. Detailed discussion of $E$ is found in [25], [46] and [48].

The following proposition due to Campbell and Jamison [25] is well-known.

**Proposition 5.2.2**

For $w \geq 0$,

1. $W^*Wf = h[E(w^2)] \circ T^{-1}f$

2. $WW^*f = w(h \circ T)E(wf)$.

**Proof.** Since $W^k f = w_k(f \circ T^k)$ and $W^* f = h_k E(w_k f) \circ T^{-k}$, we have $W^*W^k = h_k E(w_k^2) \circ T^{-k}f$, for $f \in L^2(\lambda)$ and $W^*Wf = h E(w^2) \circ T^{-1}f$ for $w \geq 0$. 

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The Aluthge transformation of $T$ introduced by Aluthge [10], is the operator $\tilde{T}$ is given by $\tilde{T} = |T|^{1/2} U |T|^{1/2}$. More generally we may form the family of operators $\{T_r : 0 < r \leq 1\}$, where $T_r = |T|^r U |T|^{1-r}$ [11].

For a composition operator $C_r$, the polar decomposition is given by $C = U |C|$ where $|C|f = \sqrt{h} f$ and $Uf = \frac{1}{\sqrt{h \circ T}} f \circ T$. In [28] Lambert has given more general Aluthge transformation for composition operators as $C_r = |C|^r U |C|^{1-r}$ as $C_r f = \left( \frac{h}{h \circ T} \right)^{r/2} f \circ T$, i.e., $C_r$ is weighted composition with weight $\pi = \left( \frac{h}{h \circ T} \right)^{r/2}$ where $0 < r < 1$. Since $C_r$ is weighted composition operator, it is easy to show that $|C_r| f = \sqrt{h} [E(\pi^2) \circ T^{-1}] f$ and $|C_r^*| f = v E(v f)$ where $v = \frac{\pi \sqrt{h \circ T}}{[E(\pi \sqrt{h \circ T})]^2}$. If $T^{-1} \Sigma = \Sigma$, then $E$ becomes identity operator and hence $C_r C_r^* f = v^4 f$.

Also we have,

$C_r^k f = \pi_k (f \circ T^k)$, where $\pi_k = \pi (\pi \circ T)(\pi \circ T^2) \cdots \cdots (\pi \circ T^{k-1})$

$(C_r^* C_r)^k f = h^k \left( E(\pi^2) \circ T^{-1} \right)^k f$

$C_r^{*k} f = h_k E(\pi_k f) \circ T^{-k}$

$C_r^{*k} C_r^k f = h_k E(\pi_k^2) \circ T^{-k} f$

$C_r^k C_r^{*k} f = \pi_k (h_k \circ T^k) E(\pi_k f)$.

5.3 $k^*$-paranormal and weighted $k^*$-paranormal composition operators

Let $H$ be Hilbert space and $B(H)$ denote Banach algebra of all bounded linear operators defined on $H$. 

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**Definition 5.3.1** [29] An operator $T$ is called $k^*$-paranormal operator for a positive integer $k$, if for every unit vector $x$ in $H$, \[\|T^k x\| \geq \|T^* x\|^k.\]

**Theorem 5.3.2** [29] For $k \geq 3$, there exists a $k^*$-paranormal operator which is not $*$-paranormal operator.

In theorem 2.2.5, we characterize $k^*$-paranormal operator as follows.

**Theorem 5.3.3** An operator $T$ is $k^*$-paranormal operator for a positive integer $k$ if and only if for any $\mu > 0$, $T^{k^*} = -k \mu^{k-1}TT^* + (k-1)\mu^k \geq 0$.

We now characterize $k^*$-paranormal composition operators and $k^*$-paranormal weighted composition operators.

**Theorem 5.3.4** For each positive integer $k$, $C_T \in B(L^2(\lambda))$ is $k^*$-paranormal if and only if $h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu^k \geq 0$ for every $\mu > 0$.

**Proof.** By theorem 5.3.3, $C_T$ is $k^*$-paranormal if and only if

$$C_T^{k^*} - k\mu^{k-1}C_TC_T^* + (k-1)\mu^k I \geq 0,$$

for every $\mu > 0$.

$$\Leftrightarrow \left\langle M_{h_k} f, f \right\rangle - k\mu^{k-1}\left\langle M_{(h \circ T)P} f, f \right\rangle + (k-1)\mu^k \left\langle f, f \right\rangle \geq 0,$$

for every $f \in L^2(\lambda)$ and $\mu > 0$.

$$\Leftrightarrow \left\langle h_k f, f \right\rangle - k\mu^{k-1}\left\langle (h \circ T)P f, f \right\rangle + (k-1)\mu^k \left\langle f, f \right\rangle \geq 0,$$

for every $f \in L^2(\lambda)$ and $\mu > 0$.

$$\Leftrightarrow \left\langle h_k \mathcal{X}_E, \mathcal{X}_E \right\rangle - k\mu^{k-1}\left\langle (h \circ T)P \mathcal{X}_E, \mathcal{X}_E \right\rangle + (k-1)\mu^k \left\langle \mathcal{X}_E, \mathcal{X}_E \right\rangle \geq 0,$$

for every characteristic function $\mathcal{X}_E$ of $E$ in $\Sigma$ such that $\lambda(E) < \infty$.

$$\Leftrightarrow \int_E \left( h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu^k \right) d\lambda \geq 0$$

for every $E$ in $\Sigma$.

$$\Leftrightarrow h_k - k\mu^{k-1}(h \circ T)P + (k-1)\mu^k \geq 0$$

a.e for every $\mu > 0$.

**Corollary 5.3.5** For each positive integer $k$, $C_T \in B(L^2(\lambda))$ is $k^*$-paranormal if and only if $h_k \geq (h^k \circ T)P$ a.e.
**Proof.** $C_T$ is $k^*$-paranormal if and only if $h_k - k\mu^{k-1} (h \circ T) P + (k - 1) \mu^k \geq 0$ a.e.

if and only if $\mu > (h \circ T) P$.

Hence $C_T$ is $k^*$-paranormal if and only if

$$h_k - k((h \circ T) P)^{k-1} (h \circ T) P + (k - 1)((h \circ T) P)^k \geq 0 \text{ a.e.}$$

$$\iff h_k - k((h \circ T) P)^k P + (k - 1)(h \circ T)^k P \geq 0 \text{ a.e.}$$

$$\iff h_k \geq (h \circ T)^k P \text{ a.e.}$$

$$\iff h_k \geq (h^k \circ T) P \text{ a.e.}$$

**Theorem 5.3.6** For each positive integer $k$, $C_T^*$ is $k^*$-paranormal if and only if

$$(h_k \circ T^k) P_k \geq h^k \text{ a.e., }$$

where $P_i'$s are the projections of $L^2(\lambda)$ onto $\operatorname{ran}(C_T^i)$.

**Proof.** By theorem 5.3.3, $C_T^*$ is $k^*$-paranormal if and only if

$$C_T^k C_T^{*-k} - k\mu^{k-1} C_T^{*-k} C_T + (k - 1)\mu^k \geq 0 \text{ for } \mu > 0.$$ 

$$\iff \left( C_T^k C_T^{*-k} - k\mu^{k-1} C_T^{*-k} C_T + (k - 1)\mu^k \right) \geq 0 \text{ for all } g \in L^2(\lambda).$$

$$\iff h_k \circ T^k P_k - k\mu^{k-1} h + (k - 1)\mu^k \geq 0 \text{ a.e. for all } \mu > 0$$

$$\iff (h_k \circ T^k) P_k \geq h^k \text{ a.e.}$$

**Corollary 5.3.7** If $C_T \in B(L^2(\lambda))$ has dense range, then $C_T^*$ is $k^*$-paranormal if and only if $h_k \circ T^k \geq h^k \text{ a.e.}$

**Proof.** Since $T$ has dense range, we have $C_T C_T^* f = (h \circ T) f$.

Hence it follows that $C_T^*$ is $k$-paranormal if and only if

$$h_k \circ T^k - k\mu^{k-1} h + (k - 1)\mu^k \geq 0 \text{ a.e. for every } \mu > 0.$$ 

e.g., $h_k \circ T^k \geq h^k \text{ a.e.}$

**Corollary 5.3.8** If $C_T \in B(L^2(\lambda))$ has dense range, then both $C_T$ and $C_T^*$ are $k^*$-paranormal if and only if $h_k \circ T^k \geq \max \left( h^k, h^k \circ T^{k+1} \right)$ a.e.
Proof. \( h_k \circ T^k \geq (h_k \circ T) \circ T^k = h_k \circ T^{k+1} \in \text{ran}(h_k) \), then both \( C_T \) and \( C_T^* \) are \( k^* \)-paranormal.

Now we characterize weighted \( k^* \)-paranormal composition operators as follows.

Theorem 5.3.9 Let \( W \in B(L^2(\lambda)) \) be the weighted composition operator with weight \( w > 0 \), induced by \( T \). Then \( W \) is \( k^* \)-paranormal if and only if

\[
h_k E(w_k^2) \circ T^{-k} - k \mu^{k-1} w(h \circ T) E(w) + (k - 1) \mu^k \geq 0 \quad \text{a.e. for } \mu > 0.
\]

Proof. Since \( W^k f = w_k (f \circ T^k) \) and \( W^{k^*} f = h_k E(w_k f) \circ T^{-k} \), we have

\[
W^{k^*} W^k f = h_k E(w_k^2) \circ T^{-k} f \quad \text{and } W^* W f = f \circ [E(w^2)] \circ T^{-1} f \quad \text{for } w \geq 0.
\]

\( W \) is \( k^* \)-paranormal

\[
\Leftrightarrow W^{k^*} W - k \mu^{k-1} W W^* + (k - 1) \mu^k I \geq 0 \quad \text{for } \mu > 0.
\]

\[
\Leftrightarrow \left\{ W^{k^*} W - k \mu^{k-1} W W^* + (k - 1) \mu^k I \right\} f, f \geq 0 \quad \text{for all } f \in L^2(\lambda).
\]

\[
\Leftrightarrow \int_E \left[ h_k E(w_k^2) \circ T^{-k} - k \mu^{k-1} w(h \circ T) E(w) + (k - 1) \mu^k \right] d\lambda \geq 0 \quad \text{a.e. for } E \in \Sigma
\]

\[
\quad \text{and } \mu > 0.
\]

\[
\Leftrightarrow h_k E(w_k^2) \circ T^{-k} - k \mu^{k-1} w(h \circ T) E(w) + (k - 1) \mu^k \geq 0 \quad \text{a.e. for } \mu > 0.
\]

Corollary 5.3.10 Let \( T^{-1} \Sigma = \Sigma \). Then \( W \) is \( k^* \)-paranormal if and only if

\[
h_k w_k^2 \circ T^{-k} - k \mu^{k-1} w^2(h \circ T) + (k - 1) \mu^k \geq 0 \quad \text{a.e. for } \mu > 0.
\]

Corollary 5.3.11 Let \( C_r \in B(L^2(\lambda)) \). Then \( C_r \) is \( k^* \)-paranormal if and only if

\[
h_k E(\pi_k^2) \circ T^{-k} - k \mu^{k-1} \pi(h \circ T) E(\pi) + (k - 1) \mu^k \geq 0 \quad \text{a.e. for } \mu > 0.
\]

Proof. Since \( C_r \) is weighted composition operator with weight \( \pi = \left( \frac{h}{h \circ T} \right)^r \), it follows that \( C_r \) is \( k^* \)-paranormal if and only if
\[
\int_{E} \left( h_k E(\pi_k^2) \circ T^{-k} - k\mu^{k-1} w(h \circ T) E(\pi) + (k - 1)\mu^k \right) |f|^2 d\lambda \geq 0 \text{ for every } E \in \Sigma.
\]
\[
\Leftrightarrow h_k E(\pi_k^2) \circ T^{-k} - k\mu^{k-1} w(h \circ T) E(\pi) + (k - 1)\mu^k \geq 0 \text{ a.e. for every } \mu > 0.
\]

**Corollary 5.3.12** Suppose \( T^{-1} \Sigma = \Sigma \). Then \( W \) is \( k \)-paranormal if and only if
\[
h_k \pi_k^2 \circ T^{-k} - k\mu^{k-1} w(h \circ T) \pi - (k - 1)\mu^k \geq 0 \text{ a.e. for every } \mu > 0.
\]

### 5.4 \( k \)-paranormal and weighted \( k \)-paranormal composition operators

In this section, we characterize \( k \)-paranormal composition operator.

**Definition 5.4.1** An operator \( T \) is called \( k \)-paranormal if
\[
\|T^{k+1}x\| \geq \|Tx\|^{k+1}
\]
for some positive integer \( k \geq 1 \) and for every \( x \in H \). Equivalently, \( T \) is called \( k \)-paranormal if
\[
\|T^{k+1}x\| \geq \|Tx\|^{k+1}
\]
for some integer \( k \geq 1 \) and for every unit vector \( x \in H \).

A paranormal operator is simply \( 1 \)-paranormal operator. Also a paranormal operator is \( k \)-paranormal for every \( k \geq 1 \).

Ando[14] has characterized paranormal operators as follows.

**Proposition 5.4.2** [14] An operator \( T \in B(H) \) is paranormal if and only if
\[
T^* T^2 - 2kT^* T + K^2 \geq 0 \text{ for every } k \in R.
\]

Generalizing this, Yuan and Gao in [91] characterize \( k \)-paranormal operators as follows.

**Theorem 5.4.3** [91] For each positive integer \( k \), an operator \( T \in B(H) \) is \( k \)-paranormal if and only if
\[
T^{*k} T^* - (1 + k) \mu^k T^* T + k\mu^{1+k} I \geq 0 \text{ for every } \mu \geq 0.
\]

Using this theorem we characterize the composition operators induced by \( k \)-paranormal operators.
Theorem 5.4.4 For each positive integer $k$, $C_T \in B(L^2(\lambda))$ is $k$-paranormal if and only if $h_{1+k} - (1 + k)\mu^k h + k\mu^{1+k} \geq 0$ a.e. for every $\mu > 0$.

Proof. By theorem 5.4.3,

$C_T$ is $k$-paranormal

$\iff C_T^{1+k} - (1 + k)\mu^k C_T^* C_T + k\mu^{1+k}I \geq 0$ for every $\mu > 0$.

$\iff \left\langle C_T^{1+k} C_T^{1+k} f, f \right\rangle - (1 + k)\mu^k \left\langle C_T^* C_T f, f \right\rangle + k\mu^{1+k} \left\langle f, f \right\rangle \geq 0$

for every $f \in L^2(\lambda)$ and $\mu > 0$.

$\iff \left\langle M_{h_{1+k}} f, f \right\rangle - (1 + k)\mu^k \left\langle M_h f, f \right\rangle + k\mu^{1+k} \left\langle f, f \right\rangle \geq 0$.

$\iff \left\langle h_{1+k} f, f \right\rangle - (1 + k)\mu^k \left\langle h f, f \right\rangle + k\mu^{1+k} \left\langle f, f \right\rangle \geq 0$.

$\iff \left\langle h_{1+k} \chi_E, \chi_E \right\rangle - (1 + k)\mu^k \left\langle h \chi_E, \chi_E \right\rangle + k\mu^{1+k} \left\langle \chi_k, \chi_k \right\rangle \geq 0$

for every characteristic function $\chi_E$ of $E$ in $\Sigma$ such that $\lambda(E) < \infty$.

$\iff \int_E \left( h_{1+k} - (1 + k)\mu^k h + k\mu^{1+k} \right) d\lambda \geq 0$ for every $E$ in $\Sigma$.

$\iff h_{1+k} - (1 + k)\mu^k h + k\mu^{1+k} \geq 0$ a.e. for every $\mu > 0$.

Corollary 5.4.5 For each positive integer $k$, $C_T \in B(L^2(\lambda))$ is $k$-paranormal if and only if $h_{1+k} \geq h^{1+k}$ a.e.

Example 5.4.6

Let $X = N$, the set of all natural numbers and $\lambda$ be the counting measure on it. Define $T : N \to N$ by

$T(1) = T(2) = 1, T(3) = 2, T(4n + m - 1) = n + 2$ for $m = 1, 2, 3, 4$ and $n \in N$.

Then for each $k \geq 3$, $h_{1+k}(n) \geq h^{1+k}(n)$ for every $n \in N$. 

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Hence $T$ is $k$-paranormal for each $k = 3, 4, 5, \cdots$

**Theorem 5.4.7** Let $C_T \in B(L^2(\lambda))$. Then for each positive integer $k$, $C_T^*$ is $k$-paranormal if and only if $h^{1+k} \circ TP_1 \leq h_{1+k}^{1+k} \circ T^{1+k} P_{1+k}$ a.e. where $P_i$'s are the projections of $L^2(\lambda)$ onto $\text{ran}(C_T^i)$.

**Proof.** By theorem 5.4.3, $C_T^*$ is $k$-paranormal if and only if

$$C_T^{1+k} C_T^{s+1+k} - (1 + k) \mu^k C_T^* C_T + k \mu^{1+k} I \geq 0$$

i.e.,

$$\langle C_T^{1+k} C_T^{s+1+k} f, f \rangle - (1 + k) \mu^k \langle C_T^* C_T f, f \rangle + k \mu^{1+k} \langle f, f \rangle \geq 0$$

for all $f \in L^2(\lambda)$.

We have

$$\langle C_T C_T^* f, f \rangle = \langle (h \circ T^*) P, f \rangle.$$ 

Thus $C_T^*$ is $k$-paranormal if and only if

$$\langle h_{1+k} \circ T^{1+k} P_{1+k} f, f \rangle - (1 + k) \mu^k \langle (h \circ TP_1) f, f \rangle + k \mu^{1+k} \langle f, f \rangle \geq 0$$

a.e.

for all $f \in L^2(\lambda)$ and all $\mu > 0$.

\[ \Leftrightarrow h_{1+k} \circ T^{1+k} P_{1+k} - (1 + k) \mu^k h \circ TP_1 + k \mu^{1+k} \geq 0 \] a.e. for every $\mu > 0$.

which is equivalent to, $h^{1+k} \circ TP_1 \leq h_{1+k}^{1+k} \circ T^{1+k} P_{1+k}$ a.e.

**Corollary 5.4.8** If $C_T \in B(L^2(\lambda))$ has dense range, then $C_T^*$ is $k$-paranormal if and only if $h_{1+k} \circ T^{1+k} \geq h^{1+k} \circ T$ a.e.

**Proof.** Since $T$ has dense range, we have $C_T C_T^* f = (h \circ T) f$.

Hence it follows that $C_T^*$ is $k$-paranormal if and only if

$$h_{1+k} \circ T^{1+k} - (1 + k) \mu^k h \circ T + k \mu^{1+k} \geq 0$$

a.e. for every $\mu > 0$.

i.e., $h_{1+k} \circ T^{1+k} \geq h^{1+k} \circ T$ a.e.
**Theorem 5.4.9** Let $W \in B(L^2(\lambda))$ be a weighted composition operator with weight $w > 0$. Then $W$ is $k$-paranormal if and only if $h_{1+k} E(w_{k+1}^2 \circ T^{-1+k}) - (1 + k) \mu_k W^* W + k \mu_1^{1+k} I \geq 0$ a.e. for every $\mu > 0$.

**Proof.** $W$ is $k$-paranormal,

\[ W^* W^{1+k} - (1 + k) \mu_k W^* W + k \mu_1^{1+k} I \geq 0 \text{ for every } \mu > 0. \]

\[ \Rightarrow \left\{ \left( W^* W^{1+k} - (1 + k) \mu_k W^* W + k \mu_1^{1+k} I \right) f, f \right\} \geq 0 \text{ for every } f \in L^2(\lambda) \]

Hence,

\[ \int_{\mathcal{E}} \left( h_{1+k} E(w_{k+1}^2 \circ T^{-1+k}) - (1 + k) \mu_k hE(w^2) \circ T^{-1} + k \mu_1^{1+k} \right) d\lambda \geq 0 \text{ a.e.} \]

for every $E \in \Sigma$ and $\mu > 0$ and so

\[ h_{1+k} E(w_{k+1}^2 \circ T^{-1+k}) - (1 + k) \mu_k hE(w^2) \circ T^{-1} + k \mu_1^{1+k} \geq 0 \text{ a.e. for every } \mu > 0. \]

Converse is also true.

**Corollary 5.4.10** Let $T^{-1}\Sigma = \Sigma$. Then $W$ is $k$-paranormal if and only if $h_{1+k} w_{k+1}^2 \circ T^{-1+k} - (1 + k) \mu_k h w^2 \circ T^{-1} + k \mu_1^{1+k} \geq 0$ a.e. for every $\mu > 0$.

**Corollary 5.4.11** Let $C_r \in B(L^2(\lambda))$. Then $C_r$ is $k$-paranormal if and only if $h_{1+k} E(\pi_{k+1}^2 \circ T^{-1+k}) - (1 + k) \mu_k hE(\pi^2) \circ T^{-1} + k \mu_1^{1+k} \geq 0$ a.e. for every $\mu > 0$.

**Proof.** Since $C_r$ is weighted composition operator with weight $\pi = \left( \frac{h}{h \circ T} \right)^r$, it follows that $C_r$ is $k$-paranormal if and only if

\[ h_{1+k} E(\pi_{k+1}^2 \circ T^{-1+k}) - (1 + k) \mu_k hE(\pi^2) \circ T^{-1} + k \mu_1^{1+k} \geq 0 \text{ a.e. for every } \mu > 0. \]

5.5 **$ek$-paranormal and weighted $ek$-paranormal composition operators**

In this section we characterize extended $k$-paranormal composition operators or $ek$-paranormal composition operators in short.
Definition 5.5.1 An operator $T$ satisfying the condition that
$$\|T^{2+k}\|^{\frac{1}{1+k}} \|T\|^{\frac{k}{1+k}} \geq \|T^2\|$$
for some integer $k \geq 1$ and for every $x \in H$ is called extended $k$-paranormal operator or in short $ek$-paranormal operator.

Theorem 5.5.2 For each positive integer $k$, an operator $T$ is $ek$-paranormal if and only if
$$T^{2+k} T^{-2} - (1+k)\mu^k T^{2} + k\mu^{1+k} T^2 T \geq 0$$
for every $\mu > 0$.

Theorem 5.5.3 Let $C_T$ be the composition operator induced by $T$. $C_T$ is $ek$-paranormal if and only if
$$h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k} h \geq 0$$
a.e. for every $\mu > 0$ and for every positive integer $k$.

Proof. $C_T$ is $ek$-paranormal for a positive integer $k$.

$$C_{2+k}^{-1} C^{-2} - (1+k)\mu^k C^2 + k\mu^{1+k} C T \geq 0$$
for every $f \in L^2(\lambda)$ and $\mu > 0$.

$$\langle h_{2+k} f, f \rangle - (1+k)\mu^k \langle h_2 f, f \rangle + k\mu^{1+k} \langle hf, f \rangle \geq 0$$
for every characteristic function $\chi_E$ of $E$ in $\Sigma$ such that $\lambda(E) < \infty$.

$$\langle h_{2+k} \chi_E, \chi_E \rangle - (1+k)\mu^k \langle h_2 \chi_E, \chi_E \rangle + k\mu^{1+k} \langle h \chi_E, \chi_E \rangle \geq 0$$
for every positive integer $k$.

$$\int_E (h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k} h) \chi_E \overline{\chi_E} d\lambda \geq 0$$
$$h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k} h \geq 0$$
a.e. for every $\mu > 0$. 

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Corollary 5.5.4 $C_T$ is $ek$-paranormal for a positive integer $k$ if and only if $h_2^{k+1} \leq h_{2+k} h^k$ a.e.

We now characterize weighted $ek$-paranormal composition operators as follows.

Theorem 5.5.5 Let $W \in B(L^2(\lambda))$ be a weighted composition operator with weight $w > 0$. Then $W$ is $ek$-paranormal if and only if

$$h_{k+2} E(w_{k+2}^2) \circ T^{-(2+k)} - (1 + k) \mu^k h_2 E(w^2) \circ T^{-2} + k \mu^{1+k} h E(w^2) \circ T^{-1} \geq 0 \text{ a.e.}$$

for every $\mu > 0$.

Proof. Since $W$ is of $ek$-paranormal,

$$W^{2+k} W^{2+k} - (1 + k) \mu^k W^{-2} W^2 + k \mu^{1+k} W W \geq 0 \text{ for every } \mu > 0.$$

$$\Rightarrow \left\{ \left( W^{2+k} W^{2+k} - (1 + k) \mu^k W^{-2} W^2 + k \mu^{1+k} W W \right) f, f \right\} \geq 0 \text{ for every } f \in L^2(\lambda)$$

and all $\mu > 0$.

As $W^k f = w_k (f \circ T^k)$ and $W^{sk} f = h_k E(w_k^2) \circ T^{-k}$,

$$W^{sk} W^k f = h_k E(w_k^2) \circ T^{-k} f \text{ and } W^* Wf = h [E(w^2)] \circ T^{-1} f \text{ for } w \geq 0.$$

Hence, $W$ is of $ek$-paranormal if and only if

$$\int_E \left( h_{k+2} E(w_{k+2}^2) \circ T^{-(2+k)} - (1 + k) \mu^k h_2 E(w^2) \circ T^{-2} + k \mu^{1+k} h E(w^2) \circ T^{-1} \right) d\lambda \geq 0 \text{ a.e.}$$

for every $E \in \Sigma$ and $\mu > 0$.

if and only if

$$h_{k+2} E(w_{k+2}^2) \circ T^{-(2+k)} - (1 + k) \mu^k h_2 E(w^2) \circ T^{-2} + k \mu^{1+k} h E(w^2) \circ T^{-1} \geq 0 \text{ a.e.}$$

for every $\mu > 0$.

Corollary 5.5.6 Let $T^{-1} \Sigma = \Sigma$. Then $W$ is $ek$-paranormal if and only if

$$h_{k+2} w_{k+2}^2 \circ T^{-(2+k)} - (1 + k) \mu^k h_2 w_2^2 \circ T^{-2} + k \mu^{1+k} h w^2 \circ T^{-1} \geq 0 \text{ a.e. for every } \mu > 0.$$

Corollary 5.5.7 Let $C_r \in B(L^2(\lambda))$. Then $C_r$ is $ek$-paranormal if and only if

$$h_{k+2} E(\pi_{k+2}^2) \circ T^{-(2+k)} - (1 + k) \mu^k h_2 E(\pi_2^2) \circ T^{-2} + k \mu^{1+k} h E(\pi^2) \circ T^{-1} \geq 0 \text{ a.e. for every } \mu > 0.$$

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CONCLUSION

We conclude this thesis with a fitting sum up. The thesis begins with a survey on Weyl’s theorem and Weyl type theorems. It has been proved that Riesz idempotent operator with respect to isolated eigen value of $k^*$-paranormal operator is self adjoint. $k^*$-paranormal operator has been characterized. Some spectral properties such as polaroid, reguloid and isoloid have been proved for $k^*$-paranormal operator. Spectral mapping theorem for essential approximate point spectrum and generalized Weyl’s theorem have been proved for algebraically $k^*$-paranormal and algebraically $k$-paranormal operators. An operator $T \in B(H)$ is said to be $m$-quasi $k$-paranormal operator for positive integers $m$ and $k$, if for all $x \in H$, $\|T^{m+k+1}x\| \|T^{m}x\|^k \geq \|T^{m+1}x\|^{k+1}$. This operator has been characterized and explained with an example that it is not normaloid. Weyl’s theorem and Weyl type theorems are investigated for algebraically $m$-quasi $k$-paranormal operator $T$. We have showed that if $T^*$ has single valued extension property, then a-Weyl’s theorem is true for $f(T)$ for every $f \in H(\sigma(T))$. Further $k^*$-paranormal composition operator, $k$-paranormal composition operator and $m$-quasi $k$-paranormal composition operator on $L^2$ space are characterized. We also have characterized weighted $k^*$-paranormal composition operator, $k$-paranormal composition operator and $m$-quasi $k$-paranormal composition operator on $L^2(\lambda)$.

On the basis of the results of this thesis, further research can be conducted whether Fuglede-Putnam theorem and Berger-Shaw’s theorem can hold for m-quasi k –paranormal operators. Also, new spectral properties namely,(Bw), (Bb) and (Bab) that have been introduced by Anuradha Gupta and Neeru Kashyap in “Variations on Weyl type theorems” in connection with Weyl type theorems, can be verified for $k$-paranormal operators.